SPECTRAL SEQUENCES

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1. Introduction

Definition 1. Let $a \geq 1$. An a-th stage spectral (cohomological) sequence consists of the following data:

- bigraded objects $E_r = \bigoplus_{p,q \in \mathbb{Z}} E_r^{p,q}$, $r \geq a$
- differentials $d_r : E_r \to E_r$ such that $d_r(E_r^{p,q}) \subseteq E_r^{p+r,q-r+1}$

satisfying $H(E_r) = E_{r+1}$, i.e.,

$$E_{r+1}^{p,q} = \frac{\ker(E_r^{p,q} \to E_r^{p+r,q-r+1})}{\operatorname{im}(E_r^{p-r,q+r-1} \to E_r^{p,q})}.$$

We usually draw the r-th stage of a spectral sequence in a tabular format with p increasing horizontally to the left and q increasing vertically to the right:

$$\begin{tikzcd}
E_0^{0,1} & E_0^{1,1} & E_0^{2,1} & E_0^{3,1} & \cdots \\
E_0^{0,0} \arrow[u] & E_0^{1,0} \arrow[u] & E_0^{2,0} \arrow[u] & E_0^{3,0} \arrow[u] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
E_1^{0,1} \arrow[r] & E_1^{1,1} \arrow[r] & E_1^{2,1} \arrow[r] & E_1^{3,1} \arrow[r] & \cdots \\
E_1^{0,0} \arrow[r] & E_1^{1,0} \arrow[r] & E_1^{2,0} \arrow[r] & E_1^{3,0} \arrow[r] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} \arrow[drr] & E_2^{3,1} & \cdots \\
E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & \cdots
\end{tikzcd}$$

2. The spectral sequence of a filtered complex

Let $K^\cdot = F^0 K^\cdot \supseteq F^1 K^\cdot \supseteq \cdots$ be a filtered complex (i.e., each object K^n in the complex $K^\cdot$ is filtered and the differentials of the complex $K^\cdot$ respect the filtration). We set $\operatorname{Gr}^p K^\cdot = F^p K^\cdot / F^{p+1} K^\cdot$. Note that the filtration on $K^\cdot$ induces a filtration $H(K^\cdot) = F^0 H(K^\cdot) \supseteq F^1 H(K^\cdot) \supseteq \cdots$ on cohomology in a natural way. Given this setup, one may prove the following.

Theorem 2. *Suppose that $K^\cdot$ is a nonnegative filtered complex (i.e., $K^n = 0$ if $n < 0$). Then there exists a spectral sequence $E_r^{p,q}$ satisfying*

- $E_0^{p,q} = \operatorname{Gr}^p K^{p+q}$
- $E_1^{p,q} = H^{p+q}(\operatorname{Gr}^p K^\cdot)$
- $E_r^{p,q} = \operatorname{Gr}^p H^{p+q}(K^\cdot)$ *for* $r = r(p,q) \gg 0$.

Sketch of proof. Set $E_0^{p,q} = \operatorname{Gr}^p K^{p+q}$. As differential of the complex $K^\cdot$ respects the filtration, it induces a map $d_0 : E_0^{p,q} \to E_0^{p,q+1}$. To realize $E_r^{p,q}$, $r \geq 1$, one defines the set

$$Z_r^{p,q} = \{x \in F^p K^{p+q} \mid dx \in F^{p+r} K^{p+q+1}\}$$

of 'cocycles modulo $F^{p+r}K^{p+q+1}$' and sets

$$E_r^{p,q} = Z_r^{p,q}/(\text{appropriate coboundaries}).$$

The differential d then induces the map $d_r : E_r \to E_r$ (it is at least evident from the above that we get a map $Z_r^{p,q} \to Z_r^{p+r,q-r+1}$). It remains to check the desired properties of these object. We omit this; for details see [3, Ch. XX, Proposition 9.1]. $\square$

The spectral sequence whose existence is asserted in the above theorem is an example of a *first quadrant spectral sequance*, by definition a spectral sequence such that $E_r^{p,q}$ is zero unless $p, q \geq 0$. It is easy to see that in a first quadrant spectral sequence, $E_r^{p,q} = E_{r+1}^{p,q} = \cdots$ if $r > \max(p, q+1)$. In the situation of the theorem, this stable value is $\operatorname{Gr}^p H^{p+q}(K^\cdot)$.

