Faithfully Flat Descent

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Introduction

Let $f_i : X_i \to Y$ be a family of morphisms of schemes. Given a quasi-coherent sheaf $\mathcal{F}$ on Y, we will be interested in reconstructing $\mathcal{F}$ from its pullbacks $f_i^*\mathcal{F}$. This problem is called a descent problem. We will see that sheaves on the X_i that come from sheaves on Y carry additional structure, called *descent data*, that will allow (in the faithfully flat and quasi-compact case) to reconstruct the original sheaf on Y. We shall give special attention to the case of a single morphism.

1 Background

1.1 Faithfully flat morphisms

Definition 1.1. *A ring homomorphism $f : A \to B$ is called* <u>*flat*</u> *if whenever a sequence $\Sigma : M_1 \to M_2 \to M_3$ of A-modules is exact, the sequence $\Sigma \otimes_A B$ is also exact. f is called* <u>*faithfully flat*</u> *if the exactness of $\Sigma \otimes_A B$ implies the exactness of Σ.*

Theorem 1.2. *Let $f : A \to B$ be a flat homomorphism. The following conditions are equivalent:*

1. *f is faithfully flat*
2. *For any A-module M, $M \otimes_A B = 0 \Rightarrow M = 0$.*
3. *The map on spectra ${}^a f : Spec(B) \to Spec(A)$ is surjective*
4. *For any maximal ideal $T < A$, $Bf(T) \neq B$.*

Proof. 1<=>2:Let $\Sigma : M_1 \xrightarrow{u} M_2 \xrightarrow{v} M_3$ be a sequence of modules, such that $v \circ u = 0$ and $\Sigma \otimes_A B$ is exact. Since f is flat, we have $(Ker(v)/Im(u)) \otimes_A B = (Ker(v) \otimes_A B)/(Im(u) \otimes_A B)$. Since $Ker(v) \otimes_A B \subseteq Ker(v \otimes id)$ and $Im(v) \otimes_A B = Im(v \otimes id)$, we get $(Ker(v)/Im(u)) \otimes_A B \subseteq (Ker(v \otimes id)/Im(u \otimes id)) = 0$ and by the assumption (1), we are done. The other implication is obvious, by considering $0 \to M \to 0$.

2=>3: Let P be a prime ideal of A, and let $k(P) = A_P/PA_P = Frac(A/P)$. Since $k(P) \neq 0$, $k(P) \otimes_A B \neq 0$. But $Spec(k(P) \otimes_A B) = {}^a f^{-1}(P)$. Indeed, $k(P) \otimes_A B$ is the quotient of the localization $B_{f(A \setminus P)}$ by the ideal generated by $f(P)$. The prime ideals of $B_{f(A \setminus P)}$ are in one to one correspondence with prime ideals $Q \triangleleft B$ such that $f^{-1}(Q) \subseteq P$. Hence the prime ideals of $k(P) \otimes_A B$ are the above Q's that in addition contain $f(P)$, and hence satisfy $f^{-1}(Q) = P$.

3=>4: In patricular, there is a prime ideal $\triangleleft B$ such that $f^{-1}(Q) = T$, hence $Bf(T) \subseteq Q \subset B$.

4=>2: Suppose $M \neq 0$, and let $0 \neq a \in M$. Since f is flat, $(Aa) \otimes_A B \subseteq M \otimes_A B$. Hence it is enough to prove $M \otimes_A B \neq 0$ assuming that $M = Aa$. But in that case, $M \simeq A/ann(a)$, and $ann(a)$ can be contained in a maximal ideal T of A. Finally, $A/ann(a) \otimes_A B \simeq B/B \cdot f(ann(a)) \twoheadrightarrow B/Bf(T) \neq 0$.

□

The above theorem motivates the following definition:

Definition 1.3. *A morphism of schemes is called faithfully flat if it is flat and surjective.*

Here is a property of faithfully flat morphisms that will prove to be useful later:

Theorem 1.4. *Let $f : A \to B$ be a faithfully flat ring homomorphism, and let M be an A-module. Then the following complex of A-modules*

$$M \xrightarrow{d_0} M \otimes_A B \xrightarrow{d_1} M \otimes_A B \otimes_A B \to ... \to M \otimes_A B^{\otimes n} \to ...$$

defined by

$$d_n(m \otimes b_0 \otimes ... \otimes b_{n-1}) = m \otimes 1 \otimes b_0 \otimes .. \otimes b_{n-1} - m \otimes b_0 \otimes 1 \otimes b_1 \otimes .. \otimes b_{n-1} + ... + (-1)^n m \otimes b_0 \otimes .. \otimes b_{n-1} \otimes 1$$

is exact.

Proof. Let us say that f has a section if there exists a homomorphism $g : B \to A$ such that $g \circ f = id$. Let us prove the theorem first in this case.

