GROTHENDIECK TOPOLOGIES

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1. Grothendieck Topologies

The machinery of sheaf cohomology defined in terms of derived functors is quite general and might at first seem to be sufficient for most purposes. However, it has some limitations that can be remedied by changing not just the cohomology but the underlying structure, the topology on the space of interest. What are these limitations?

For a variety over $\mathbb{C}$, we have two kinds of cohomology: sheaf cohomology as a variety and sheaf cohomology as a complex manifold. For example, as a complex manifold, we can take $\dim_{\mathbb{Q}} H^r(X, \mathbb{Q})$ and obtain the Betti numbers. On the other hand, for any irreducible algebraic variety $H^r(X, \mathcal{F})$ is zero for $r > 0$ for any constant sheaf (since every open set is connected, $\mathcal{F}$ is itself flasque). If we want to have a cohomology theory analogous to that on a complex manifold, this is obviously unsatisfactory.

Second, in the Zariski topology, a map between varieties can be an isomorphism of tangent spaces without being a local isomorphism. Recall a local isomorphism is a map such that for every point in the image, there is an open neighborhood whose preimage is a disjoint union of open sets isomorphic by way of the given map to the image neighborhood. So, for example, the map $x \mapsto x^2$ from $\mathbb{A}^1 \setminus \{0\}$ to itself is an isomorphism on tangent spaces but is not a local isomorphism. In the complex analytic setting, there are local inverses, but it is necessary to delete a line, which leaves a neighborhood which is not Zariski open. So if we wish to have a local inverse, we must introduce more open sets. In an algebraic context it is not feasible to introduce smaller open sets, so we instead introduce open sets that are in some sense a different shape. In particular, to solve this problem, we can go to the étale topology, where $\mathbb{A}^1 \setminus \{0\}$ serves as an open neighborhood by way of the map $x \mapsto x^2$; allowing this as a neighborhood clearly makes the map into a local isomorphism.

As a third example of the limitations of the Zariski topology, let us look at fibre bundles. When the fibre bundle is a vector bundle, the Zariski topology is sufficiently fine for neighborhoods to exist on which the fibre bundle looks like a product; however, more exotic kinds of bundle may need a different topology. For a simple example, let G be the multiplicative group $\{1, -1\}$. Then the map $x \mapsto x^2$ makes $\mathbb{A}^1 \setminus \{0\}$ into a fiber bundle over $\mathbb{A}^1 \setminus \{0\}$ with an action of G by multiplication. Clearly no Zariski neighborhood U is going to make this bundle look like $U \times G \to U$. If we again allow the covering $\mathbb{A}^1 \setminus \{0\} \xrightarrow{x \mapsto x^2} \mathbb{A}^1 \setminus \{0\}$ as an open set, we immediately obtain a local trivialization.

We will therefore proceed by redefining topologies and sheaves; as one might expect this requires some work to establish familiar results, and the abstraction can be overwhelming. However, the essential idea is to enlarge our idea of what an open set is; rather than being simply subsets of the global space, they will be sets along with an embedding into the global space, much like coordinate systems for a differentiable manifold. This embedding need not be a simple inclusion; it can usually include some "twisting". This turns out to be useful, for example, in trivializing a fiber bundle: instead of simply finding a subset of the global space on which the bundle is trivial, we can "untwist" a neighborhood in the global space in order to obtain a local trivialization. As a side benefit, our theory will be almost independent of the nature of the global space, so that it becomes possible to work with problems like the moduli problem for nonsingular curves where the global space cannot be easily defined.

1.1. Definition of a Grothendieck Topology.

Definition 1.1. A *Grothendieck topology* is a category $\mathcal{C}$ whose objects we call "open sets" and a set S of collections of maps we call "coverings". A covering of U is a collection of morphisms $\{p_\alpha : U_\alpha \to U\}$ in $\mathcal{C}$. These objects must satisfy the following:

(1) Fibred products exist in $\mathcal{C}$.
(2) If p is an isomorphism, then $U' \xrightarrow{p} U$ is a covering; if

$$\left\{U_\alpha \xrightarrow{p_\alpha} U\right\}$$

is a covering, and for all α

$$\left\{U_{\alpha,\beta} \xrightarrow{q_{\alpha,\beta}} U_\alpha\right\}$$

is also a covering, then

$$\left\{U_{\alpha,\beta} \xrightarrow{p_\alpha \circ q_{\alpha,\beta}} U\right\}$$

is a covering.
(3) If $\left\{U_\alpha \xrightarrow{p_\alpha} U\right\}$ is a covering, and $V \to U$ is a morphism, then we can produce a covering of V:

$$\left\{V \times_U U_\alpha \xrightarrow{q_\alpha} V\right\}$$

where q_α is the projection onto the first factor of the fibred product.

Remark 1.2. The existence of fibred products means that whenever we have a diagram of the form

$$\begin{tikzcd} X \arrow[dr] & & Y \arrow[dl] \\ & Z & \end{tikzcd}$$

an object $X \times_Z Y$ exists such that all maps to X and Y factor uniquely through $X \times_Z Y$:

$$\begin{tikzcd}
V \arrow[rr, dashed, "\exists!"] \arrow[dr] \arrow[drrr] & & X \times_Z Y \arrow[dl] \arrow[dr] & \\
& X \arrow[dr] & & Y \arrow[dl] \\
& & Z &
\end{tikzcd}$$

Example 1.3. Let G be a group, and consider the category of sets with an action of G. Suppose we have a diagram

$$\begin{tikzcd}
X \arrow[dr, "f"'] & & Y \arrow[dl, "h"] \\
& Z &
\end{tikzcd}.$$

Then I claim that the fibred product $X \times_Z Y$ is the G-set

$$\{(x, y) | x \in X, y \in Y, f(x) = h(y)\}$$

with the action $g(x, y) = (gx, gy)$. The map to X is projection on the first coordinate, and the map to Y is projection on the second coordinate. The universal property is clear.

Remark 1.4. This generalization can appear bizarre, but the analogy with ordinary topologies can help clarify the issue. Open sets, in this context, are not simply subsets of a global space; they can be viewed as something like coordinate patches on the space, so that there might be many with the same image.

While the definition makes no mention of a global space X, in most applications there will be such an object lurking somewhere, and open sets will come equipped with morphisms of some sort to this object. When no such object appears naturally, the goal of using this definition of topology will often be to work as if such an object existed. If such an object does exist, it will appear as the final object in the category $\mathcal{C}$.

Generally, the morphisms in $\mathcal{C}$ should be viewed as the inclusions of open sets; we are allowing open sets to be more general, so there may be more than one way for an open set to be included in another. The intersection of two open sets would naively be their product in $\mathcal{C}$; however, for the intersection to make sense, the two open sets must be embedded in a common, larger, open set; thus the fibred product acts as the intersection of two open sets.

Similarly, to take the union of an arbitrary collection of open sets, it would be necessary to embed them in a common, larger, open set; as a result we have no real equivalent to the ability to create arbitrary unions. Useful topologies will generally already contain suitable open sets. What is then necessary is to know when a family of open sets covers a given open set; here again, we must specify how each open set is embedded in the given open set. This knowledge about covers will clearly be essential if we are to work with sheaves on these topologies.