3. Convergence of spectral sequences

Let $E_r^{p,q}$ be a spectral sequence, and suppose that for every pair (p,q), the term $E_r^{p,q}$ stabilizes as $r \to \infty$ (a first quadrant spectral sequence, for example). Denote this stable value by $E_\infty^{p,q}$. Let H^n be a collection of objects with *finite* filtrations

$$0 \subseteq F^s H^n \subseteq \cdots \subseteq F^t H^n = H^n.$$

We say that $E_r^{p,q}$ converges to $H^\cdot$, and write $E_r^{p,q} \Rightarrow H^{p+q}$, if

$$E_\infty^{p,q} = F^p H^{p+q}/F^{p+1}H^{p+q} = \operatorname{Gr}^p H^{p+q}.$$

Example 3. Suppose $K^\cdot$ is a nonnegative filtered complex. Then by Theorem 2, we have a convergent spectral sequence

$$E_1^{p,q} = H^{p+q}(\operatorname{Gr}^p K^\cdot) \Rightarrow H^{p+q}(K^\cdot).$$

Given this definition of convergence, one is led immediately to ask to what extent the limit of a spectral sequence is determined by the sequence itself. The next two lemmas give some basic though useful results in this direction.

Lemma 4. *Suppose* $E_r^{p,q} \Rightarrow H^{p+q}$.

(1) *If* $E_\infty^{p,q} = 0$ *unless* $q = q_0$*, then* $H^n = E_\infty^{n-q_0,q_0}$.
(2) *If* $E_\infty^{p,q} = 0$ *unless* $p = p_0$*, then* $H^n = E_\infty^{p_0,n-p_0}$.

Proof. We prove (1), the proof of (2) being similar. Consider the filtration

$$0 \subseteq \cdots \subseteq F^{n-q_0+1}H^n \subseteq F^{n-q_0}H^n \subseteq F^{n-q_0-1}H^n \subseteq \cdots \subseteq H^n.$$

Since $E_\infty^{p,q} = 0$ unless $q = q_0$, the only nonzero quotient of this filtration is $F^{n-q_0}H^n/F^{n-q_0+1}H^n = E_\infty^{n-q_0,q_0}$. Thus, $F^{n-q_0+1}H^n = 0$ and $F^{n-q_0}H^n = H^n$, implying

$$H^n = F^{n-q_0}H^n = H^n/F^{n-q_0+1}H^n = E_\infty^{n-q_0,q_0}.$$

$\square$

The next result shows that in certain cases the form of the filtration on the limit object is determined by the spectral sequence.

Lemma 5. *Suppose $E_r^{p,q} \Rightarrow H^{p+q}$.*

(1) *If $E_r^{p,q}$ is a first quarter spectral sequence, then H^n has a filtration of the form*
$$0 = F^{n+1}H^n \subseteq F^n H^n \subseteq \cdots \subseteq F^1 H^n \subseteq F^0 H^n = H^n.$$
(2) *If $E_r^{p,q}$ is a third quarter spectral sequence, then H^{-n} has a filtration of the form*
$$0 = F^1 H^{-n} \subseteq F^0 H^{-n} \subseteq \cdots \subseteq F^{-n+1}H^{-n} \subseteq F^{-n}H^{-n} = H^{-n}.$$

A third quarter spectral sequence $E_r^{p,q}$ is one in which $E_r^{p,q} = 0$ unless $p, q \le 0$.