Define $s_n : M \otimes_A B^{\otimes n} \to M \otimes_A B^{\otimes n-1}$ by $s_n(m \otimes b_0 \otimes ... \otimes b_{n-1}) = g(b_0)m \otimes b_1 \otimes .. \otimes b_{n-1}$. This definition makes sense since for any $a \in A$, $am \otimes b_0 \otimes .. \otimes b_{n-1} =$

$m \otimes f(a)b_0 \otimes .. \otimes b_{n-1}$ and the definition agrees on both sides. One can check that $sd + ds = id$, hence our complex is homotopy contractible and in particular exact.

In the general case, let us apply $\otimes_A B$ to our complex. The resulting complex can be seen as the complex attached to the module $M \otimes_A B$ and the ring homomorphism $B \overset{1\otimes id}{\to} B \otimes B$. This ring homomorphism has a section $x \otimes y \mapsto xy$, hence the latter complex is exact. By faithful flatness, we are done. $\square$

1.2 Pullbacks

Let us recall the basic definitions regarding pushing sheaves back and forth along morphisms. Suppose a morphism of schemes $f : X \to Y$ is given, then given a sheaf $\mathcal{F}$ on X, its *direct image* $f_*\mathcal{F}$ is the sheaf on Y given by

$$f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}(U)).$$

Given a sheaf of modules $\mathcal{G}$ on Y one can define its *inverse image, or pullback* $f^*\mathcal{G}$. It is a sheaf of modules on X. The functor f^* is left-adjoint to the functor f_*. see [ChK] for more details.

Definition 1.5. *Let $f : X \to Y$ be a morphism of schemes, and let $\mathcal{F}, \mathcal{G}$ be sheaves over X, Y respectively. A map of sheaves $\phi : \mathcal{G} \to \mathcal{F}$ above f is a morphism of sheaves $f^*\mathcal{G} \to \mathcal{F}$ (Note that by the adjointness property, giving such morphism is equivalent to giving a morphism $\mathcal{G} \to f_*F$). Given $X \overset{f}{\to} Y \overset{g}{\to} Z$, and maps of sheaves $\mathcal{H} \overset{\psi}{\to} \mathcal{G} \overset{\phi}{\to} \mathcal{F}$ over f and g respectively, we have a map $f^*\psi : f^*g^*\mathcal{H} \to f^*\mathcal{G}$, and we define the composition of ϕ and ψ to be the map*

$$\phi \circ f^*\psi : (gf)^*\mathcal{H} \to \mathcal{F}. \tag{1}$$

1.3 Quasi-Coherent sheaves of modules

Just as we attach to a ring A the scheme $Spec(A)$, it is natural to attach to an A-module M a sheaf of $O_{spec(A)}$-modules $\tilde{M}$ on $Spec(A)$. It is defined as follows: For an open set $U \subset A$, $\tilde{M}(U)$ is defined as the set of sections $f : \{P : P \in U\} \rightrightarrows \{M_P : P \in U\}$ such that locally, f is given by m/d for some $m \in M$ and $d \in A$.

A sheaf of O_X-modules $\mathcal{F}$ on a scheme X is called quasi-coherent if for any affine subset $U = spec(A) \subseteq X$, there exists an A-module M such that $\mathcal{F}|_U \simeq \tilde{M}$. This condition can be equally checked on an affine cover of X only. (see [H], p.113).

For a noetherian scheme X, the sheaf $\mathcal{F}$ is called coherent if in addition, the modules M can be taken to be finitely generated. Again, it is enough to verify this property for an affine cover. However, for non-noetherian schemes this is not the case, and one calls a sheaf $\mathcal{F}$ on X coherent if there exists an affine cover for which the property holds.

We will need the following proposition ([H], p. 110):

Proposition 1.6. *Let $f : A \to B$ be a homomorphism of rings, and let ${}^a f : specB \to spec(A)$ be the corresponding map on spectra. Then:*

1. *The map $M \mapsto \tilde{M}$ gives an equivalence of categories between the categoty of A-modules and the category of quasi-coherent $O_{spec(A)}$-modules.*
2. *If M and N are two A-modules then $(\widetilde{M \otimes N}) \simeq \tilde{M} \otimes_{O_{spec(A)}} \tilde{N}$*
3. *For any B-module N we have ${}^a f_*(\tilde{N}) \simeq ({}_A N)$ where ${}_A N$ means N considered as an A-module*
4. *For any A-module M, $f^*(\tilde{M}) \simeq (\widetilde{M \otimes_A B})$.*

2 Faithfully flat descent

2.1 Glueing of sheaves (Zariski descent)

We begin our journey into fpqc descent with the problem of reconstructing a sheaf from its restrictions to the various sets of an open cover.