Finally, we must be able to restrict a cover to lie within a smaller open set; viewing the fibred product as intersection, this is what the final axiom guarantees.

Example 1.5. Let X be a topological space. Then let $\mathcal{C}$ be the category of open sets on X, with a unique morphism from U to V if $U \subseteq V$. For each collection $\mathcal{U}_U$ of open sets whose union is an open set U, we have a covering $\{U_\alpha \to U\}$. We will denote this X_{top}; when X is a scheme, we will denote this X_{Zar}.

Example 1.6. Let G be a finite group and let $\{*\}$ be a one-pointed set. Let the collection of open sets be the collection of finite G-sets. Let the inclusion maps be arbitrary G-maps, and $\{f_\alpha : S_\alpha \to S\}$ is a covering if $S = \bigcup_\alpha f_\alpha(S_\alpha)$. We must verify that fibred products exist in this category. Let $f : S \to U$ and $h : T \to U$ be morphisms. What is the fibred product $S \times_U T$? Let $W = \{(s,t) | s \in S, t \in T, f(s) = h(t)\}$, with the action $g(s,t) = (gs, gt)$. Then the projection maps provide G-linear maps to both S and T, and it is clear that any object mapping to S and T can be factored through W; therefore W is the fibred product.

In this category we have the final object $\{*\}$, and one can view all the other objects as open sets that cover $\{*\}$ in various ways. We will call this topology $\mathbb{T}_G$.

We will see more examples in Section 2.

1.2. **Sheaves on a Grothendieck topology.** In the language of Example 1.5, a presheaf of the familiar sort is simply a contravariant functor from the underlying category to some given category. We will define a presheaf in exactly the same way when we are dealing with a Grothendieck topology, and we will use the same notation, with some caution.

Definition 1.7. Let $(\mathcal{C}, S)$ be a Grothendieck topology, and let $\mathcal{D}$ be another category. Then a *presheaf* of objects in $\mathcal{D}$ on $(\mathcal{C}, S)$ is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$.

Given a presheaf $\mathcal{F}$ and any open set (in the new sense) U, we will call the elements of $\mathcal{F}(U)$ sections, and given a morphism $U \xrightarrow{p} V$ we will write $\mathcal{F}(p)(s)$ as $s|_V$, with the proviso that there may be more than one morphism from U to V, in which case the notation $s|_V$ is somewhat ambiguous.

The condition for a presheaf to be a sheaf can be expressed concisely in categorical language: whenever $U = \bigcup_\alpha U_\alpha$ the diagram

$$\mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \cap U_\beta)$$

is exact. Here, the two arrows on the right denote the restriction maps obtained from the two inclusion maps of the intersection, and the exactness condition is not exactly what it seems — it amounts to saying that the image of the first map must be *precisely* those vectors on which the two restriction maps agree.

Definition 1.8. A presheaf $\mathcal{F}$ on a Grothendieck topology is a *sheaf* if for every cover $\{U_\alpha \to U\}$ the following diagram is exact:

$$\mathcal{F}(U) \to \prod_\alpha \mathcal{F}(U_\alpha) \overset{\to}{\to} \prod_{\alpha,\beta} \mathcal{F}(U_\alpha \times_U U_\beta)$$

This sounds nice, but what does it actually mean? Suppose that we have a cover $\left\{U_\alpha \overset{r_\alpha}{\to} U\right\}$ of U and a collection of sections s_α such that on $U_\alpha \times_U U_\beta$ we have $\mathcal{F}(p_1)(s_\alpha) = \mathcal{F}(p_2)(s_\beta)$. Then the sheaf axiom tells us that there exists a unique $s \in \mathcal{F}(U)$ such that $\mathcal{F}(r_\alpha)(s) = s_\alpha$. In particular, this diagram should in a sense be exact not only in the middle but also on the left.

Example 1.9. If X is a topological space, then a sheaf on X_{top} is precisely a sheaf in the usual sense, so it will not be essential to have new terminology for these objects.

Example 1.10. Let G be a finite group. What is a sheaf on $\mathbb{T}_G$? Let $\mathcal{F}$ be any sheaf on $\mathbb{T}_G$. Consider G as a G-set with G acting on the on the left, and denote this $\langle G\rangle$. Then the action of G on the right makes G into a group of automorphisms of the G-set $\langle G\rangle$. But the functor $\mathcal{F}$ then makes G a group of automorphisms of $\mathcal{F}(\langle G\rangle)$, that is, $\mathcal{F}(\langle G\rangle)$ becomes a G-set; call it M. Then we will see that $\mathcal{F}$ is canonically determined by M.

First suppose S is a G-set on which G acts transitively. Then choose an element a of S; then $p : g \mapsto ag$ is a surjective map from $\langle G\rangle$ to S. This map is taken to a restriction map by $\mathcal{F}$. Let H be the stabilizer in G of $p(e)$. Then S is isomorphic to G/H, and in particular every element of H acts trivially on G/H. Thus every element of H acts trivially on $\mathcal{F}(G/H)$, so by G-linearity, $\mathcal{F}(p)(G/H)$ is invariant under the action of H. Given any G-set T, we will write the H-invariants as T^H; then we know $\mathcal{F}(p)(G/H) \subseteq \mathcal{F}(\langle G\rangle)^H$.

Since p is surjective, $\left\{\langle G\rangle \overset{p}{\to} G/H\right\}$ is a covering, and we can apply the sheaf axiom. Since this covering has only one open set, it tells us that

$$\mathcal{F}(G/H) \to \mathcal{F}(G) \overset{\to}{\to} \mathcal{F}(\langle G\rangle \times_{G/H} \langle G\rangle)$$

In particular, it tells us that $\mathcal{F}(p)$ is injective, so that $\mathcal{F}(G/H)$ is isomorphic to some subset of $\mathcal{F}(\langle G\rangle)^H$.

Looking at the other side of the diagram, what is $\langle G\rangle \times_{G/H} \langle G\rangle$? Recalling the definition of the fibred product, we see that this is $\{(g_1, g_2) | g_1, g_2 \in G, g_1H = g_2H\}$; the two projection maps are simply projection onto the first and second coordinates, which we will denote p_1 and p_2. However, this is isomorphic, by way of the map $(g, h) \mapsto (g, gh)$, to $G \times H$ with G acting only on the first coordinate. Here the projection maps are $p_1 : (g, h) \mapsto g$ and $p_2 : (g, h) \mapsto gh$.

Now, $G \times H$ is the disjoint union of copies of G, so we have a surjective family of maps $\{f_h : g \mapsto (g, h), h \in H\}$. This is a covering, so the sheaf axiom again gives us an exact sequence. The fibred product of any two distinct copies of G is the empty set. Thus the sheaf axiom for this covering becomes

$$\mathcal{F}(G \times H) \to \prod_{h \in H} \mathcal{F}(\langle G \rangle) \rightrightarrows \prod_{h \in H} \mathcal{F}(\langle G \rangle \times_{G \times H, f_h} \langle G \rangle).$$

However, the two maps on the right are identical, so exactness simply means that $\mathcal{F}(G \times H)$ is isomorphic to $\prod_{h \in H} \mathcal{F}(G)$. The product in this case means the product in the category of sets, i.e., the Cartesian product, and the action of G is elementwise.

