Weil Cohomologies

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1. The work of Weil and his followers: Dwork, Artin, Grothendieck, Deligne, etc.

1.1. Weil conjectures. Let X be a smooth, proper variety over $\mathbb{F}_q$. We define the zeta function of X by the formula:

$$Z(X/\mathbb{F}_q, T) = \exp(\sum_{n\geq 1} N_n T^n/n),$$

where N_n is the number of points of X with coordinates in the field $\mathbb{F}_{q^n}$. Weil proved the following properties of $Z(X/\mathbb{F}_q, T)$ in some special cases (X a curve, an abelian variety, or a Fermat hypersurface):

- $Z(X/\mathbb{F}_q, T)$ is a *rational* function of T, i.e.

$$Z(X/\mathbb{F}_q, T) = \frac{P_1(T)P_3(T)\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)},$$

  where the P_i's have integer coefficients, and $P_i(T) = \prod_{j=1}^{b_i}(1-\alpha_{ij}T)$, for algebraic integers α_{ij} such that $|\alpha_{ij}| = q^{i/2}$, the so-called *Riemann hypothesis.*
- The *functional equation*: the $\alpha_{i,j}$ are carried bijectively to the $\alpha_{2n-i,j}$ under $T \mapsto \frac{1}{Tq^n}$.
- The Betti numbers, defined to be the degrees of the polynomials $P_i(T)$, match the topological Betti numbers of a lift of X in characteristic zero.

Weil also observed that most of these properties would follow formally for general varieties X from the existence of a good cohomology theory, what is now called a *Weil cohomology theory.*

1.2. Definition of a Weil cohomology theory. Let X be a smooth, proper variety over $\mathbb{F}_q$.

Definition 1.1. A *cohomology functor* is a contravariant functor

$$X \mapsto H^*(X)$$

from the category of irreducible, smooth, proper varieties X over finite fields to the category of graded anticommutative algebras H^* over a coefficient field K of characteristic zero.

Definition 1.2. A cohomology functor is a *Weil cohomology* ([**5**]) if it satisfies the following list of properties:

- Finiteness: the $H^i(X)$ are finite dimensional vector spaces over K. They are zero except in the range $[0, 2n]$, where $n = \dim(X)$.
- Poincaré duality. The cup-product pairing

$$H^i(X) \times H^{2n-i}(X) \longrightarrow H^{2n}(X),$$

  is a perfect pairing for each $i, 0 \leq i \leq 2n$; moreover, $H^{2n}(X)$ is 1-dimensional.
- Künneth formula: For each X and Y, the projections induce an isomorphism:

$$H^*(X) \otimes H^*(Y) \xrightarrow{\cong} H^*(X \times Y).$$

- Cohomology class of a cycle: for each X, let $C^i(X)$ denote the group of algebraic cycles of codimension i. Then there is a group homomorphism
$$\lambda_X : C^i(X) \longrightarrow H^{2i}(X),$$
called the "cycle map" that is functorial, multiplicative (i.e. compatible with the Künneth formula) and sends a 0-cycle to its degree.
- Weak Lefschetz theorem: Given X, there is an integer $d_0 = d_0(X)$ such that if $f : Y \hookrightarrow X$ is any smooth hypersurface section of X of degree $d \geq d_0$, then $f^* : H^i(X) \longrightarrow H^i(Y)$ is an *isomorphism* for $i \leq n-2$, and is injective for $i = n-1$.
- Hard Lefschetz theorem: Let $L \in H^2(X)$ denote the class of a hyperplane. Then for $i \leq n$, the map $L^i : H^{n-i}(X) \longrightarrow H^{n+i}(X), a \mapsto a \otimes L^{\otimes i}$ is an isomorphism.
- Lefschetz trace formula: Let $f : X \longrightarrow X$ be a morphism with isolated fixed points, and for each fixed point $x \in X$, assume that the action of $1 - df$ on Ω^1_X is injective (i.e. has multiplicity one). Let $L(f, X)$ be the number of fixed points of f. Then
$$L(f, X) = \sum_i (-1)^i \mathrm{Tr}(f^*; H^i(X)),$$
where f^* is the induced map on cohomology.
- Riemann hypothesis: The polynomial $P^i(X/\mathbb{F}_q, T) = \det(1 - TF|H^i(X))$ lies in $\mathbb{Z}[T]$, and its reciprocal zeroes, the eigenvalues of Frobenius, all have complex absolute value $q^{i/2}$.

Example 1.3. The ℓ-adic, cristalline, rigid cohomologies are Weil cohomologies, e.g. $H^i(X, \mathbb{Q}_\ell) := \varprojlim H^i_{et}(X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Z}/\ell^r\mathbb{Z}) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is a Weil cohomology.

