ERRATA AND COMPLEMENT FOR "SUPERSPECIAL ABELIAN VARIETIES, THETA SERIES AND THE JACQUET-LANGLANDS CORRESPONDENCE"

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1. Introduction

This short note corrects some minor mistakes contained in my Ph.D. thesis (typically, a missing factor 2 here or there) and describes an interesting link to the Doi-Naganuma lifting in the ramified case which was obfuscated by one of those mistakes. The main results of my thesis are to be found in the published versions of [1] and [2].

2. Chapter I.

2.1. **Section 1.4. Traverso's Boundedness Conjecture. p.34-36.** It is best to simply ignore Section 1.4 and Subsection 1.5.1., as they are not used in the rest of the thesis. For readers interested in the truncation conjectures of Traverso, Section 1.4. is superseded by [3] (which contains also a polarized variant of the integral truncation conjecture in the supersingular case) and complemented by [4] (which establishes the rational truncation conjecture in general).
For interested readers, note that Lemma 1.4.4 is wrong. A weaker assertion holds:

Proposition 2.1. *Let $\mathbb{D}_1$ be determined by its truncation modulo p^{n_1} up to isomorphism. Let $\mathbb{D}_2 \longrightarrow \mathbb{D}_1$ be an isogeny whose kernel is annihilated by p^{n_2}. Then $\mathbb{D}_2$ is determined by its truncation modulo $p^{2n_2+n_1}$ up to isomorphism.*

This affects in particular Corollary 1.4.7, where the correct statement should in particular replace $n+1$ with $2n+1$. Ibidum for Theorem 1.5.3., where the final bound is 3, not 2 (even though the optimal bound proves to be 2 indeed).
As those statements are not used anywhere in the thesis (and are proven elsewhere), we skip the details.

3. Chapter II.

3.1. **Section 2.8. The ramified case.** Our results in the ramified case are significantly more striking in their corrected form. In the middle of page 114, $\mathfrak{p}$ was mistaken for $\mathfrak{p}^2$. As a consequence, in Theorem 2.8.5., the possible levels are thus $\mathfrak{p}^{g-2j}$, and not $\mathfrak{p}^{g-j}$. E.g., for $g=2$, $p=\mathfrak{p}^2$, the levels are either $\mathfrak{p}^2$ or 1.
In Remark 2.8.6, the parenthesis should be erased (there is no "extra condition"), but the levels are $\mathfrak{p}^{g-2j}$.
Question 2.8.11 should be reformulated with $g-j$ replaced by $g-2j$.
Proposition 2.8.12. can be generalized to all degrees: the superspecial points of level 1 (g even) or $\mathfrak{p}$ (g odd) having maximal endomorphism orders.

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As a consequence of the correction of this mistake, a link can be established in the ramified case between Hilbert modular varieties modulo p and quadratic base change (i.e., the Doi-Naganuma lift). We sketch this for the convenience of the reader in the simplest case i.e., restricting to $L = \mathbb{Q}(\sqrt{p})$, $p = 1 \mod 4$, $h^+(L) = 1$. Cf. [2] for details.

The Doi-Naganuma lift is an embedding of the space of modular forms of level p with non-trivial quadratic character:

$$S_2(\Gamma_0(p), \chi_p) \hookrightarrow S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L)).$$

Note that $B_{p,\infty} \otimes L \cong B_{\infty_1,\infty_2}$. Denote by $x \mapsto \overline{x}$ the quaternion involution, and by $x \mapsto x^*$ the $\mathbb{Q}$-automorphism of B_{∞_1,∞_2} extending the non-trivial Galois involution $\sigma : L \longrightarrow L$ and which is trivial on $B_{p,\infty}$.
Let $V := \{x \in B_{\infty_1,\infty_2} | x^* = \overline{x}\}$. With the reduced norm, it is a quadratic space over $\mathbb{Q}$ of discriminant p (not p^2 !).

Proposition 3.1. ([5]) *There is a **bijection** between: left $\mathcal{O}$-ideal classes Λ for a symmetric maximal order of B_{∞_1,∞_2} and proper similitude classes of lattices with reduced discriminant p given by the map $\Lambda \mapsto \overline{\Lambda^*}\Lambda \cap V$.*

$$\begin{tikzcd}
\mathcal{O}\text{-ideal classes } \Lambda \arrow[r, "\textbf{Bijection}", "\Lambda \mapsto \overline{\Lambda^*}\Lambda \cap V"'] \arrow[d, "\text{theta series}"'] & \text{Lattices of disc. } p \arrow[d, "\text{theta series}"] \\
S_{2,2}(\mathrm{SL}_2(\mathcal{O}_L)) & \text{Theta span in } S_2(\Gamma_0(p), \chi_p) \arrow[l, "Doi-Naganuma"']
\end{tikzcd}.$$

Remark 3.2. ***Warning:*** *from the diagram, we see that we need the basis problem for Nebentypus to hold (there exists a well-known necessary and sufficient condition, which fails for $p = 389$). Alternatively, one can avoid weight two by using spherical harmonics* ([6]).

All parts of the bijection allow a geometric interpretation, that we outline below. In a nutshell, the singular points of the Hilbert modular surface provide all desired modular forms.

Quaternion algebra	**Abelian surfaces with $\mathcal{O}_L$-action**
Quaternion involution	Dual
Norm form	$\mathcal{O}_L$-degree $\|\| - \|\|_{\mathcal{O}_L}$
Galois induced involution *	$1_{\mathrm{End}(E)} \otimes \sigma$
$V := \{x \in B_{\infty_1,\infty_2} \| x^* = \overline{x}\}$	$\{x \in B_{\infty_1,\infty_2} \| (1 \otimes \sigma)(x) \cdot x = \|\|x\|\|_{\mathcal{O}_L}\}$
Symmetric maximal order $\mathcal{O}$	$\mathrm{End}_{\mathcal{O}_L}(E \otimes_{\mathbb{Z}} \mathcal{O}_L/H) = \mathcal{O}, H \cong \alpha_p$
$\mathcal{O}$-ideal classes	$\mathrm{Hom}_{\mathcal{O}_L}(E \otimes_{\mathbb{Z}} \mathcal{O}_L/H, A)$

Remark 3.3. *Relatively **quadratic** base change follows the same pattern: in particular, there is a* **canonical chain** *of α_p-isogenies from $E \otimes \mathcal{O}_L$.*

References

[1] Nicole, M.-H., Superspecial Abelian Varieties and the Eichler Basis Problem for Hilbert Modular, Forms, preprint, 2006.
[2] Nicole, M.-H., Hilbert modular varieties over totally ramified primes: Revisiting Eichler, Manin and Doi-Naganuma, in preparation.

[3] Nicole, M.-H., Vasiu, A., Minimal Truncations of Supersingular p-divisible Groups, 9 pages, arxiv math.NT/0606777.

[4] Nicole, M.-H., Vasiu, A., Traverso's Isogeny Conjecture for p-divisible Groups, 7 pages, arxiv math.NT/0606780.

[5] Ponomarev, P., The Doi-Naganuma lifting of quaternary theta series. *Math. Ann.* **255** (1981), no. 4, 443–452.

[6] Waldspurger, J.-L. Formes quadratiques à 4 variables et relèvement. *Acta Arith.* **36** (1980), no. 4, 377–405.

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