Proof. Again we only prove (1), the proof of (2) being similar. We must show that $F^{n+k}H^n = 0$ if $k > 0$ and $F^k H^n = H^n$ if $k \le 0$. Consider the "left tail" of the (finite!) filtration of H^n,
$$0 \subseteq \cdots \subseteq F^{n+2}H^n \subseteq F^{n+1}H^n \subseteq F^n H^n.$$
As $E_r^{p,q}$ is a first quarter spectral sequence,
$$F^{n+k}H^n/F^{n+k+1}H^n = \begin{cases} E_\infty^{n+k,-k} = 0 & \text{if } k > 0 \\ E_\infty^{n,0} & \text{if } k = 0. \end{cases}$$
Therefore, $F^{n+k}H^n = 0$ if $k > 0$. One shows that $F^k H^n = H^n$ if $k \le 0$ in a similar fashion by considering the "right tail" of the filtration. □

4. The spectral sequence of a double complex

In this section, we treat one of the most common ways spectral sequences arise – from a double complex.

Definition 6. A double complex M consists of a bigraded object $M = \bigoplus_{p,q\in\mathbb{Z}} M^{p,q}$ together with differentials $d : M^{p,q} \to M^{p+1,q}$ and $\delta : M^{p,q} \to M^{p,q+1}$ satisfying $d^2 = \delta^2 = d\delta + \delta d = 0$.

Example 7. Let R be a ring $(P^\cdot, d_P)$ and $(Q^\cdot, d_Q)$ be complexes of R-modules. Define a double complex $M = P^\cdot \otimes_R Q^\cdot$ by $M^{p,q} = P^p \otimes_R Q^q$, $d = d_P$, and $\delta = (-1)^p d_Q : P^p \otimes_R Q^q \to P^p \otimes_R Q^{q+1}$.

To each double complex M, we attach a (single) complex $\operatorname{Tot} M$ called its *total complex* defined by
$$\operatorname{Tot}^n M = \bigoplus_{p+q=n} M^{p,q}.$$
The differential D on this total complex is given by $D = d + \delta$. Notice that $D^2 = (d+\delta)^2 = d^2 + \delta^2 + d\delta + \delta d = 0$, i.e., $(\operatorname{Tot} M, D)$ is a complex.

There are two canonical filtrations on the total complex $\operatorname{Tot} M$ of a double complex M given by
$$'F^p \operatorname{Tot}^n M = \bigoplus_{\substack{r+s=n \\ r\ge p}} M^{r,s} \quad \text{and} \quad ''F^q \operatorname{Tot}^n M = \bigoplus_{\substack{r+s=n \\ s\ge q}} M^{r,s}.$$
By the theorem of Section 2, the filtrations $'F^p \operatorname{Tot}^n M$ and $''F^p \operatorname{Tot}^n M$ determine spectral sequcences $'E_r^{p,q}$ and $''E_r^{p,q}$, respectively. One observes easily that $'E_0^{p,q} =$

$M^{p,q}$ and checks (perhaps not so easily) that the differential ${}'E_0^{p,q} \to {}'E_0^{p,q+1}$ arising from the construction of the spectral sequence of a filtration is simply given by δ. Thus, ${}'E_1^{p,q} = H_\delta^q(M^{p,\cdot})$. The differential ${}'E_1^{p,q} \to {}'E_1^{p+1,q}$ is induced by d, viewed as a homomorphism of complexes $M^{p,\cdot} \to M^{p+1,\cdot}$, implying that ${}'E_2^{p,q}$ is equal to the p-th homology group of the complex $(H_\delta^q(M^{p,\cdot}), d)$. We will write this simply (though slightly ambiguously) as $H_d^p(H_\delta^q(M))$. One may obtain similar expressions for ${}''E_r^{p,q}$, $r = 0, 1, 2$. We summarize these important results in the following theorem.

Theorem 8. *Let M be a double complex with total complex* $\operatorname{Tot} M$*. Then there exist two spectral sequences ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$ (corresponding to the two canonical filtrations on* $\operatorname{Tot} M$*) such that*

$$\begin{aligned} {}'E_0^{p,q} &= M^{p,q}, & {}'E_1^{p,q} &= H_\delta^q(M^{p,\cdot}), & {}'E_2^{p,q} &= H_d^p(H_\delta^q(M)), \\ {}''E_0^{p,q} &= M^{q,p}, & {}''E_1^{p,q} &= H_d^q(M^{\cdot,p}), & {}''E_2^{p,q} &= H_\delta^p(H_d^q(M)). \end{aligned}$$

Further, if M is a first or third quadrant double complex, then both ${}'E_r^{p,q}$ and ${}''E_r^{p,q}$ converge to $H^{p+q}(\operatorname{Tot} M)$.