Thus we have the exact diagram

$$\mathcal{F}(G/H) \to \mathcal{F}(G) \rightrightarrows \prod_{h \in H} \mathcal{F}(\langle G \rangle).$$

Now, what do the maps $\mathcal{F}(p_1)$ and $\mathcal{F}(p_2)$ look like? Recall that for any h, we have a map f_h of G-sets $\langle G \rangle \to G \times H$. by our construction of the isomorphism, we have $\mathcal{F}(f_h) = \pi_h$, the projection on the h component. Now, $p_1 \circ f_h$ is the identity map, and $p_2 \circ f_h$ is multiplication by h. Thus $\mathcal{F}(f_h) \circ \mathcal{F}(p_1)$ is the identity. Similarly $\mathcal{F}(f_h) \circ \mathcal{F}(p_h)$ is multiplication by h. Recalling that $\mathcal{F}(f_h) = \pi_h$, we see that $\mathcal{F}(p_1)(s) = (s, s, \ldots, s)$ and $\mathcal{F}(p_2)(s) = (h_1 s, h_2 s, \ldots, h_n s)$. Then clearly the values s on which they agree are precisely the invariants and we see that $\mathcal{F}(G/H) = \mathcal{F}(\langle G \rangle)^H$.

Now suppose S is not a transitive G-set. Then S is a disjoint union of such objects, and $\{S_i \hookrightarrow S\}$ is a covering, for which the sheaf axiom implies that the unique possibility for $\mathcal{F}(S) = \prod_i \mathcal{F}(S_i)$.

It is therefore clear that specifying a sheaf of sets is equivalent to specifying a single G-set. If we look at sheaves of abelian groups, then we get an equivalence of categories between (finite) G-modules and sheaves on $\mathbb{T}_G$. In this context, the global sections functor is the functor $M \mapsto M^G$. So if we have a sheaf cohomology on $\mathbb{T}_G$, we will have

$$H^i(\mathbb{T}_G, \mathcal{F}) = H^i(G, M).$$

In other words, sheaf cohomology here gives rise to group cohomology.

1.3. **Continuous functions.** We will also need to generalize the idea of a continuous map. Since we have no concept of points, we are only concerned with how a continuous map acts on open sets.

Definition 1.11. Let T_1 and T_2 be Grothendieck topologies with final objects. Then a *continuous map from T_1 to T_2* is a covariant functor from the category of open sets of T_2 to the category of open sets of T_1 such that:

(1) It takes the final object to the final object.
(2) It takes fibre products in T_2 to fibre products in T_1.
(3) It takes coverings in T_2 to coverings in T_1.

Remark 1.12. Note that a continuous map F from T_1 to T_2 is a functor in the other direction, from the open sets of T_2 to those of T_1; for the sake of tradition, it is written $F^{-1}(U)$ (the participants in SGA defined the "continuous map" to be in the same direction as the functor, but this was abandoned in the print version).

Example 1.13. If X and Y are topological spaces, and $f : X \to Y$ is a continuous function then f^{-1} provides a functor from Y_{top} to X_{top}; it is easily verified that this is a continuous function (in the sense of Grothendieck topologies) from X to Y. However, not every continuous function of Grothendieck topologies arises from a continuous function of the underlying spaces.

Define a functor F^{-1} from the Zariski open sets of $\mathbb{C}$ to those of $\mathbb{C} \times \{0, 1\}$ by taking the empty set to the empty set and any other open set U to $U \times \{0\} \cup \mathbb{C} \times \{1\}$. Clearly the final object is taken to the final object, and coverings become coverings. Intersections are also clearly preserved, since no two nonempty Zariski open sets can have empty intersection. So F is a continuous function in the new sense. Suppose there is some continuous function f in the old sense such that $F^{-1} = f^{-1}$. Then suppose $f((0, 1)) = z$. Then $f^{-1}(\mathbb{C} \setminus \{z\})$ must not contain $\{(0, 1)\}$. But $\mathbb{C} \setminus \{z\}$ is nonempty, so its preimage must contain $\mathbb{C} \times \{1\}$. So F^{-1} cannot arise from a continuous function in the old sense. It turns out that if Y includes all its generic points, this cannot occur.

Example 1.14. Let G_1 and G_2 be finite groups and let ϕ be a homomorphism. Then G_1 acts on any G_2-set by way of ϕ, and any G_2-linear map of G_2-sets then becomes a G_1-linear map of the resulting G_1-sets. Clearly this functor, which we will denote ϕ^{-1}, takes the final object to the final object, and since the criterion for a covering is purely set-theoretic, it preserves coverings. Moreover, since in both categories the fibred product is the same subset of the Cartesian product, ϕ^{-1} also preserves fibred products, and therefore ϕ is a continuous function from $\mathbb{T}_{G_1}$ to $\mathbb{T}_{G_2}$.

Example 1.15. If one has an open set U in a Grothendieck topology, one can construct the induced topology as follows:

(1) The open sets are morphisms $V \to U$.
(2) Its morphisms are commutative diagrams:

$$\begin{tikzcd} V_1 \arrow[rr] \arrow[dr] & & V_2 \arrow[dl] \\ & U & \end{tikzcd}$$

(3) A set of morphisms

$$\begin{tikzcd} V_\alpha \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & U & \end{tikzcd}$$

is a covering if $\{V_\alpha \to V\}$ is a covering.

The induced topology always has a final object, namely the identity map on U.

If the original topology has a final object X we obtain a canonical "inclusion" map i where $i^{-1}(V)$ is the projection $p_2 : V \times_X U \to U$.

As in the case of sheaves on a topological space, a continuous map F induces a push-forward map defined by

$$F_*(\mathcal{F})(U) = \mathcal{F}(F^{-1}(U)).$$

More advanced properties of sheaves are more difficult to establish: for example, stalks cannot be defined without some concept of points; as we can see from the topology $\mathbb{T}_G$, it is not clear that every topology has such a concept. Many do, and in those cases we will be able to construct stalks by just such a limiting procedure.

2. The Étale Topology

In the introduction we described the need for a topology that was finer than the Zariski topology but that was nevertheless algebraic. We will define étale maps and use them to construct this new topology.

Definition 2.1. *étale*, adj., Dont le niveau est stationnaire. *Mer étale* (Dictionnaire Univerale Francophone En Ligne)

The intention is for étale to generalize the notion of local isomorphism or unramified cover, familiar from complex analysis. There, we were able to use small neighborhoods to define these maps; in an algebraic context we will use these maps to define small neighborhoods, now in the setting of Grothendieck topologies.

2.1. Étale morphisms.

In the interest of definiteness, we will define étale morphisms first for nonsingular varieties, then for (possibly) singular varieties, and finally for schemes. In each case the definition will be a generalization of the previous definitions, growing progressively more abstract and algebraic.

2.1.1. *Varieties.*

Definition 2.2. Let X and Y be nonsingular varieties,[1] and let $f : X \to Y$ be a morphism. Then we say f is *étale at* $x \in X$ if the induced map $df : T_x X \to T_{f(x)} Y$ is an isomorphism. We say that f is *étale* if it is étale at every point of X.

Remark 2.3. If X and Y are complex varieties, then one can view them as complex manifolds, and this condition clearly means that f is a local isomorphism. Thus any covering of compact Riemann surfaces is an example of an étale morphism in this sense.

