Cris is for Crystalline

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1. Cristalline topology and divided powers

Goal: explain the definition of the crystalline site

There are two distinct aspects to the definition of the crystalline site: some geometric data and some algebraic data. The algebra, or rather PD-algebra (for **P**uissances **D**ivisées, or divided powers) is a remedy to the non-existence of the usual logarithm in characteristic p. The geometry that arises is nilgeometry, the geometry of infinitesimal thickenings.

1.1. The logarithm in characteristic p. It is a truism in the arithmetic of global fields that function fields are easier to deal with than number fields: the existence of additional structures (e.g. the Frobenius map $x \mapsto x^p$) gives us more handles, more tools (e.g. the Cartier isomorphism), to prove theorems. One aspect in which positive characteristic is more difficult than characteristic zero is the pathological behaviour of the usual differential calculus ($d(x^p) = px^{p-1} = 0$ in char. p), or, equivalently, the non-existence of a well-behaved logarithm function. Nonetheless, one can cook up useful exponential and logarithm functions in characteristic p, under certain restrictive hypotheses. I know of essentially only one way of circumventing the division by zero problem in characteristic p, and it involves truncating a suitable infinite series giving the logarithm. A primitive version of it comes as follows: Let A be an associative $\mathbb{F}_p$-algebra. Put

$$N := \{z \in A | z^p = 0\},$$

and

$$U := \{z \in A | z^p = 1\}.$$

We may define exp and log by:

$$exp(z) = \sum_{i=0}^{p-1} \frac{z^i}{i!} \text{ and } log(x) = -\sum_{i=1}^{p-1} \frac{(1-z)^i}{i}.$$

Then $exp : N \longrightarrow U$ and $log : U \longrightarrow N$ are inverses of each other.

Remark 1.1. If $\alpha, \beta \in N$ such that $\alpha\beta = \beta\alpha$, then $\alpha + \beta \in N$, but in general

$$exp(\alpha + \beta) \neq exp(\alpha) \cdot exp(\beta).$$

If $\alpha^i \beta^{p-i} = 0$ whenever $1 \leq i \leq p-1$, then

$$exp(\alpha + \beta) = exp(\alpha) \exp(\beta).$$

In this toy version, we avoid the difficulty of $p = 0$ by truncating brutally the familar Taylor expansions at the term where the denominator is p. This trick has the virtue of simplicity, but it reduces drastically the domain and the image of our familiar functions. A more fruitful and supple trick is to add a convenient replacement $\gamma_n(x)$ for every problematic term $\frac{x^n}{n!}$ in the Taylor expansion of the exponential. We explore this idea systematically in the next section.

1.2. Divided powers. In this sequel, all rings $A \ni 1_A$ are commutative.

DEFINITION 1.2. Let A be a commutative ring and J an ideal of A. A divided power structure on J is a family of maps $\gamma_n : J \longrightarrow J, \mathbb{N} \ni n \geq 1$ such that

(1)
$$\gamma_1(x) = x \text{ for all } x \in J,$$

(2)
$$\gamma_n(x+y) = \gamma_n(x) + \sum_{i=1}^{n-1} \gamma_{n-i}(x)\gamma_i(y) + \gamma_n(y) \text{ for all } x \in J, y \in J,$$

(3)
$$\gamma_n(xy) = x^n \gamma_n(y) \text{ for all } x \in A, y \in J,$$

(4)
$$\gamma_m \circ \gamma_n(x) = \frac{(mn)!}{(n!)^m m!} \gamma_{mn}(x),$$

(5)
$$\gamma_m(x)\gamma_n(x) = \frac{(m+n)!}{m!n!} \gamma_{m+n}(x).$$

The first and last axiom give in particular the formula

$$x^n = (\gamma_1(x))^n = n!\gamma_n(x).$$

We also write $x^{[n]}$ for $\gamma_n(x)$.

EXAMPLE 1.3. We discuss a few canonical examples of PD structures.

- If A is a $\mathbb{Q}$-algebra, there is a unique PD-structure on any ideal $J \subset A$. If A has characteristic zero (i.e. is not torsion), there is at most one PD-structure. This is clear from $x^n = n!\gamma_n(x)$.
- If $mA = 0$, a necessary condition for a PD-structure to exist on an A-ideal I is that for all $x \in I$, $x^m = 0$. This is also clear from $x^m = m!\gamma_m(x)$.
- Suppose there is $m \in \mathbb{N}$ such that $(m-1)!$ is invertible in A, and $I^m = 0$. Then there exists at least one (possibly many !) PD-structure on I: define $\gamma_n(x) = (n!)^{-1}x^n$ for $n < m$, and 0 otherwise. The conditions are satisfied if $I^2 = 0$ or if, in characteristic p, $I^p = 0$. Eyal mentioned that Deligne and Pappas ([**5**]) found useful to bootstrap this idea: let k be a field of characteristic p, $A = k[t]/(t^{p^i})$, and $I = (t^{p^{i-1}})$, $m = p$, and let $i = 1, 2, \ldots$ The curious feature is that at each step, the necessary condition that $x^p = 0$ is a limiting factor: one can go from k to $k[t]/(t^n)$ for $1 \leq n \leq p$, but if $n = p+1$, say, the ideal (t) cannot have a PD-structure, one must change the ideal, and repeat the procedure at each n that is a power of p.
- An example to see that PD-structures are not unique in general: let $x \in \mathbb{Z}/2^i\mathbb{Z}$. There exists a PD-structure γ on the maximal ideal $2\mathbb{Z}/2^i\mathbb{Z}$ such that $\gamma_2(2) = x$ iff $x = 2 + a \cdot 2^{i-1}$, $a \in \{0, 1\}$.
- Let R be a discrete valuation ring in characteristic 0 with residue field of characteristic p. The ramification index e is *defined* by the equation
$$pR = \mathfrak{m}^e,$$
where $\mathfrak{m}$ is the maximal ideal of R (for example, $R = W(k)$ has ramification index 1, since $pW(k)$ is the maximal ideal). Then $\mathfrak{m}$ admits a PD-structure iff $e \leq p-1$.