Example 9. Let R be a ring. In what all follows, all tensor products are taken over R. We shall use the spectral sequences attached to a double complex to show that for R-modules A and B, we have equality of left derived functors $L_p(-\otimes B)(A) = L_p(A \otimes -)(B)$. Let

$$\cdots \xrightarrow{d} P^{-2} \xrightarrow{d} P^{-1} \xrightarrow{d} P^0 \to A \quad \text{and} \quad \cdots \xrightarrow{\delta} Q^{-2} \xrightarrow{\delta} Q^{-1} \xrightarrow{\delta} Q^0 \to B$$

be projective resolutions of A and B, respectively, and consider the double complex $P^\cdot \otimes Q^\cdot$. We use negative indexing on these spectral sequences in order to stay in a cohomological, as opposed to a homological, situation. Then ${}'E_1^{p,q} = H_\delta^q(P^p \otimes Q^\cdot)$. Since P^p is a projective and hence flat R-module, one can show in an elementary fashion that $H_\delta^q(P^p \otimes Q^\cdot) = P^p \otimes H_\delta^q(Q^\cdot)$. Since $Q^\cdot$ is a projective resolution of B, we have

$${}'E_1^{p,q} = \begin{cases} P^p \otimes B & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$${}'E_2^{p,q} = \begin{cases} L_{-p}(-\otimes B)(A) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In like manner, one computes

$${}''E_2^{p,q} = \begin{cases} L_{-p}(A \otimes -)(B) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Using Lemma 4 and the fact that both of these (third quadrant!) spectral sequences converge to $H^{p+q}(\operatorname{Tot} P^\cdot \otimes Q^\cdot)$, we have

$$\begin{aligned} L_{-p}(-\otimes B)(A) = {}'E_2^{p,0} = {}'E_\infty^{p,0} = H^p(\operatorname{Tot} P^\cdot \otimes Q^\cdot) \\ = {}''E_\infty^{p,0} = {}''E_2^{p,0} = L_{-p}(A \otimes -)(B). \end{aligned}$$

Example 10. Let M be an R-module and $P^\cdot$ be a nonpositive complex of *flat* R-modules (i.e., $P^p = 0$ if $p > 0$). Again, all tensor products and Tor's are taken over R. The relation between $H^*(P^\cdot \otimes M)$ and $H^*(P^\cdot) \otimes M$ is encoded by a spectral sequence.

Proposition 11. *Let M and $P^\cdot$ be as above. Then there exists a third quadrant spectral sequence*

$$E_2^{p,q} = \operatorname{Tor}_{-p}(H^q(P^\cdot), M) \Rightarrow H^{p+q}(P \otimes M).$$

Proof. Take a projective resolution $\cdots \xrightarrow{\delta} Q^{-2} \xrightarrow{\delta} Q^{-1} \xrightarrow{\delta} Q^0 \to M$ of M. We consider the double complex $P^\cdot \otimes Q^\cdot$ and its associated spectral sequences. By the same argument as in the previous example,

$$'E_2^{p,q} = \begin{cases} H^p(P^\cdot \otimes M) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here, we have used the flatness of $P^\cdot$. Therefore, by Lemma 4,

$$H^{p+q}(\operatorname{Tot} P^\cdot \otimes Q^\cdot) = {'E}_\infty^{p,q} = {'E}_2^{p+q,0} = H^{p+q}(P^\cdot \otimes M),$$

and we have identified the limit term of the spectral sequence whose existence the proposition asserts. Now let us consider the spectral sequence $''E_r^{p,q}$. We have

$$''E_1^{p,q} = H^q(P^\cdot \otimes Q^p) = H^q(P^\cdot) \otimes Q^p,$$

as Q^p is projective and hence flat. Therefore,

$$''E_2^{p,q} = H_\delta^p(H^q(P^\cdot) \otimes Q^p) = L_{-p}(H^q(P^\cdot) \otimes -)(M) = \operatorname{Tor}_{-p}(H^q(P^\cdot), M).$$

Since $''E_r^{p,q} \Rightarrow H^{p+q}(\operatorname{Tot} P^\cdot \otimes Q^\cdot) = H^{p+q}(P^\cdot \otimes M)$, we are done. □

This spectral sequence contains much useful information, the extraction of which is the topic of the next section.