Proposition 2.1. *Let X be a topological space, let $\{U_i\}$ be an open cover of X. To give a sheaf on X is equivalent to giving for each i a sheaf $\mathcal{F}_i$ on U_i, together with isomorphisms $\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \tilde{\to} \mathcal{F}_j|_{U_i \cap U_j}$ such thay (i) $\phi_{ii} = id$ and (ii) for any i, j, k we have $\phi_{ik} = \phi_{jk}\phi_{ij}$ on $U_i \cap U_j \cap U_k$.*

Proof. Given the sheaves $\{\mathcal{F}_i\}$, let us define the sheaf $\mathcal{F}$ in the following way: For an open subset W of X, we let $\mathcal{F}(W)$ be the set of all $(t_i \in \mathcal{F}_i(W \cap U_i))_i$ such that for every i, j we have

$$\phi_{ij}(t_i|_{W \cap U_i \cap U_j}) = t_j|_{W \cap U_i \cap U_j}. \tag{2}$$

The restriction map $F(W) \to F(W')$ sends $(t_i \in \mathcal{F}_i(W \cap U_i))_i$ to $(t_i|_{W' \cap U_i})_i$. Now in the verification of the sheaf axioms, let $\{W_\alpha\}$ be an open cover of an open set W and suppose given elements $t_\alpha \in \mathcal{F}(W_\alpha)$ such that $t_\alpha|_{W_\alpha \cap W_\beta} = t_\beta|_{W_\alpha \cap W_\beta}$, then $t_\alpha = (t_{\alpha,i} \in \mathcal{F}_i(W_\alpha \cap U_i))_i$,

$$\phi_{ij}(t_{\alpha,i}|_{W_\alpha \cap U_i \cap U_j}) = t_{\alpha,j}|_{W_\alpha \cap U_i \cap U_j} \tag{3}$$

, and $t_{\alpha,i}|_{W_\alpha \cap W_\beta \cap U_i} = t_{\beta,i}|_{W_\alpha \cap W_\beta \cap U_i}$. We wish to define an element $t \in \mathcal{F}(W)$, given by $(t_i \in \mathcal{F}_i(W \cap U_i))_i$, so that for any α, $t|_{W_\alpha} = t_\alpha$, that is, $t_i|_{W_\alpha \cap U_i} = t_{i,\alpha}$. Such t_i exist and are unique by the sheaf axiom for $\mathcal{F}_i$. We have to verify that (2) holds. restricting both sides to W_α, this follows from (3) and the sheaf axioms for $\mathcal{F}_i$. But now, a mystery arises: the cocycle condition $\phi_{ik} = \phi_{jk}\phi_{ij}$ was never used! Indeed, we can glue the sheaves $\mathcal{F}_i$ to a sheaf without having this condition. However, if we want the outcoming sheaf $\mathcal{F}$ to restrict back to $\mathcal{F}_i$ on U_i, this property must hold, as can be easily checked.

□

By the local property of a sheaf being quasi coherent, the above proposition remains true if we take X to be a scheme and consider glueing of quasi coherent sheaves, or of coherent sheaves when X is noetherian.

2.2 Specifying local data: Sieves vs. Čech approaches

Let us review the statement of proposition 2.1: Given a topological space X and a cover $\mathcal{U} = (U_i)$ of X, we say that <u>to give a sheaf on X $\mathcal{U}$-locally</u> is to give for each U_i a sheaf $\mathcal{F}_i$ on it, together with the maps ϕ_{ij} satisfying (i) and (ii). One can actually form a category $Sheaves_{\mathcal{U}}(X)$ whose objects are the sheaves on X given $\mathcal{U}$-locally, and restate proposition 2.1 as an equivalence of categories between this category and the category of sheaves on X. This point of view will be called, for evident reasons, <u>the Čech point of view</u>.

On the other hand, one can get rid of the need for the maps ϕ_{ij} by specifying the restrictions of $\mathcal{F}$ not only to U_i, but to any open subset contained in one of the U_i's. That is <u>the point of view of sieves.</u>

Definition 2.2. *Let X be a topological space, and let $Op(X)$ be the set of open subsets of X. <u>A sieve on X</u> is a family $\mathcal{U} \subseteq Op(X)$ such that $U \in U$, $Op(X) \ni V \subset U \Rightarrow V \in \mathcal{U}$. A sieve $\mathcal{U}$ is called <u>covering</u> if $\bigcup_{U \in \mathcal{U}} U = X$. Given any family $(U_i)_i \subset Op(X)$, <u>the sieve generated by the family</u> is $\{U \in Op(X) : \exists i : U \subseteq U_i\}$.*

Definition 2.3. *Let X be a topological space and let $\mathcal{U}$ be a sieve on X. <u>A sheaf given $\mathcal{U}$-locally</u> is a collection of sheaves $\mathcal{F}_U : U \in \mathcal{U}$ together with isomorphisms $Res^U_V : \mathcal{F}_U|_V \xrightarrow{\sim} \mathcal{F}_V$ for any $U \supseteq V$, $U, V \in \mathcal{U}$, such that whenever $U \supseteq V \supseteq W$ we have $Res^V_W(Res^U_V|_W) = Res^U_W$.*

Again, regarding a sheaf given $\mathcal{U}$-locally as a diagram of sheaves over the directed set $\mathcal{U}$, we can form the category of sheaves given $\mathcal{U}$-locally. We can now give a "sieve version" of proposition 2.1:

Proposition 2.4. *Let $\mathcal{U}$ be a covering sieve on a topological space X. The functor from the category of sheaves on X to the category of sheaves on X given $\mathcal{U}$-locally, associating with any sheaf on X the sheaves $(\mathcal{F}_U = F|_U)_{U\in\mathcal{U}}$ and the identity maps $F|_U|_V \to F|_V$ is an equivalence of categories.*

The point of view of sieves seems to be more convenient when generalizing to other categories.