1.3. Some formal properties.

1.3.1. *Where we give the formula for the zeta function.* Some parts of the Weil conjectures follow formally from properties of Weil cohomologies: the Lefschetz trace formula implies the rationality of the zeta function, and using this, Poincaré duality implies the functional equation. On the other hand, the proof of the statement about Betti numbers requires a smooth base change theorem and a comparison theorem. We prove in this subsection that the Lefschetz trace formula applied to the Frobenius operator gives the formula of the zeta function (see [**2**, Appendic C]). Among all elements of the algebraic closure of $\mathbb{F}_p$, the elements of $\mathbb{F}_q$ are singled out as the fixed points of the Frobenius morphism $x \mapsto x^q$. The analogous statement is true for points on X: a point of $\overline{X} := X \otimes \overline{\mathbb{F}}_q$ is defined over $\mathbb{F}^r_q$ iff it is fixed under $Frob^r_q$. Thus $F := Frob_q$ is an endomorphism of X over $\mathbb{F}_q$, and $N_r = Fix(F^r)$; thus
$$Z(X/\mathbb{F}_q, T) = \exp(\sum (T^r/r) Fix(F^r)).$$
The Lefschetz fixed point formula then gives us
$$N_r = \sum_{i=0}^{2g} (-1)^i \mathrm{Tr}(F^{r*}; H^i(X)),$$
substituted in the zeta function formula, this gives:
$$Z(X/\mathbb{F}_q, T) = \prod_{i=0}^{2g} \Big(exp(\sum_{r=1}^{\infty} \mathrm{Tr}(f^{r*}; H^i(X) \frac{T^r}{r}) \Big)^{(-1)^i}.$$

LEMMA 1.4. *([**2**, Lemma 4.1]) Let ϕ be an endomorphism of a finite-dimensional vector space V over a field K. Then we have an identity of formal power series in t, with coefficients in K,*

$$exp(\sum_{r=1}^{\infty} \mathrm{Tr}(\phi^r; V)\frac{T^r}{r}) = \det(1 - \phi T; V)^{-1}.$$

PROOF. If $\dim(V) = 1$, then ϕ is multiplication by a scalar $\lambda \in K$, and it says that

$$exp(\sum_{r=1}^{\infty} \lambda^r \frac{T^r}{r}) = \frac{1}{1 - \lambda T}.$$

In general, we use induction on $\dim(V)$. We may assume that K is algebraically closed. Hence ϕ has an eigenvector, so we have an invariant subspace $V' \subseteq V$. We use the exact sequence:

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V/V' \longrightarrow 0$$

and the fact that both sides of the above equation are multiplicative for short exact sequences of vector spaces. By induction, this gives the result.

THEOREM 1.5. *Let X be smooth, proper over $\mathbb{F}_q$ of dimension n. Then*

$$Z(X/\mathbb{F}_q, T) = \frac{P_1(T)P_3(T)\cdots P_{2n-1}(T)}{P_0(T)P_2(T)\cdots P_{2n}(T)},$$

where $P_i(T) = \det(1 - f^*T; H^i(\overline{X}, \mathbb{Q}_\ell))$ and f^* is the map on cohomology induced by the Frobenius $f : \overline{X} \longrightarrow \overline{X}$.

As Hartshorne notes, this shows that $Z(T)$ is a quotient of polynomials with $\mathbb{Q}_\ell$ coefficients. Since we know by definition that it is a power series in $\mathbb{Q}[[T]]$, thence it follows that $Z(T)$ is a rational function (by the equality $\mathbb{Q}[[T]] \cap \mathbb{Q}_\ell(T) = \mathbb{Q}(T)$). $\square$

1.3.2. *Where we see that some properties of Weil cohomologies follow automatically from others.* Katz and Messing have proven, assuming the existence of *a* Weil cohomology defined as we did above, that if a cohomology theory satisfies Poincaré duality, the weak Lefschetz theorem and the formula for the zeta function, it automatically satisfies the hard Lefschetz theorem and the Riemann hypothesis.
We explain what they mean by a cohomology theory, and refer to their 4 pages paper ([**4**] for the proof. A cohomology theory à la Katz-Messing is a cohomology functor $\mathcal{H}$ in our sense, such that the following hold:

- Poincaré duality. Let $X/\mathbb{F}_q$ be as above, with $n = \dim(X)$. Then $\mathcal{H}^{2n}(X)$ is one-dimensional, $\mathcal{H}^i(X) \otimes \mathcal{H}^{2n-i}(X) \longrightarrow \mathcal{H}^{2n}(X)$ is a perfect pairing, and Frobenius F relative to $\mathbb{F}_q$ acts as multiplication by q^n (this implies that F is an automorphism of each $\mathcal{H}^i(X)$).
- Weak Lefschetz Theorem.
- The zeta-function formula: let $\mathcal{P}^i(X/\mathbb{F}_q, T) = \det(1 - TF|\mathcal{H}^i(X))$. Then the zeta function $Z(X, T)$ is given by the formula:

$$Z(X/\mathbb{F}_q, T) = \prod_{i=0}^{2n} (\mathcal{P}^i(X/\mathbb{F}_q, T))^{(-1)^{i+1}}.$$

Deligne had shown earlier that the Hard Lefschetz Theorem and the Riemann hypothesis hold for ℓ-adic ($\ell \neq p$) étale cohomology using his sophisticated monodromy techniques.