5. Getting information out of spectral sequences

5.1. **A universal coefficients theorem.**

Consider again the situation of Example 10 in the special case where the ring R is a principal ideal domain. In this case, $E_2^{p,q} = 0$ unless $p = 0, 1$. In such a situation, one may apply the following lemma to obtain an exact sequence from the spectral sequence of Proposition 11.

Lemma 12. *Suppose $E_2^{p,q} \Rightarrow H^{p+q}$ is a third quarter spectral sequence, and that $E_2^{p,q} = 0$ unless $p = 0, -1$. Then we have an exact sequence*

$$0 \to E_2^{0,-n} \to H^{-n} \to E_2^{-1,-n+1} \to 0,$$

for all $n \geq 0$.

Proof. By Lemma 5, the filtration on H^{-n} has the form

$$0 = F^1H^{-n} \subseteq F^0H^{-n} \subseteq \cdots \subseteq F^{-n+1}H^{-n} \subseteq F^{-n}H^{-n} = H^{-n}.$$

Combining this with the fact that

$$F^{-k}H^{-n}/F^{-k+1} = E_\infty^{-k,-n+k} = E_2^{-k,-n+k} = 0$$

for $2 \leq k \leq n$, it follows that $F^{-1}H^{-n} = H^{-n}$ and $F^0H^{-n} = E_2^{0,-n}$. Now substitute these values in the exact sequence

$$0 \to F^0H^{-n} \to F^{-1}H^{-n} \to E_2^{-1,-n+1} \to 0.$$

□

Proposition 11 and Lemma 12 yield the *universal coefficients theorem for homology.*

Corollary 13. *Let R be a principal ideal domain, and let M, and $P^\cdot$ be as in proposition 11. Then for all $n \geq 0$, we have an exact sequence*

$$0 \to H^{-n}(P^\cdot) \otimes_R M \to H^{-n}(P^\cdot \otimes_R M) \to \mathrm{Tor}_1^R(H^{-n+1}(P^\cdot), M) \to 0.$$

5.2. **Edge maps and terms of low degree.** Let $E_2^{p,q} \Rightarrow H^{p+q}$ be a first quadrant spectral sequence. By Lemma 5, $F^{n+1}H^n = 0$, implying

$$E_\infty^{n,0} = F^nH^n/F^{n+1}H^n = F^nH^n \hookrightarrow H^n.$$

For each $r \geq 2$, the differential leaving $E_r^{n,0}$ is the zero map. Therefore, we have surjections $E_r^{n,0} \twoheadrightarrow E_{r+1}^{n,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{n,0}$. The composition

$$E_r^{n,0} \twoheadrightarrow E_\infty^{n,0} \hookrightarrow H^n$$

is called an edge map, as the terms $E_r^{n,0}$ lie along the bottom edge of the r-th stage diagram. Similarly, $E_\infty^{0,n} = F^0H^n/F^1H^n = H^n/F^1H^n$ by Lemma 5, giving a surjection $H^n \twoheadrightarrow E_\infty^{0,n}$. As the differentials mapping into $E_r^{0,n}$ are all zero, we have inclusions $E_r^{0,n} \hookleftarrow E_{r+1}^{0,n} \hookleftarrow \cdots \hookleftarrow E_\infty^{0,n}$. The composition

$$H^n \twoheadrightarrow E_\infty^{0,n} \hookrightarrow E_r^{0,n}$$

is also called an edge map. When $r = 2$ and n is small, we can be more specific about the kernels and images of these edge maps.

Proposition 14 (Exact sequence of terms of low degree)**.** *Let $E_2^{p,q} \Rightarrow H^{p+q}$ be a first quadrant spectral sequence. Then the sequence*

$$0 \to E_2^{1,0} \xrightarrow{e} H^1 \xrightarrow{e} E_2^{0,1} \xrightarrow{d} E_2^{2,0} \xrightarrow{e} H^2$$

is exact, where d is the E_2-stage differential and the arrows labelled e are the edge maps described above.