If X or Y is singular, the tangent space no longer contains much information about the structure at a point: the curves $y^2 = x^2(x+1)$ and $y^2 = x^3$ have isomorphic tangent spaces at $(0,0)$ but are very different there. We have a more informative object, the tangent cone.

Definition 2.4. Let $f(X_1, \ldots, X_n)$ be a polynomial. Then we can write f uniquely as the sum of homogeneous polynomials $f_r(X_1, \ldots, X_n)$ of degree r. Let t be the smallest integer such that f_t is nonzero. Then define $L(f)$ to be f_t.

Let X be a variety in $\mathbb{A}^n$ passing through zero, defined by the ideal I. Then the *tangent cone to* X *at zero* is the scheme defined by the homogeneous polynomials $\langle L(f) | f \in I \rangle$.

Example 2.5. The tangent cone to the curve $y^2 = x^2(x+1)$ at zero is $y^2 = x^2$, a pair of lines. On the other hand, the tangent plane to this curve at zero is the whole plane $\mathbb{A}^2$.

Example 2.6. Observe that even for a variety, we may obtain a nonreduced scheme as the tangent cone: the tangent cone to $y^2 = x^3$ at the origin is (and should be) the line $y = 0$ with multiplicity two.

[1] A variety is, for our purposes, an integral separated scheme of finite type over an algebraically closed field.

Unfortunately, this definition is not obviously intrinsic, so we formulate it in a more intrinsic way.

Definition 2.7. Let R be a local ring with maximal ideal $\mathfrak{m}$. Define the *graded ring associated to* R to be

$$\mathrm{gr}(R) = \bigoplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1}.$$

Proposition 2.8. *Let X be a variety in $\mathbb{A}^n$ passing through zero. Then the coordinate ring of the tangent cone to X at zero is isomorphic to* $\mathrm{gr}(\mathcal{O}_{X,0})$.

Proof. Let X be $Z(I)$, and let $\mathfrak{m}$ be the maximal ideal of zero in $k[X_1, \ldots, X_n]/I$. Then

$$\begin{aligned}
\mathrm{gr}(\mathcal{O}_{X,0}) &= \bigoplus_i (\mathfrak{m}\mathcal{O}_{X,0})^i/(\mathfrak{m}\mathcal{O}_{X,0})^{i+1} \\
&= \bigoplus_i \mathfrak{m}^i/\mathfrak{m}^{i+1} \\
&= \bigoplus_i \langle X_1, \ldots, X_n\rangle^i / \left\langle \langle X_1, \ldots, X_n\rangle^{i+1}, I \cap \langle X_1, \ldots, X_n\rangle^i \right\rangle \\
&= \bigoplus_i \langle X_1, \ldots, X_n\rangle^i / \left\langle \langle X_1, \ldots, X_n\rangle^{i+1}, \{L(f) | f \in I, \deg f = i\} \right\rangle \\
&= \bigoplus_i \{x | x \in k[X_1, \ldots, X_n]/\langle L(f) | f \in I\rangle, \deg x = i\} \\
&= k[X_1, \ldots, X_n]/\langle L(f) | f \in I\rangle.
\end{aligned}$$

□

We can now make the following invariant definition:

Definition 2.9. Let X be a variety and let $x \in X$. Then the *tangent cone to X at x* is $\mathrm{Spec}(\mathrm{gr}(\mathcal{O}_{X,x}))$.

Proposition 2.10. *Let A be a local ring and $\widehat{A}$ be its completion. Then* $\mathrm{gr}(A)$ *is isomorphic to* $\mathrm{gr}(\widehat{A})$.

Proof. To see this, simply observe that if $\mathfrak{m}$ is the maximal ideal of A, then $A/\mathfrak{m}^i$ is isomorphic to $\widehat{A}/\mathfrak{m}^i\widehat{A}$. □

Proposition 2.11. *Let $f : A \to B$ be a local homomorphism; then f induces maps from* $\mathrm{gr}(A)$ *to* $\mathrm{gr}(B)$ *and from $\widehat{A}$ to $\widehat{B}$. Then f induces an isomorphism from* $\mathrm{gr}(A)$ *to* $\mathrm{gr}(B)$ *if and only if f induces an isomorphism from $\widehat{A}$ to $\widehat{B}$.*

One direction is now obvious; to see the other, see (for example) [AM69, Lemma 10.23].

Thus we can capture the isomorphism of tangent cones by looking at the completed local rings, which motivates the following definition:

Definition 2.12. Let X and Y be arbitrary varieties, and let $f : X \to Y$ be a morphism. Then we say f is *étale at $x \in X$* if the induced map from $\widehat{\mathcal{O}}_{Y,f(x)}$ to $\widehat{\mathcal{O}}_{X,x}$ is an isomorphism.

One can view this definition as a way of saying that an étale map is a local isomorphism when one interprets local as "infinitesimal neighborhood" rather than "Zariski neighborhood".

2.1.2. *Flat morphisms.* If one wants to broaden the definition of étale to apply to schemes in general, which we do, we will need some more algebra.

To begin with, we will define flat morphisms. One can think of a morphism as defining a family of schemes: for every point in the image, the fiber is a scheme. Of course, not every family gets along equally well, and to have a family that is well-behaved, we expect to need some conditions on the morphism. It turns out that a good condition to apply is flatness, which will result in certain numerical invariants being constant along fibers.

Recall that a ring homomorphism $A \to B$ is flat if and only if the functor $B \otimes_A -$ is exact.

Definition 2.13. Let $f : X \to Y$ be a morphism of schemes. Then f is flat at x if the induced homomorphism $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat.

This is not at all edifying, but hopefully a few propositions will clear the murk slightly.

Proposition 2.14. *Let $f : X \to Y$ be a flat morphism of schemes of finite type over a field k. Let x be any point of X, and let $X_{f(x)}$ be the fibre over X. Then*

$$\dim_x(X_{f(x)}) = \dim_x X - \dim_{f(x)} Y,$$

where $\dim_x(X)$ means the dimension of $\mathcal{O}_{X,x}$.

See [Har77, Thm. III.9.5] for a proof.

In particular, if we let x and $f(x)$ be closed points, and X and Y be varieties, then this means that the fibres always have the same dimension.

Example 2.15. Let k be a field. Then any $k \to A$ is flat, since a k-algebra is just a vector space, therefore a free k-module, therefore flat.

In a sense, flatness means that the numerical invariants of the fibres are constant:

Proposition 2.16. *If T is an integral noetherian scheme and X is a closed subscheme of $\mathbb{P}^n_T$, then for $t \in T$ we can compute the Hilbert polynomial ϕ_t of the fibre X_t in $\mathbb{P}^n_{k(t)}$. Then X is flat over T if and only if ϕ_t is independent of t.*

See [Har77, Thm. III.9.9] for a proof.

Example 2.17. The map $x \mapsto x^n$ from $\mathbb{A}^1$ to $\mathbb{A}^1$ is a flat map. We can see this from the fact that $\mathbb{A}^1$ is nonsingular.

Example 2.18. The blow-up of a surface at a closed point is not flat: the preimage of the point is the projectivization of the tangent cone, which has dimension at least one, while the preimage of any other point is a single point.