2. Cohomology of an abelian variety

The Riemann hypothesis implies that the polynomials appearing in the zeta function formula are defined over $\mathbb{Z}$ and are *independent of* ℓ. We can therefore compute them with any cohomology theory we like (e.g. ℓ-adic, cristalline) satisfying the axioms Katz and Messing singled out. We shall use the cristalline cohomology, but the same theorems hold for the ℓ-adic cohomology.

The Dieudonné module $\mathbb{D}(A(p))$ of the p-divisible group of A can be identified with the first crystalline cohomology group $H^1_{cris}(A)$; it is a free $W(k)$-module of rank $2g$. Moreover, $H^*(A/W(k))$ is torsionfree: the canonical arrow

$$\wedge H^1(A/W(k)) \longrightarrow H^*(A/W(k))$$

is an isomorphism.

THEOREM 2.1. *Let A be an abelian variety. Then $H^i_{cris}(A) = \wedge^i H^1_{cris}(A)$.*

PROOF. An abelian variety A lifts to characteristic zero to a formal scheme $\mathfrak{A}/W(k)$. By [**3**, III, 7.1] (see also Pete's notes),

$$\wedge H^1_{DR}(\mathfrak{A}/W(k)) \xrightarrow{\cong} H^*_{DR}(\mathfrak{A}/W(k)),$$

and by the comparison theorem between de Rham and cristalline cohomology, we get the result.

$\square$

2.1. The zeta function of a curve and its Jacobian. There is a canonical way to associated to a smooth, proper curve C of genus g an abelian variety $Jac(C)$ of dimension g called the *Jacobian* of the curve. The cohomology of an abelian variety is entirely determined by its first cohomology group. Moreover, the first cohomology group of a curve and its Jacobian are isomorphic:

THEOREM 2.2. *Let C be a smooth proper curve. Let $Jac(C)$ be its Jacobian variety. Then $H^1(C) \xrightarrow{\cong} H^1(Jac(C))$.*

PROOF. See [**3**, 3.11.2] for crystalline cohomology. $\square$

This allows us to compute the zeta function of a Jacobian variety. Note, though, that it is not true in general that any abelian variety is the Jacobian of a curve. The locus T^0_g of Jacobian varieties of algebraic curves of genus g in the moduli space $\mathcal{A}_{g,1}$ of (principally polarized) abelian varieties of dimension g is called the *Torelli locus*. If T_g is the Zariski closure of T^0_g in $A_{g,1}\otimes\mathbb{C}$, then for $g \leq 3$, $T_g = \mathcal{A}_{g,1}$.If $g \geq 4$, then $T_g \neq \mathcal{A}_{g,1}$. This follows essentially from the dimensions: the Torelli theorem indicates that the dimension of T_g is $3g-3$; the dimension of $\mathcal{A}_{g,1}$ is $\frac{g(g+1)}{2}$ (these two numbers are equal if and only if $g \leq 3$). Suppose that C is a curve of genus g over a finite field. Computing $N_1, \ldots, N_g$ is enough to determine N_r, for all $r \geq 1$, and we can in theory compute the zeta function of C if we are given an explicit formula. Suppose that $Z(C,T) = \frac{P_1(T)}{(1-T)(1-qT)}$, where $P_1(T) = \prod_{i=1}^{2g}(1-a_iT)$ is a polynomial of degree $2g = H^1(C)$ with coefficients in $\mathbb{Z}$, the *characteristic polynomial.*

PROPOSITION 2.3. *([**7**, Section 14]) The number of points of A over $\mathbb{F}_{q^m}$ is*

$$\prod_{i=1}^{2g}(1-a_i^m).$$

PROOF. Replace $Frob$ (resp. q, resp. a_i) by $Frob^m$ (resp. q^m, resp. a_i^m) in the formula :

$$|A(\mathbb{F}_q)| = \deg(F - id) = P_{Frob}(T) = \prod(1 - a_i).$$

$\square$

COROLLARY 2.4. *Define $P_r(T)$ as the characteristic polynomial of Frobenius acting on $\wedge^i H^i(Jac(C))$. We can write $P_r(T) = \prod(1 - a_{i,r}T)$, where the $a_{i,r}$ run through the products $a_{i_1}a_{i_2}\dots a_{i_r}, 0 < i_1 < \cdots i_r \leq 2g$, a_i a (reciprocal) root of $P_1(T)$. Then*

$$Z(Jac(C), T) = \frac{P_1(T)\cdots P_{2g-1}(T)}{P_0(T)P_2(T)\cdots P_{2g-2}(T)P_{2g}(T)}.$$

PROOF. Take the logarithm on each side, and use the identity:

$$-log(1 - T) = T + \frac{T^2}{2} + \frac{T^3}{3} + \cdots .$$

$\square$

REMARK 2.5. The computation in the proof relies on the fact that the endomorphisms induced by Frobenius are semisimple. This is true for curves, abelian varieties, and a few other varieties, but still unknown in general; it is implied by the so-called *standard conjectures*.

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