2.3 Statement of the problem

Suppose that your scheme X is covered by *morphisms* $\phi_i : U_i \to X$, $i = 1, .., n$ (i.e. $X = \bigcup \phi_i(U_i)$). We would like to reconstruct a quasi-coherent sheaf $\mathcal{F}$ on X from its pullbacks $\phi_i^*\mathcal{F}$ on U_i. Grothendieck's theorem of faithfully flat descent guarantees that this is possible, provided that the morphisms ϕ_i are flat, and that we specify enough additional information on the pullback (called descent data), conveniently packed using the sieves point of view.

Definition 2.5. *Let X be a scheme, and let $\mathbf{G}$ be a class of schemes over X stable under fiber products over X. A subclass $\mathcal{U} \subset \mathbf{G}$ is called a sieve over X if for any $U \in \mathcal{U}$, and $V \in \mathbf{G}$ such that a morphism $V \to U$ over X evists, V is in $\mathcal{U}$. The sieve over X generated in $\mathbf{G}$ by a family $\{U_i\} \subset \mathbf{G}$ is the sieve of all the schemes in $\mathbf{G}$ having a morphism to one of the U_i's over X.*

Definition 2.6. *Let $\mathcal{U}$ be a sieve over a scheme X. A quasi-coherent module on X given $\mathcal{U}$-locally is a collection of quasi coherent modules $(E_U)_{U\in\mathcal{U}}$, together with isomorphisms $\rho_\phi : E_V \tilde{\to} \phi^*E_U$ given for any morphism $\phi : V \to U$ ($U, V \in \mathcal{U}, \phi \in Mor(\mathbf{G})$), such that given a composition $W \xrightarrow{\psi} V \xrightarrow{\phi} U$, the following diagram commutes:*

$$\begin{tikzcd} E_W \arrow[rr, "\rho_{\phi\circ\psi}"] \arrow[dr, "\rho_\psi"'] & & \psi^*\phi^*R_U \\ & \psi^*E_V \arrow[ur, "\psi^*\rho_\phi"'] & \end{tikzcd}$$

(cf. section 1.2).

Theorem 2.7. *(Fpqc descent).Let $\mathcal{U}$ be a covering sieve over X, generated by a finite number of flat and quasi-compact X-schemes. Giving a quasi-coherent module on X is equivalent to giving it U-locally.*

2.4 The case of a single covering scheme

Suppose now our sieve $\mathcal{U}$ is generated by a single scheme $U \to X$. We will show that the data coming from all the possible morphisms $V \to U$ is encapsulated in the data coming from the morphisms

$$U^* : U \times_X U \times_X U \begin{array}{c} \xrightarrow{p_{23}} \\ \xrightarrow{p_{13}} \\ \xrightarrow{p_{12}} \end{array} U \times_X U \begin{array}{c} \xrightarrow{p_1} \\ \xrightarrow{p_2} \end{array} U.$$

Remark: The multitude of arrows stands for several morphisms in each step. Here the arrows satisfsy the 3 commutativity relations:

$$\begin{aligned} p_1 p_{12} &= p_1 p_{13} \\ p_2 p_{12} &= p_1 p_{23} \\ p_2 p_{23} &= p_2 p_{13} \end{aligned} \tag{4}$$

In general, a diagram $A \begin{array}{c} \to \\ \to \\ \to \end{array} B \begin{array}{c} \to \\ \to \end{array} C$ is called <u>cartesian</u> if its arrows satisfy these relations.

Now the descent data gives us quasi-coherent sheaves $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ over $U, U \times_X U$ and $U \times_X U \times_X U$ respectively, together with isomorphisms $f_i : p_i^* \mathcal{F}_1 \tilde{F}_2$,$f_{ij} : p_{ij}^* \mathcal{F}_2 \tilde{F}_3$, that fit in the diagram

$$\mathcal{F}^* : \mathcal{F}_1 \begin{array}{c} \to \\ \to \end{array} \mathcal{F}_2 \begin{array}{c} \to \\ \to \\ \to \end{array} \mathcal{F}_3$$

this is a cartesian diagram of maps over the maps of U^*, in the sense of section 1.2.