Example 2.19. Let Y be the curve $y^2 = x^2(x+1)$. The normalization morphism $f : \tilde{Y} \to Y$ is not flat.

There is a higher-level proof of this in [Har77, Ex. III.9.7.1], but it is instructive to do this hands-on. We have $Y = \operatorname{Spec} A$, where $A = k[x,y] = k[X,Y]/(Y^2 - X^2(X+1))$. The fraction field of A is $k(t)$, where $t = \frac{y}{x}$, and the integral closure of A in $k(t)$ is $k[t]$. The inclusion of A into $k[t]$ is given by $x \mapsto t^2 - 1$, $y \mapsto t^3 - t$, which induces the normalization morphism $f : \tilde{Y} = \operatorname{Spec} k[t] \to Y$. We want to show that $k[t]$ is not a flat A-module. For this we use the criterion: an A-module M is flat if and only if for every finitely generated ideal I of A, the map $I \otimes M \to M$ is injective [Har77, Prop. II.9.1A(a)].

Let $I = \langle x, y \rangle$. The map in question is

$$\begin{array}{rcl} \varphi : \langle x, y \rangle \otimes k[t] & \to & k[t] \\ x \otimes t^n & \mapsto & (t^2 - 1)t^n \\ y \otimes t^n & \mapsto & (t^3 - t)t^n \end{array}$$

Now consider the element $x \otimes t - y \otimes 1 \in \langle x, y \rangle \otimes k[t]$. It is clearly nonzero, but its image via φ is $(t^2-1)t - (t^3 - t) = 0$, so φ is not injective and f is not flat.

It turns out (as one might expect) that one-parameter families are easier to check for compatibility:

Proposition 2.20. *Let $f : X \to Y$ be a morphism of schemes, with X reduced and Y integral and regular of dimension 1. Then f is flat if and only if every irreducible component of X dominates Y.*

See [Har77, Prop. III.9.7].

Example 2.21. Let $Y = \operatorname{Spec} k[t]$, $X = \operatorname{Spec} k[x,y,t]/(ty - x^2)$ with the natural morphism $f : X \to Y$. Both X and Y are integral, and f is surjective so it's flat. Note that the fiber X_t above $t \neq 0$ is integral, but $X_0 = \operatorname{Spec} k[x,y]/(x^2)$ is a double line.

This example shows part of the motivation for considering schemes rather than varieties: we have a perfectly reasonable family of schemes, one of which happens to be non-reduced.

Example 2.22. Similarly, let $Y = \operatorname{Spec} k[t]$ and $X = \operatorname{Spec} k[x,y,t]/(t - xy)$. Again X and Y are integral and f is surjective so f is flat. The fiber X_t above $t \neq 0$ is integral, but $X_0 = \operatorname{Spec} k[x,y]/(xy)$ is reducible (union of two lines).

Example 2.23 (An arithmetic example)**.** Consider the morphism $f : X = \operatorname{Spec} \mathbb{Z}[x,y]/(x^2 - y^2 - 5) \to Y = \operatorname{Spec} \mathbb{Z}$. It is flat by the same argument as before. It's interesting to consider the fibers:

- The fiber above (0) is $\operatorname{Spec} \mathbb{Q}[x,y]/(x^2 - y^2 - 5)$, a nonsingular conic.
- The fiber above (p), $p \neq 2, 5$ is $\operatorname{Spec} \mathbb{F}_p[x,y]/(x^2 - y^2 - 5)$, also a nonsingular conic.
- The fiber above (2) is $\operatorname{Spec} \mathbb{F}_2[x,y]/(x+y+1)^2$, a double line.
- The fiber above (5) is $\operatorname{Spec} \mathbb{F}_5[x,y]/(x-y)(x+y)$, a union of two lines.

We will need a slightly stronger concept as well, later on:

Definition 2.24. Let B be an A-algebra. Then B is *faithfully flat* if B is flat and for any nonzero M, $B \otimes_A M$ is nonzero.

Remark 2.25. We will need this property because B is faithfully flat if and only if $\operatorname{Spec} B \to \operatorname{Spec} A$ is flat and surjective [Mil98, p. 35].

2.1.3. *Unramified morphisms.* When dealing with coverings of Riemann surfaces, we relax the conditions for a covering space and allow both ramified and unramified coverings; in this context, a (finite) ramified cover fails to be a local isomorphism at some discrete collection of points, where the projection map is locally isomorphic to $x \mapsto x^n$ for some $n > 1$. The terminology comes from number theory; the function field of a covering appears as an extension of the function field of the base space, and this extension is ramified if and only if the covering is ramified.

It turns out that to define étale maps we will need a similar condition, phrased in algebraic terms, to ensure that our étale maps are the proper generalization of local isomorphisms.

Definition 2.26. Let $f : X \to Y$ be a morphism of finite type, and let $x \in X$, $y = f(x) \in Y$. Two equivalent conditions for f to be unramified at x:

- $\mathfrak{m}_{Y,y}\mathcal{O}_{X,x} = \mathfrak{m}_{X,x}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ is a finite separable extension of $\mathcal{O}_{Y,y}/\mathfrak{m}_{Y,y}$.
- $\Omega_{X/Y}$ is zero at x.

For the equivalence of these conditions, see [Har77, Exer. III.10.3].

We can describe more clearly what $\Omega_{X/Y}$ is when X and Y are affine:

Proposition 2.27. *Let A be a ring, and let $B = A[X_1, \dots, X_n]/\langle f_1, \dots, f_m\rangle$ be an A-algebra. Then $\Omega_{B/A}$ is the B-module with generators*

$$dX_1, \dots, dX_n$$

and relations

$$df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial X_j} dX_j = 0.$$

See [Har77, Rem. II.8.9.2] and the definitions in [Har77, III.8].

Example 2.28. Let's consider again the normalization map $\operatorname{Spec} k[t] \to \operatorname{Spec} k[X,Y]/(Y^2 - X^2(X+1))$. We want to show that $\Omega_{B/A} = 0$, where $A = k[X,Y]/(Y^2 - X^2(X+1))$, $B = k[t]$ and the homomorphism $A \to B$ is given by $x \mapsto t^2 - 1$, $y \mapsto t^3 - t$. By definition

$$\Omega_{B/A} = \{db, b \in B | d(b+b') - db - db' = 0, \\ d(bb') - bdb' - b'db = 0, da = 0\}.$$

In our case $\Omega_{B/A}$ is generated by dt, but we also have $dx = dy = 0$, i.e.

$$\begin{aligned} dx &= d(t^2 - 1) = 2tdt = 0, \\ dy &= d(t^3 - t) = (3t^2 - 1)dt = 0. \end{aligned}$$

Since there are no values of t for which $2t = 3t^2 - 1 = 0$, we conclude that $dt = 0$ so the morphism is everywhere unramified.

Example 2.29. Let k be a field, and let K be another field, finitely generated over k. I claim K/k is unramified if and only if K/k is a separable algebraic extension.