Proof. By Lemma 5, $F^2H^1 = 0$. Therefore,

$$E_2^{1,0} = E_\infty^{1,0} = F^1H^1/F^2H^1 = F^1H^1 \hookrightarrow H^1,$$

and the edge map $E_2^{1,0} \to H^1$ is injective with image F^1H^1. The edge map $H^1 \to E_2^{0,1}$ is the composition $H^1 \to E_3^{0,1} = E_\infty^{0,1} \hookrightarrow E_2^{0,1}$. Therefore, $\ker(H^1 \to E_2^{0,1}) = \ker(H^1 \to E_\infty^{0,1}) = F^1H^1$, proving exactness at H^1. Also, the image of H^1 in $E_2^{0,1}$ is precisely $E_3^{0,1} = \ker(E_2^{0,1} \xrightarrow{d} E_2^{2,0})$, proving exactness at $E_2^{0,1}$. As

$$E_3^{2,0} = F^2H^2/F^3H^2 = F^2H^2 \hookrightarrow H^2,$$

we have $\ker(E_2^{2,0} \to H^2) = \ker(E_2^{2,0} \to E_3^{2,0}) = \mathrm{im}(E_2^{0,1} \to E_2^{2,0})$, completing the verification. □

6. The Grothendieck spectral sequence

Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories with enough injectives and let $\mathcal{A} \xrightarrow{G} \mathcal{B} \xrightarrow{F} \mathcal{C}$ left exact covariant functors (so we may form their right derived functors). It is natural to ask if there is a relationship between right derived functors of FG and those of F and G. Under a certain technical hypothesis, this relationship exists and is encoded in the Grothendieck spectral sequence.

Theorem 15 (The Grothendieck spectral sequence)**.** *Suppose that GI is F-acyclic for each injective object I of $\mathcal{A}$ (i.e., $R^pF(GI) = 0$ if $p > 0$). Then for each object A of $\mathcal{A}$, there exists a spectral sequence*

$$E_2^{p,q} = (R^pF)(R^qG)(A) \Rightarrow R^{p+q}(FG)(A).$$

Let $C^\cdot$ be a complex and let $C^\cdot \to I^{\cdot,0} \to I^{\cdot,1} \to \cdots$ be an injective resolution (i.e., each term of each complex $I^{\cdot,j}$ is injective).

$$\begin{array}{ccccccc}
I^{0,1} & \longrightarrow & I^{1,1} & \longrightarrow & I^{2,1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
I^{0,0} & \longrightarrow & I^{1,0} & \longrightarrow & I^{2,0} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
C^0 & \longrightarrow & C^1 & \longrightarrow & C^2 & \longrightarrow & \cdots
\end{array}$$

From this array, we can extract complexes

$$\begin{aligned}
&Z^p(C^\cdot) \to Z^p(I^{\cdot,0}) \to Z^p(I^{\cdot,1}) \to \cdots, \\
&B^p(C^\cdot) \to B^p(I^{\cdot,0}) \to B^p(I^{\cdot,1}) \to \cdots, \quad \text{and} \\
&H^p(C^\cdot) \to H^p(I^{\cdot,0}) \to H^p(I^{\cdot,1}) \to \cdots.
\end{aligned}$$

We shall say that $I^{\cdot,\cdot}$ is a *fully injective resolution* of $C^\cdot$ if the above complexes are injective resolutions of $Z^p(C^\cdot)$, $B^p(C^\cdot)$, and $H^p(C^\cdot)$, respectively. Such things exist:

Lemma 16. *Suppose $\mathcal{A}$ has enough injectives. Then any complex in $\mathcal{A}$ has a fully injective resolution.*

Proof. [3, Ch. 20, Lemma 9.5] □

Sketch of proof of Theorem 15. Let A be an object of $\mathcal{A}$, and let $0 \to A \to C^\cdot$ be an injective resolution of A. Let $I^{\cdot,\cdot}$ be a fully injective resolution of $GC^\cdot$. We examine the spectral sequences associated to the double complex $FI^{\cdot,\cdot}$. We have

$$'E_1^{p,q} = H^q(FI^{p,\cdot}) = R^qF(GC^p) = \begin{cases} (FG)C^p & \text{if } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

as GC^p is F-acyclic. Therefore,

$$'E_2^{p,q} = \begin{cases} H^p((FG)C^\cdot) = R^p(FG)(A) & \text{if } q = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and consequently, $''E_2^{p,q} \Rightarrow H^{p+q}(\operatorname{Tot} FI^{\cdot,\cdot}) = R^p(FG)(A)$, by Lemma 4.