We claim that this diagram determines the whole module given $\mathcal{U}$-locally. Indeed, for any $V \in U$ there exists a morphism $\phi : V \to U$ and we can set $E_V = \phi^* \mathcal{F}_1$. The point is that this definition depends on the morphism ϕ, while E_U should depend on V only. But given two morphisms over X $\phi_1, \phi_2 : V \to U$, we have a unique morphism $\phi_1 \times \phi_2 : V \to U \times_X U$ commuting with the p_i and the ϕ_i, and one can identify

$$\phi_1^* \mathcal{F}_1 = (p_1 \circ (\phi_1 \times \phi_2))^* \mathcal{F}_1 = (\phi_1 \times \phi_2)^* p_1^* \mathcal{F}_1 \cong (\phi_1 \times \phi_2)^* \mathcal{F}_2 \cong \phi_2^* \mathcal{F}_1.$$

These identifications are compatible with each other. This can be verified (for 3 morphisms ϕ_i) in the same way, using $\mathcal{F}_3$. With this definition of the E_V's it is now easy to define the isomorphisms $\rho_\phi : E_V \xrightarrow{\sim} \phi^* E_U$ and verify the compatibility relations.

We shall call such a diagram *descent data* for quasi coherent modules with respect to $U \to X$.

It is sometimes more convenient to write the conditions of descent data in the following way:

Lemma 2.8. *Giving descent data is equivalent to giving a quasi-coherent sheaf $\mathcal{F}_1$ on U and an isomorphism*

$$\phi : p_1^* \mathcal{F}_1 \xrightarrow{\sim} p_2^* \mathcal{F}_1$$

such that the following diagram is commutative:

$$\begin{array}{ccc} p_{12}^* p_1 \mathcal{F}_1 & \to p_{12}^* p_2^* \mathcal{F}_1 = p_{23}^* p_1^* \mathcal{F}_1 \to & p_{23}^* p_2^* \mathcal{F}_1 \\ \| & & \| \\ p_{13}^* p_1^* \mathcal{F}_1 & \longrightarrow & p_{13}^* p_2^* \mathcal{F}_1 \end{array}$$

namely:

$$p_{13}^* \phi = p_{23}^* \phi \circ p_{12}^* \phi \tag{5}$$

Proof. Given ϕ, define $\mathcal{F}_2 = p_2^* \mathcal{F}_1$,$\mathcal{F}_3 = p_{13}^* \mathcal{F}_2$, $f_1 = \phi$,$f_2 = id$,$f_{13} = id$,$f_{12} = p_{23}^* \phi : p_{12}^* \mathcal{F}_2 = p_{12}^* p_2^* \mathcal{F}_1 = p_{23}^* p_1^* \mathcal{F}_1 \to p_{23}^* p_2^* \mathcal{F}_1 = p_{13}^* p_2^* \mathcal{F}_1$,$f_{23} = id : p_{23}^* p_2^* \mathcal{F}_1 \to p_{13}^* p_2^* \mathcal{F}_1$.

Comparing (4) with (1) we can write explicitely the expected commutativity relations of the cartesian diagram:

$$\begin{aligned} f_{12} p_{12}^* f_1 &= f_{13} p_{13}^* f_1 \\ f_{12} p_{12}^* f_2 &= f_{23} p_{23}^* f_1 \\ f_{23} p_{23}^* f_2 &= f_{13} p_{13}^* f_2 \end{aligned}$$

plugging in the definitions of the f_i and f_{ij} we get that the first equation is exactly the condition on ϕ and the other two are obvious.

In the other direction, given the descent data $(\mathcal{F}_t, f_i, f_{ij})$ we can define in a similar fashion $\phi = f_2^{-1} \circ f_1$ and verify that the condition on ϕ is satisfied. □

2.5 Descent of modules

Suppose now that in addition, X and U are affine. Let $X = spec(A)$, $U = spec(B)$ and let the morphism come from a homomorphism $f : A \to B$. By theorem 1.2 (and since flatness is a local property), f is faithfully flat.

By the results of the previous section, the quasi-coherent X-module given $\mathcal{U}$-locally is determined by a diagram of quasi-coherent modules M^* above the diagram of X-schemes U^*. But in the affine case, using proposition 1.6, this is equivalent to giving a cartesian diagram of modules

$$M^* : M_1 \begin{array}{c} \stackrel{\sigma_1}{\rightarrow} \\ \stackrel{\sigma_2}{\rightarrow} \end{array} M_2 \begin{array}{c} \stackrel{\sigma_{23}}{\rightarrow} \\ \stackrel{\sigma_{13}}{\rightarrow} \\ \stackrel{\sigma_{12}}{\rightarrow} \end{array} M_3$$

over the diagram of rings

$$B^* : B \begin{array}{c} \rightarrow \\ \rightarrow \end{array} B \otimes_A B \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \end{array} B \otimes_A B \otimes_A B$$

(Here again, giving a ring homomorphism $S \to T$ and modules N, M over S, T respectively, a map $N \to M$ over the map $S \to T$ is a homomorphism of T-modules $N \otimes_S T \to M$, or equivalently a homomorphism of S-modules $N \to_S M$),

such that all the arrows in M^* induce isomorphisms (e.g. $\sigma_0 : M_1 \otimes_B (B \otimes_A B) = M_1 \otimes_A B \stackrel{\sim}{\to} M_2$).