Write $K = k(T_1, \dots, T_n)[X]/\langle f \rangle$. By definition, unramified implies finite type, and transcendental extensions are not of finite type. However, in this case, we can show that $\Omega_{K/k} = 0$ implies that the morphism is of finite type. This is a localization of $A = k[T_1, \dots, T_n, X]/\langle f \rangle$ by some set S. Then we can use [Har77, Prop. II.8.2A] and Proposition 2.27 to show that $\Omega_{K/k}$ is the K-module generated by $dT_1, \dots, dT_n, dX$ with the relation $df = 0$. Then this can be zero only if $n = 0$, so we see K must be algebraic over k. Consider dX. This will be zero if there exist polynomials $g(X)$ and $h(X)$ such that $1 = g(X)f(X) + h(X)f'(X)$; this occurs if and only if f has no repeated roots, i.e., if K is a separable algebraic extension of k.

2.1.4. *Étale for schemes.*

Definition 2.30. Let $f : X \to Y$ be a morphism of schemes, and let $x \in X$. Then f is *étale at x* if it is flat and unramified at x.

Of course, it is not clear that flat and unramified together are enough to assure we have a map that generalizes a local isomorphism. However, we have:

Proposition 2.31. *Let $f : X \to Y$ be a morphism of schemes of finite type over k. For every $x \in X$, choose $k(x) \subseteq \widehat{\mathcal{O}_{X,x}}$ and $k(f(x)) \subseteq \widehat{\mathcal{O}_{Y,f(x)}}$ such that $k(f(x)) \subseteq k(x)$ under the natural map $\widehat{\mathcal{O}_{Y,f(x)}} \to \widehat{\mathcal{O}_{X,x}}$. Then f is étale if and only if for every x we have $k(x)$ is a separable algebraic extension of $k(f(x))$, and the natural map*

$$\widehat{\mathcal{O}_{Y,f(x)}} \otimes_{k(f(x))} k(x) \to \widehat{\mathcal{O}_{X,x}}$$

is an isomorphism.

See [Har77, Ex. II.10.4] for details.

Example 2.32. This proposition allows us to easily draw examples from the category of varieties. So, for example, the map $x \to x^n$ is étale when x is not zero and n is prime to the characteristic of the ground field.

Example 2.33. We have already seen that the normalization map of the curve $y^2 = x^2(x+1)$ is unramified but not flat, and hence not étale at 0. We can see, however, that it is étale everywhere else.

Example 2.34. We have seen that the map $x \mapsto x^n$ on $\mathbb{A}^1$ is flat but not unramified. However, away from zero we can see that it is étale from the criteria for varieties.

Proposition 2.35. *Étale morphisms have the following properties, suggested by analogy with local isomorphisms:*

(1) *Open immersions are étale.*
(2) *The composition of two étale morphisms is étale.*
(3) *Given $U \to X$ étale and $Y \to X$, the map $U \times_X Y \to Y$ is also étale.*
(4) *If $\phi \circ \psi$ and ϕ are étale, then so is ψ.*

See [SGA71, Prop. I.4.6].

Finally, just as for varieties, we could express the condition for étale in terms of the Jacobian of the map, we can do the same when dealing with schemes; of course the statement and the proof are much more complicated. Here is a special case:

Proposition 2.36. *A morphism $f : X \to Y$ is étale if and only if for each $x \in X$ we have open affine neighborhoods $\operatorname{Spec} C$ of x and $\operatorname{Spec} A$ of $f(x)$ such that*

$$C = A[T_1, \dots, T_n]/\langle f_1, \dots, f_n\rangle, \quad \det\left(\frac{\partial f_i}{\partial T_j}\right) \in C^\times$$

If Y is normal, and f is étale then f is locally a standard étale morphism*, of the form $\operatorname{Spec} C \to \operatorname{Spec} A$, where A is an integral domain and $C = B_b$ for some $B = A[T]/\langle P(T)\rangle$, $P(T)$ irreducible in the quotient field of A, and $b \in B$. Conversely, if $P'(T)$ is a unit in B_b then this map is étale.*

See [Art73, Ch. 1].

Example 2.37. Let k be a field. What are the étale maps $X \to \operatorname{Spec} k$? Recalling Example 2.15, we see that such a map is automatically flat. Now suppose $X = \operatorname{Spec} A$ is affine. Then A must be a k-algebra of dimension zero. This implies that A is an Artinian ring, which implies that A is a finite product of finite separable field extensions of k ([AM69, Thm. 8.5, 8.7]).

Example 2.38. If A is a Henselian ring (see Definition 2.60; for example, a complete local ring) with residue field k then there is an equivalence of categories between finite étale A-algebras and finite étale k-algebras given by $B \mapsto B \otimes_A k$. See [Mil80, Prop. 4.4].

To see the analogy with coverings of complex manifolds more clearly, we will state some further results:

Definition 2.39. Let $f : X \to Y$ be locally of finite type. Then the annihilator of $\Omega_{X/Y}$ is a sheaf of ideals on X which we denote $\mathfrak{d}_{X/Y}$ and call the *different* of X over Y. The closed subscheme it defines is called the *branch locus* of f. Its complement is the maximal open set where f is unramified.

Remark 2.40. In many cases, of course, the branch locus will be all of X, for example, when X has positive relative dimension. However, in some cases we have a result:

Proposition 2.41. *Suppose $f : X \to Y$ is faithfully flat and finite. Then the branch locus is either empty or of pure codimension* 1*.*

It turns out that flatness is also an open condition, so that for any $f : X \to Y$ locally of finite type there is a maximal U such that $f|_U$ is étale ([Mum99, III.10 Thm. 3]).

2.2. **The étale topology.** Now that we have defined étale maps, we will use them to construct a topology on a scheme that has more open sets than the Zariski topology. This topology will help solve the problems we discussed in the introduction, but some remain; a few questions will require more exotic topologies.

Definition 2.42. Let X be a scheme. Then the étale topology $X_{\text{ét}}$ on X is defined in the following way:

- An open set is an étale map from some scheme U to X.
- A morphism of open sets from $U \to X$ to $V \to X$ is a morphism from U to V making the following diagram commute:

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & V \arrow[dl] \\ & X & \end{tikzcd}$$

- A collection of maps $\left\{U_\alpha \stackrel{p_\alpha}{\to} U\right\}$ is a covering if $U = \cup p_\alpha(U_\alpha)$.

Remark 2.43. Since an open immersion is an étale map, every Zariski-open set is also open in the étale topology. In fact, we have a continuous function F from $X_{\text{ét}}$ to X_{Zar}, which will allow us to compare the two topologies.

Remark 2.44. Examining Proposition 2.35 we see that a morphism of open sets must be an étale map as well.

2.3. **Variants on the étale topology.** One can devise many variations on the étale topology.

The finite-étale topology: Here the open sets are arbitrary étale morphisms, but the coverings are restricted: $\{U_i \to U\}$ must not only be a surjective family of X-morphisms, each map must be finite.

The flat topology: Open sets are simply maps to X, and coverings are surjective families in which each map is a flat X-morphism of finite type. This is denoted X_{Fl}.

The big étale topology: Open sets are arbitrary morphisms to X, coverings are surjective families of étale X-morphisms. This is denoted $X_{\text{Ét}}$.

There are obvious continuous maps

$$X_{\text{Fl}} \to X_{\text{Ét}} \to X_{\text{ét}} \to X_{\text{Zar}}.$$

2.4. **Sheaves in the étale topology.** Recall that we have already defined presheaves and sheaves on an arbitrary topology. We will look into what this definition means in this context.