To complete the proof, it suffices to show that $''E_2^{p,q} = (R^pF)(R^qG)(A)$. Evidently, $''E_1^{p,q} = H^q(FI^{\cdot,p})$. Using the fact that everything in sight is injective, one may show that $H^q(FI^{\cdot,p}) = FH^q(I^{\cdot,p})$. Therefore, $''E_2^{p,q} = H^p(FH^q(I^{\cdot,p}))$. Now $H^q(I^{\cdot,p})$ is an injective resolution of $H^q(GC^\cdot) = R^qG(A)$ by the full injectivity of $I^{\cdot,\cdot}$. Therefore,

$$''E_2^{p,q} = H^p(FH^q(I^{\cdot,p})) = (R^pF)(R^qG)(A),$$

as desired. □

Example 17 (Base change for Ext). Let $R \to S$ be a ring homomorphism (endowing S with an R-module structure) and let A be an S-module. Let $G = \mathrm{Hom}_R(S, -)$ and $F = \mathrm{Hom}_S(A, -)$, and consider the diagram

$$R\text{-modules} \xrightarrow{G} S\text{-modules} \xrightarrow{F} \text{Abelian Groups}.$$

F and G are left exact, covariant functors, and

$$FG(B) = \mathrm{Hom}_S(A, \mathrm{Hom}_R(S, B)) = \mathrm{Hom}_R(A \otimes_S S, B) = \mathrm{Hom}_R(A, B),$$

i.e., $FG = \mathrm{Hom}_R(A, -)$. If I be an injective R-module, then

$$\mathrm{Hom}_S(-, \mathrm{Hom}_R(S, I)) = \mathrm{Hom}_R(- \otimes_S S, I) = \mathrm{Hom}_R(-, I).$$

So $\mathrm{Hom}_S(-, \mathrm{Hom}_R(S, I))$ is exact and $\mathrm{Hom}_R(S, -)$ sends injectives to injectives. Therefore, Theorem 15 implies that there exists a spectral sequence

$$E_2^{p,q} = \mathrm{Ext}_S^p(A, \mathrm{Ext}_R^q(S, B)) \Rightarrow \mathrm{Ext}_R^{p+q}(A, B).$$

Now suppose S is a *projective* R module. In this case, $\mathrm{Hom}_R(S, -)$ is exact, implying $\mathrm{Ext}_R^q(S, B) = R^q \mathrm{Hom}_R(S, -)(B) = 0$ if $q > 0$. That is, the above spectral sequence collapses at the E_2-stage. Therefore, by Lemma 4, we obtain the identity

$$\mathrm{Ext}_S^p(A, \mathrm{Hom}_R(S, B)) = \mathrm{Ext}_R^p(A, B)$$

for any S-module A and R-module B.

7. The Leray spectral sequence

In this section, we construct the Leray spectral sequence, an essential tool in modern algebraic geometry. Its construction is an application of Theorem 15. Let $f : X \to Y$ be a continuous map of topological spaces. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Sheaves}_X \arrow[rr, "f_*"] \arrow[dr, "\Gamma(X,-)"'] & & \mathrm{Sheaves}_Y \arrow[dl, "\Gamma(Y,-)"] \\ & \mathrm{Ab} & \end{tikzcd}$$

where Ab, f_* and Γ denote the category of abelian groups, sheaf pushforward, and the sections functor, respectively. It is well known that the category of sheaves of abelian groups on a topological space has enough injectives. Also, f_* and $\Gamma(Y, -)$ are left exact, covariant functors. We claim that f_* sends injectives to injectives (and thus, to $\Gamma(Y, -)$-acyclics). To see this, we shall use the following useful lemma.