Starting with an A-module N, we obtain from it a diagram $M^*(N)$ by putting

$$M_1(N) = N \otimes_A B,\ M_2(N) = N \otimes_A B \otimes_A B,\ M_3(N) = N \otimes_A B \otimes_A B \otimes_A B$$

The whole question of faithfully flat descent now boils down to the following algebraic question: Given a B-module M_1, we wish to know whether there exists an A-module N such that $M_1 \simeq N \otimes_A B$. What is the additional structure on such M_1 that will allow us to recover N? The answer is: If M_1 comes from an module N, then it is equipped with descent data, namely the diagram $M^*(N)$ (Note that the special thing about $M^*(N)$ is not the modules $M_i(N)$, which are determined already by M_1, but the mappings of the cartesian diagram which can be defined only using the "A-model" N). In case $f : A \to B$ is faithfully flat, this descent data is exactly what it takes to recover N:

Theorem 2.9. *Let $f : A \to B$ be a faithfully flat ring homomorphism. The functor $N \to M^*(N)$ is an equivalence of categories between Mod_A and the category of cartesian diagrams M^* over B^* whose arrows induce isomorphisms.*

Proof. The above functor has an inverse functor, the functor which attaches to

$$M^* : M_1 \begin{array}{c} \stackrel{\sigma_1}{\rightarrow} \\ \stackrel{\sigma_2}{\rightarrow} \end{array} M_2 \begin{array}{c} \stackrel{\sigma_{23}}{\rightarrow} \\ \stackrel{\sigma_{13}}{\rightarrow} \\ \stackrel{\sigma_{12}}{\rightarrow} \end{array} M_3$$

The A-module $ker(M_1 \overset{\rightarrow}{\rightarrow} M_2) = \{x \in M_1 : \sigma_1(x) = \sigma_2(x)\}$. To verify this we need to show two isomorphisms:

1. For an A-module N, $ker(N \otimes_A B \overset{\rightarrow}{\rightarrow} N \otimes_A B \otimes_A B) \simeq N$

2. For a diagram M^* as above, $ker(M_1 \overset{\rightarrow}{\rightarrow} M_2) \otimes_A B \simeq M_1$.

the first isomorphism results from theorem 1.4. For the second isomorphism, consider the exact sequence of A-modules

$$ker(M_1 \overset{\rightarrow}{\rightarrow} M_2) \rightarrow M_1 \overset{\rightarrow}{\rightarrow} M_2$$

(note that we can replace the two arrows pointing at M_2 by one arrow which represents their difference, and then we get an exact sequence in the usual sense. In the rest of this proof, we shall regard a pair of arrows this way).

By flatness, the following sequence is also exact:

$$ker(M_1 \overset{\rightarrow}{\rightarrow} M_2) \otimes_A B \rightarrow M_1 \otimes_A B \overset{\rightarrow}{\rightarrow} M_2 \otimes_A B.$$

By theorem 1.4 (used for $B \rightarrow B \otimes B$), we also have the exact sequence

$$M_1 \rightarrow M_1 \otimes_B (B \otimes_A B) \overset{\rightarrow}{\rightarrow} M_1 \otimes_B (B \otimes_A B \otimes_A B)$$

the two sequences can be fitted into a commutative diagram in which two of the vertical maps are isomorphisms: in the middle column it is the evident one, and in the right columns we have two isomorphisms resulting from tensoring $\sigma_1, \sigma_2 : M_1 \otimes_B (B \otimes B) \overset{\rightarrow}{\rightarrow} M_2$ with B. By the five lemma, we have the second isomorphism. $\square$

2.6 Treatment of the general case

To conclude the proof of theorem 2.7, we have to reduce the general case to the case treated in the last section (in which, the scheme X is affine and the sieve $\mathcal{U}$ is generated by a single affine scheme). We shall proceed in several steps:

1. Reduction to the case where $\mathcal{U}$ is generated by a single scheme: Suppose that $\mathcal{U}$ is generated by $U_1, .., U_n$. We let $U = \amalg_i U_i$. It is easy to define a quasi-coherent sheaf on X locally with respect to the sieve generated by U. Since this sieve is more rich than $\mathcal{U}$ and our definition agrees with the previous one on $\mathcal{U}$, we are done.

2. Reduction to the case where X is affine: Our covering is given now by a morphism $\phi : U \to X$. For any affine open subset J of X, we have a faithfully flat covering of J, $\phi^{-1}(J) \to J$. By restriction, we have descent data on this covering, and assuming our theorem true in the case where X is affine, we get a quasi-coherent sheaf $\mathcal{F}_J$ on J. Now use Zariski descent to glue the sheaves to a sheaf on X.

3. Reduction to the case where $\mathcal{U}$ is generated by affine schemes: We now take a step back, and reduce back to the case where $\mathcal{U}$ may be generated by several schemes, but this time they will be affine: We cover U by affines A_j. We get descent data with respect to the sieve generated by the A_j, hence we have a quasi-coherent sheaf on X whose pullback to the A_j's is the desired one. The pullback of this sheaf to U is also the desired one because of the uniqueness of Zariski descent on U.