Recall that a sheaf must specify an object for each open set as well as a restriction map between any two objects, and it must satisfy an exactness condition for all coverings. In this topology, the coverings are families of étale maps whose images cover the open set in question. So verifying the sheaf axiom could be non-trivial. Fortunately we have:

Proposition 2.45. *A presheaf $\mathcal{F}$ on the étale site is a sheaf if and only if it satisfies the sheaf condition for every Zariski open cover and for every cover by a single étale map of the form $\{\phi : V \to U\}$ with V and U both affine.*

For the proof see [Mil80, II.1.5].

2.4.1. *Examples.* We will now see a few examples of sheaves on the étale topology.

Whenever we have an étale morphism $f : X \to Y$ (or, in fact, any morphism of schemes) we obtain a homomorphism from $\Gamma(Y, \mathcal{O}_Y)$ to $\Gamma(X, \mathcal{O}_X)$. So the global sections functor is a contravariant functor on the underlying category for the étale topology, that is, a presheaf.

Proposition 2.46. *This presheaf is a sheaf.*

We will use Proposition 2.45 to prove this. We see immediately that for a Zariski cover, the presheaf is just the structure sheaf, and so the sheaf axiom holds. So we need only verify it for surjective étale maps of affine schemes. Such a map always corresponds to a faithfully flat homomorphism $A \to B$; the sheaf condition then becomes that the sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{b \mapsto 1\otimes b - b \otimes 1} B \otimes_A B$$

is exact.

Proposition 2.47. *Let $A \to B$ be a faithfully flat homomorphism. Then*

$$0 \longrightarrow A \longrightarrow B \xrightarrow{b \mapsto 1\otimes b - b \otimes 1} B \otimes_A B$$

is exact.

See [Mil98, Prop. 6.8].

Definition 2.48. Let X be a scheme. Then the *structure sheaf* $\mathcal{O}_{X_{\text{ét}}}$ for $X_{\text{ét}}$ is the global sections functor.

Example 2.49. Let Λ be a set, and let U be (the domain of) an arbitrary open set in $X_{\text{ét}}$. Let $\pi_0(U)$ denote the set of connected components of U. Then define $\mathcal{F}(U) = \Lambda^{\pi_0(U)}$, and use the obvious restriction maps. Then $\mathcal{F}$ is clearly a sheaf, called the constant sheaf on $X_{\text{ét}}$ defined by Λ.

Example 2.50. Let Z be any scheme. Then the functor $\mathrm{Hom}_X(-, Z)$ is a presheaf of sets. Clearly this presheaf satisfies the sheaf axiom for Zariski coverings. Thus we need only show that for $A \to B$ faithfully flat,

$$Z(A) \to Z(B) \underset{\to}{\overset{\to}{}} Z(B \otimes_A B)$$

is exact. When $Z = \mathrm{Spec}(C)$ is affine, we need only show that

$$\mathrm{Hom}_A(C, A) \to \mathrm{Hom}_A(C, B) \underset{\to}{\overset{\to}{}} \mathrm{Hom}(C, B \otimes_A B),$$

which follows immediately from Proposition 2.47. The case where Z is not affine is relatively straightforward.

If Z has a group structure, one obtains a sheaf of groups.

Example 2.51. Let $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_X$-modules (in the usual, Zariski sense). Then if $\phi : U \to X$ is any étale map, then $\phi^*\mathcal{M}$ is a coherent sheaf of $\mathcal{O}_U$-modules. For example, if $U = \mathrm{Spec}\, B$ and $X = \mathrm{Spec}\, A$ are affine, then $\mathcal{M}$ is defined by an A-module M. Then $\phi^*\mathcal{M}$ corresponds to $B \otimes_A M$. Then we can make a presheaf $U \mapsto \Gamma(U, \phi^*\mathcal{M})$ on $X_{\text{ét}}$, which we denote $\mathcal{M}^{\text{ét}}$. One can check that this is indeed a sheaf; the proof resembles that of Proposition 2.47, applied to modules. One checks that (as we would hope) $(\mathcal{O}_X)^{\text{ét}} = \mathcal{O}_{X_{\text{ét}}}$.

Example 2.52. Let us examine more closely the sheaf $(\Omega_{X/Z})^{\text{ét}}$ for some morphism $X \to Z$. Let $f : Y \to X$ be étale. Then we have an exact sequence

$$0 \to f^*\Omega_{X/Z} \to \Omega_{Y/Z} \to \Omega_{Y/X} \to 0.$$

But since f is étale, $\Omega_{Y/X}$ is zero, and we obtain an isomorphism between $f^*\Omega_{X/Z}$ and $\Omega_{Y/Z}$. We see that $(\Omega_{X/Z})^{\text{ét}}$ is exactly $(U \to X) \mapsto \Omega_{U/X}$.

2.4.2. *Stalks.* In the classical situation, we defined stalks by taking the limit over neighborhoods of a fixed point. A neighborhood was simply an open set that contained the point. However, in a general Grothendieck topology, even if one has a concept of a point, an open neighborhood might contain it in more than one way. To see this, suppose $\phi : U \to X$ is an étale map. If we want to say it is a neighborhood of x, clearly x should be in its image. However, we might well want to restrict our attention to the induced topology on U. Now we need to select a point u on U mapping to x, as there may be many choices.

If X is a scheme, our problems are still greater: we will frequently have a field k, not necessarily the field of definition of X, and we will want to ask for the k-points of X. A common situation asks for the k^{sep}-points of X, where X is some k-scheme. These will generally not correspond to points in the underlying topological space of X. For example, consider the $\mathbb{C}$-points of $\operatorname{Spec} \mathbb{R}[x]$: these are, as one might expect, in bijection with $\mathbb{C}$, but for any prime ideal of $\mathbb{R}[x]$ not coming from a real point, there are two complex points, corresponding to the two embeddings of the residue field in $\mathbb{C}$.

Definition 2.53. let k be a field, and let X be a scheme. Then a *k-point of X* is a morphism from $\operatorname{Spec} k$ to X. If k is separably closed, this is called a *geometric point of X*.

Remark 2.54. Note that, by this definition, generic points also give rise to geometric points for a larger field: in $\mathbb{A}^1/k$, the generic point is a geometric point for $(k(X))^{\text{sep}}$.

When X is a variety (and so defined over some algebraically closed field k) the k-points are naturally identified with the closed points of the underlying topological space.

We have seen that we will need to specify how a point is embedded in a neighborhood.

Definition 2.55. An étale neighborhood of a k-point x of X is a pair (U, u) such that $x = U \circ u$, that is, the image of u under the étale map U is x:

$$\begin{tikzcd} & U \arrow[d] \\ \operatorname{Spec} k \arrow[ur, "u"] \arrow[r, "x"] & X \end{tikzcd}$$

A morphism of étale neighborhoods $(U,u) \to (V,v)$ is a morphism of étale open sets $U \to V$ which takes u to v:

$$\begin{tikzcd} & U \arrow[dr] & & V \arrow[dl] \\ \operatorname{Spec} k \arrow[ur, "u"] \arrow[urrr, "v"'] & & X & \end{tikzcd}$$

For closed points on varieties, this means exactly what one would expect. For schemes, it respects the need to specify how a point is embedded in U as well as how it is embedded in X.