Lemma 18. *Let $G : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$ be covariant functors. Further suppose that F is right adjoint to G and that G is exact. Then F sends injectives to injectives.*

Proof. Let I be an injective object of $\mathcal{B}$. We must show that $\mathrm{Hom}(FI, -)$ is exact. Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$. The the by the exactness of G, the sequence $0 \to GA' \to GA \to GA'' \to 0$ is exact. By the injectivity of I, the sequence $0 \to \mathrm{Hom}(GA'', I) \to \mathrm{Hom}(GA, I) \to \mathrm{Hom}(GA', I) \to 0$ is exact. The adjointness property of F and G completes the argument. $\square$

If is a fact that f_* is right adjoint to the sheaf inverse image functor f^{-1} – an exact functor (see [2, Ch. 2, Exercise 1.18]). Therefore, by the lemma, f_* sends injectives to injectives, and by Theorem 15, for any sheaf $\mathcal{F}$ on X, there exists a spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

with exact sequence of terms of low degree

$$0 \to H^1(Y, f_*\mathcal{F}) \to H^1(X, \mathcal{F}) \to \Gamma(Y, R^1 f_*\mathcal{F}) \to H^2(Y, f_*\mathcal{F}) \to H^2(X, \mathcal{F}).$$

The above spectral sequence is called the Leray spectral sequence. For a nice application of this spectral sequence to geometry, see [1], where it is used to analyze certain birational isomorphisms between surfaces.

8. Group cohomology and the Hochschild-Serre spectral sequence

Let G be a finite group and let A be a G-module (equivalently, a $\mathbb{Z}[G]$-module). Let G-mod and Ab denot the categories of G-modules and abelian groups, respectively. We consider the G-invariants functor

$$\mathrm{inv}_G : G\text{-mod} \to \mathrm{Ab}, \quad A \mapsto A^G$$

which to a G-module A associates its subgroup of elements invariant invariant under the action of G. It is easy to see that inv_G is a left exact, covariant functor, so we may take its right derived functors. We define $H^n(G, A) = R^n\mathrm{inv}_G(A)$ and call this the *n-th cohomology group of G with coefficients in A.*

If H is a normal subgroup of G and A is a G-module (and thus, also an H-module, then A^H naturally has the structure of a G/H-module. In fact, the diagram

$$\begin{tikzcd} G\text{-mod} \arrow[rr, "\mathrm{inv}_H"] \arrow[dr, "\mathrm{inv}_G"'] & & G/H\text{-mod} \arrow[dl, "\mathrm{inv}_{G/H}"] \\ & \mathrm{Ab} & \end{tikzcd}$$

is easily seen to be commutative. In order to apply Theorem 15 to this situation, we shall verify that inv_H sends injectives to injectives. Every G/H-module is also a G-module in a natural way. Let $\rho : G/H\text{-mod} \to G\text{-mod}$ be this functor, easily checked to be exact. It follows trivially that inv_H is right adjoint to ρ and therefore, by Lemma 18, the functor inv_H preserves injectives. Thus, by Theorem 15, there exists a spectral sequence

$$E_2^{p,q} = H^p(G/H, H^q(H, A)) \Rightarrow H^{p+q}(G, A)$$

with exact sequence of terms of low degree

$$0 \to H^1(G/H, A^H) \stackrel{\mathrm{inf}}{\to} H^1(G, A) \stackrel{\mathrm{res}}{\to} H^1(H, A)^{G/H} \stackrel{t}{\to}$$
$$H^2(G/H, A^H) \stackrel{\mathrm{inf}}{\to} H^2(G, A) \stackrel{\mathrm{res}}{\to} H^2(H, A)^{G/H}.$$

The maps inf, res, and t are called 'inflation', 'restriction', and 'transgression', respectively. Working with a more down-to-earth description of these cohomology groups in terms of certain cocycles and coboundaries, one may explicitly describe these maps; see [3, Ch. XX, Exercise 6].

References

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[3] S. Lang, *Algebra*, Addison-Weseley Publishing Company, 1993.