4. Reduction to the case where $\mathcal{U}$ is generated by a single affine scheme: Finally, we repeat step 1 with affine covering schemes: We can now suppose that X is affine, and $\mathcal{U}$ is generated by affine schemes flat over X, say $U_1, .., U_n$, $U_i = Spec(R_i)$. We let $U = \amalg_i U_i = spec(R_1 \times .. \times R_n)$. Now U is affine, and again it is easy to define a quasi-coherent sheaf on X locally with respect to the sieve generated by U. Since this sieve is more rich than $\mathcal{U}$ and our definition agrees with the previous one on $\mathcal{U}$, we are done.

3 Examples

3.1 Galois cohomology and Hilbert 90

Let L/K be a finite Galois field extension with Galois group G. We would like to study descent of quasi-coherent modules with respect to the map $Spec(L) \to Spec(K)$, or equivalently descent of modules with respect to $K \hookrightarrow L$. Modules over K, L are just *vector spaces*, and we have an isomorphism

$$L \otimes_K L \xrightarrow{\sim} \bigoplus_{\sigma \in G} L \tag{6}$$

$$x \otimes y \mapsto (x \cdot \sigma(y))_\sigma$$

(Proof: by taking a primitive element $\theta \in L$, let f be the minimal polynomial of θ, then

$$L \otimes_K L \cong (K[x]/(f)) \otimes_K L \cong L[x]/(f) = L[x]/\prod_{\sigma \in G}(x - \sigma(\theta)) \cong \bigoplus_{\sigma \in G} L$$

and the isomorphism from $(K[x]/(f)) \otimes L$ to both sides take $\bar{g} \otimes r$ to $g(\theta) \otimes r$ on one hand, and $(g(\sigma(\theta)) \cdot r)_{\sigma \in G}$ on the other.)

Let us see what it means to give descent data with respect to this homomorphism: we have a diagram of modules

$$V_1 \underset{\xrightarrow{\phi_2}}{\xrightarrow{\phi_1}} V_2 \begin{array}{c}\xrightarrow{\phi_{23}} \\ \xrightarrow{\phi_{13}} \\ \xrightarrow{\phi_{12}}\end{array} V_3$$

over

$$L \underset{\xrightarrow{\sigma_2}}{\xrightarrow{\sigma_1}} L \otimes_K L \begin{array}{c}\xrightarrow{\sigma_{23}} \\ \xrightarrow{\sigma_{13}} \\ \xrightarrow{\sigma_{12}}\end{array} L \otimes_K L \otimes_K L$$

Satisfying the requirements of descent data for modules stated above. In particular, we can take $V_2 = V_1 \otimes_{L,\sigma_1} (L \otimes_K L)$ (Making the isomorphism induced by ϕ_1 an identification). By (6), we have

$$V_2 = \bigoplus_{\sigma \in G} V_1$$

viewed as an $L \otimes_K L$-module by $(x \otimes y) \cdot (v_\sigma)_{\sigma \in G} = (x\sigma(y)v_\sigma)_{\sigma \in G}$. Now, ϕ_2 is given by a homomorphism $V_1 \to V_2$ in which V_2 has L-module structure through σ_2. We can write

$$\phi_2(v) = (\phi_\sigma(v))_{\sigma \in G}$$

where the $\phi_\sigma : V_1 \to V_1$ satisfy:

$$\phi_\sigma(\lambda v) = \sigma(\lambda)\phi_\sigma(v).$$

Further, we can write the condition on ϕ_2 as a cocycle condition, resulting from (5):

$$\phi_2 \otimes_{\sigma_{1,3}} (L \otimes_K L \otimes_K L) = (\phi_2 \otimes_{\sigma_{2,3}} (L \otimes_K L \otimes_K L)) \circ (\phi_2 \otimes_{\sigma_{1,2}} (L \otimes_K L \otimes_K L))$$

both homomorphisms from $V_1 \otimes_{L,\sigma_2} (L \otimes_K L) \otimes_{L\otimes L,\sigma_{1,3}} (L \otimes_K L \otimes_K L) = V_1 \otimes_{L,\sigma_{1,3}\circ\sigma_2} (L \otimes_K L \otimes_K L)$ to $V_1 \otimes_{L,\sigma_{1,3}\circ\sigma_1} (L \otimes_K L \otimes_K L)$. Writing it in coordinates, we get the condition:

$$\forall \sigma, \tau \in G, \phi_{\sigma\tau} = \phi_\sigma \phi_\tau.$$

Thus, the descent data is equivalent to a *semilinear operation* of G on V_1. Let us take now the K-vector space $V_1' = V_1^G = ker(V_1 \underset{\phi_2}{\overset{\phi_1}{\rightrightarrows}} V_2)$. By our previous discussion, we have

$$V_1' \otimes_K L = V_1 \tag{7}$$

.