Observe that the étale neighborhoods of a k-point form a directed set[2]. In fact, if (U,u) is a connected neighborhood of x and X is a variety, then Proposition 2.35 tells us there is a unique way it is embedded in any larger neighborhood.

Definition 2.56. Let X be a scheme, let $\mathcal{F}$ be a sheaf on $X_{\text{ét}}$, and let x be a k-point of X. Then define

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U)$$

where the limit is taken over the directed set of étale neighborhoods of x.

What this means is that an element of $\mathcal{F}_x$ is an étale neighborhood (U,u) of x and an element $f \in \mathcal{F}(U)$. We consider (U,u,f) and (U',u',f') equivalent if there exists (V,v) such that we have maps $\iota : (V,v) \to (U,u)$ and $\iota' : (V,v) \to (U',u')$, and the associated restriction maps take f and f' to the same element: $\mathcal{F}(\iota)(f) = \mathcal{F}(\iota')(f')$.

We also know that, since fibred products exist, there is a common neighborhood mapping to any pair of neighborhoods. Thus if $\mathcal{F}$ is a sheaf of groups or rings, then $\mathcal{F}_x$ is also a group or ring.

We know that $\mathcal{F}$ also induces a sheaf on X_{Zar} which we could denote by $\mathcal{F}^{\text{Zar}}$. Since any Zariski neighborhood is also an étale neighborhood, any element of $(\mathcal{F}^{\text{Zar}})_x$ is also an element of $\mathcal{F}_x$, and we have a homomorphism from $(\mathcal{F}^{\text{Zar}})_x$ to $\mathcal{F}_x$. Similarly, if (U,u) is any étale neighborhood of x, we get a homomorphism from $((\mathcal{F}|_U)^{\text{Zar}})_x$ to $\mathcal{F}_x$, and

$$\mathcal{F}_x = \varinjlim ((\mathcal{F}|_U)^{\text{Zar}})_x.$$

That is, the stalk in the étale topology is the direct limit of the stalks in the Zariski topology over all étale neighborhoods.

To give all this a little more definiteness, we will let X be a variety and discuss in more detail the stalks of the structure sheaf $\mathcal{O}_{X_{\text{ét}}}$.

If (U,u) is any étale neighborhood, we know that $\mathcal{O}_{X_{\text{ét}}}|_U = \mathcal{O}_{U_{\text{ét}}}$, and so $(\mathcal{O}_{X_{\text{ét}}}|_U)^{\text{Zar}} = \mathcal{O}_U$. So we have

$$\mathcal{O}_{X_{\text{ét}},x} = \varinjlim \mathcal{O}_{U,u}.$$

If $(U,u) \to (V,v)$ is a map of étale neighborhoods, then we know that the morphism involved is étale, and so the map $\mathcal{O}_{V,v} \to \mathcal{O}_{U,u}$ is a flat (hence injective) homomorphism of local rings with Krull dimension $\dim X$. Also the map $\widehat{\mathcal{O}_{V,v}} \to \widehat{\mathcal{O}_{U,u}}$ is always an isomorphism. This allows one to prove:

[2]Here we are assuming that the category of étale open sets is a small category; it can be made into one with some care. We will not worry about such considerations.

Proposition 2.57. *The ring $\mathcal{O}_{X_{ét},x}$ is a local Noetherian ring of Krull dimension $\dim X$, and $\widehat{\mathcal{O}_{X_{ét},x}} = \widehat{\mathcal{O}_{X,x}}$.*

The proof ([Mil98, Prop. 4.1]) requires an argument of Nagata to prove that the result is Noetherian.

We will see that this ring shares many nice properties of the completed local ring $\widehat{\mathcal{O}_{X,x}}$. One can view the similarities between $\mathcal{O}_{X_{ét},x}$ and $\widehat{\mathcal{O}_{X,x}}$ as an indication that the étale site is "local enough" for many purposes.

Proposition 2.58. *If $\phi : X \to Y$ is étale at x, then the induced map $\mathcal{O}_{X_{ét},\phi(x)} \to \mathcal{O}_{X_{ét},x}$ is an isomorphism.*

We see that ϕ itself serves as an étale neighborhood of x, and any étale neighborhood of x is also an étale neighborhood of $\phi(x)$. Moreover, given any étale neighborhood of $\phi(x)$, there is an étale neighborhood of x mapping to it, so by the direct limit construction, the result follows.

Proposition 2.59. *Let $d = \dim X$ and let X be nonsingular at x. Then there is a map $X \to \mathbb{A}^d$ which is étale at x.*

See [Mil98, Prop. 4.9].

In other words, in the étale topology, all nonsingular points are locally like $\mathbb{A}^d$.

Can the ring $\mathcal{O}_{X_{ét},x}$ be described any more clearly? It turns out that the answer is yes, although we will be forced to digress into some algebra.

Definition 2.60. A local ring A with maximal ideal $\mathfrak{m}$ and residue field k is said to be *Henselian* if it satisfies the following condition:

Let f be a monic polynomial in $A[T]$ and let $\bar{f}$ denote its image in $k[T]$. Then any factorization of $\bar{f}$ into two relatively prime monic polynomials g_0 and h_0 yields a factorization of f into the product of two monic polynomials g and h such that $g_0 = \bar{g}$ and $h_0 = \bar{h}$.

Remark 2.61. The polynomials g and h thus obtained are unique and coprime.

Remark 2.62. Every complete local ring is Henselian. In fact, the usual proof of Hensel's lemma works in any complete local ring.

One reason to bring all this algebra up is:

Proposition 2.63. *For any point x, $\mathcal{O}_{X_{ét},x}$ is Henselian.*

See [Mil98, Thm. 4.5].

Proposition 2.64. *The local ring for the étale topology at the origin in $\mathbb{A}^d$ is the set of all elements in $k[[T_1, \dots, T_d]]$ which are the roots of some polynomial in $k[T_1, \dots, T_d, X]$.*

Observe that this says, for example, that there is locally a function $\sqrt{1+T}$, which is just what one would have expected if the étale topology is to resolve the issues listed in the introduction.

One can characterize the ring $\mathcal{O}_{X_{ét},x}$ more precisely.

Definition 2.65. Let A be a local ring. Then $A \to B$ is the Henselization of A if B is Henselian and for any $A \to C$ where C is Henselian, the map factors uniquely through B.

Let A be a local ring with residue field k. Let K be a finite separable extension of k, and fix an embedding $k \hookrightarrow K$ (i.e., fix a K-point of $\operatorname{Spec} A$). Then a strict K-Henselization of A is a Henselian local ring C with a map $C \to K$ such that every local map

$$\begin{tikzcd} A \arrow[rr] \arrow[dr] & & B \arrow[dl] \\ & K & \end{tikzcd}$$

with B a Henselian local ring factors through C. If K is a separable closure of k, this is called the *strict Henselization* of A.

Proposition 2.66. *Let x be a geometric point of X. Then the ring $\mathcal{O}_{X_{\acute{e}t},x}$ is the strict Henselization of $\mathcal{O}_{X,x}$.*

See [Mil98, Cor. 4.14].

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