In particular, let $V_1 = Lv$ be one dimensional, then our semilinear operation is determined by constants $c(\sigma) \in L^*$ so that $\phi_\sigma(v) = c(\sigma)v$. For $\sigma, \tau \in G$ our conditions imply (and, in fact, are eqiuvalent to)

$$c(\sigma\tau)v = \phi_{\sigma\tau}(v) = \phi_\sigma(c(\tau)v) = \sigma(c(\tau))c(\sigma)v$$

this means that the cochain $(c(\sigma))_{\sigma\in G}$ is a cocycle in $C^1(G, L^*)$. By (7), v is a multiple of a G-invariant vector v': Say $v' = \mu v$, $\mu \in L^*$. We have $v' = \phi_\sigma(v') = \phi_\sigma(\mu v) = \sigma(\mu)\phi_\sigma(v) = \sigma(\mu)c(\sigma)v = \sigma(\mu)c(\sigma)\mu^{-1}v'$. Hence, $\forall \sigma \in G, c(\sigma) = \mu\sigma(\mu)^{-1}$ and we see that c is a coboundary. Hence, we have proved Hilbert's theorem 90:

Corollary 3.1. *We have $H^1(G, L^*) = 0$.*

3.2 Galois Descent

In this section we generalize the discussion of the previous one. We wish to describe the descent procedure from U to X in the case where $X = U/\Gamma$, Γ is a finite group, and p is the projection from U to X.

Definition 3.2. *A faithfully flat morphism of schemes $p : U \to X$ is called a Galois covering if there exists a finite group of X-automorphisms Γ such that the morphism*

$$\coprod_{\gamma\in\Gamma} \gamma \times id : \coprod_{\gamma\in\Gamma} U \to U \times_X U$$

is an isomorphism.

Let $p : U \to X$ be such a covering. For a quasi-coherent module $\mathcal{F}$ on U, if $\mathcal{F} = p^*\mathcal{G}$, then for any $\gamma \in \Gamma$, since $p\gamma = p$, we have an isomorphism $f_\gamma : \mathcal{F} = p^*\mathcal{G} \tilde{\to} \gamma^*p^*\mathcal{G} = \gamma^*\mathcal{F}$. Moreover, since $(\gamma_1\gamma_2)^* = \gamma_2^*\gamma_1^*$, we have the cocycle condition

$$\gamma_2^* f_{\gamma_1} \circ f_{\gamma_2} = f_{\gamma_1\gamma_2}.$$

A similar analysis to that of section 3.1 shows that giving isomorphisms f_γ that satisfy this cocycle condition is equivalent to giving descent data for quasi-coherent modules with respect to $U \to X$.

Let us now switch to an analytic category and consider the covering of the modular curve $X(N) = \mathcal{H}/\Gamma(N)$ by the complex upper half-plane $\mathcal{H}$. Let us consider the question of constructing line bundles on $X(N)$. We start with the trivial line bundle $\mathcal{H}\times\mathbf{C} \to \mathcal{H}$ on $\mathcal{H}$ and look for descent data that will give us a line bundle on $\mathcal{H}/\Gamma(N)$. In this setting, given a line bundle $E \to X(N)$, its pullback to $\mathcal{H}$ is the line bundle $E \times_{X(N)} \mathcal{H} \to \mathcal{H}$. For any $\gamma \in \Gamma(N)$, we have a commutative diagram

$$\begin{array}{ccc} E \times_{X(N)} \mathcal{H} & \stackrel{id\times\gamma}{\to} & E \times_{X(N)} \mathcal{H} \\ \downarrow & & \downarrow \\ \mathcal{H} & \stackrel{\gamma}{\to} & \mathcal{H} \end{array}$$

If the line bundle $E \times_{X(N)} \mathcal{H}$ is isomorphic to the line bundle $\mathcal{H} \times \mathbf{C}$ then we get a diagram

$$\begin{array}{ccc} \mathcal{H} \times \mathbf{C} & \stackrel{\phi_\gamma}{\to} & \mathcal{H} \times \mathbf{C} \\ \downarrow & & \downarrow \\ \mathcal{H} & \stackrel{\gamma}{\to} & \mathcal{H} \end{array}$$

the map ϕ_γ must be given by $\phi_\gamma(\tau, c) = (\gamma(\tau), j(\gamma,\tau)\cdot c)$ where $j(\gamma,\tau) \in \mathbf{C}^*$. Moreover, since the upper diagram defines an action of Γ on $E \times \mathcal{H}$ (as a set), the functions ϕ_γ should define an action of $\Gamma(N)$ on $\mathcal{H} \times \mathbf{C}$. We get the cocycle condition

$$j(\gamma_1\gamma_2, \tau) = j(\gamma_1, \gamma_2(\tau))j(\gamma_2, \tau).$$

We have thus recovered the definition of a *factor of automorphy.* Given the factor j, we can define the line bundle on $X(N)$ as $(H \times \mathbf{C})/\Gamma(N) \to \mathcal{H}/\Gamma(N)$ with $\Gamma(N)$ acting on $H \times \mathbf{C}$ via the functions ϕ_γ.

In particular, taking $j(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau) = (c\tau + d)^k$, we get the line bundle whose global sections are the modular forms of weight k for $\Gamma(N)$.

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