PROOF. Since R has characteristic zero, there is at most one PD-structure. It suffices to check that $\gamma_n(x) = \frac{x^n}{n!}$ lies in $\mathfrak{m}$. The following lemma is standard:

LEMMA 1.4. *Let* $\mathbb{N} \ni n = \sum a_i p^i$, *with* $0 \leq a_i < p$. *Then*

$$ord_p(n!) = \frac{1}{p-1} \sum a_i(p^i - 1).$$

Pick a uniformizer $\pi \in \mathfrak{m}$. We have:

$$ord_\pi(\gamma_n(\pi)) = n - ord_\pi(n!) = n - e ord_p(n!)$$

$$= \sum a_i p^i - \frac{e}{p-1} \sum a_i(p^i - 1) = \frac{1}{p-1} \sum a_i[p^i(p-1-e)+e] = (\frac{p-1-e}{p-1})n + e\frac{\sum a_i}{(p-1)}.$$

Thus, $\gamma_n(\pi)$ lies in (π) for all $n \geq 1$ iff $e \leq p-1$. $\square$

- If $J \subset I$, j is a sub-PD-ideal of I if for all $n \geq 1$, for all $x \in J$, $x^{[n]} \in J$. For example, if we write $\mathfrak{m} = pW(k)$, then $\mathfrak{m}^n$ is a sub-PD-ideal, and therefore there is a natural induced PD-structure on $pW_n(k)$. Beware: $pW_n(k)$ admits other PD-structures.

DEFINITION 1.5. Let A be a ring and (J, γ_n) a PD-ideal. We suppose that $x^{[n]}$ (resp. $(n-1)!x^{[n]}$) is zero for $n >> 0$, for all $x \in A$. Then we can define

$$exp : J \longrightarrow 1 + J,\ exp(x) = \sum_{n \geq 0} x^{[n]},$$

$$log : 1 + J \longrightarrow J,\ log(1+x) = \sum_{n \geq 1} (-1)^{n-1}(n-1)!x^{[n]},$$

One immediate advantage of this definition is that *exp* and *log* are (group) isomorphisms and inverses of each other:

$$J^+ \underset{log}{\overset{exp}{\rightleftarrows}} (1+J)^\times.$$

Define $J^{[n]}$ to be the ideal generated by the monomials $x_1^{[a_1]} x_2^{[a_2]} \cdots x_r^{[a_r]}$ such that $\sum a_i \geq n$. An ideal J is PD-nilpotent if there is a n such that $J^{[n]} = 0$. We can relax this condition and still have a logarithm function: it suffices to have n such that $(n-1)!J^{[n]} = 0$ (Berthelot's condition).

We now address the issue of comparing PD-structures.

DEFINITION 1.6. Let (A, J, γ) and (A', J', γ') be two PD-rings. A PD-homomorphism $\phi : (A, J, \gamma) \longrightarrow (A', J', \gamma')$ is a ring homomorphism $\phi : A \longrightarrow A'$ such that $\phi(J) \subset J'$ and for all $x \in J$, $\phi(\gamma_n(x)) = \gamma_n'(\phi(x))$.

DEFINITION 1.7. If (A, J, γ) is a PD-ring and if $\phi : A \longrightarrow B$ is a ring homomorphism, we say that γ *extends* to B if there exists a PD-structure γ' on JB such that:

$$\phi : (A, J, \gamma) \longrightarrow (B, JB, \gamma')$$

is a PD-homomorphism. There is at most one PD-structure satisfying this condition.

REMARK 1.8. As the reader suspects, the extended PD-structure does not always exists (see [**1**, 1.7, p.35]).

We quote two results of Berthelot giving necessary conditions for PD-structures to extend.

PROPOSITION 1.9. *([**8**, p.72]) Let (A, J, γ) be a PD-ring. If J is principal, γ can always be extended.*

PROPOSITION 1.10. *([**8**, p.72]) Let (A, J, γ) be a PD-ring and $J \otimes_A B \xrightarrow{\cong} JB$ (e.g. B is a flat A-algebra). Then γ extends to JB.*

EXAMPLE 1.11. The truncated Witt vectors $W_n(k)$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$. Note that $\mathbb{Z}/p^n\mathbb{Z}$ admits many different PD-structures, hence so does $W_n(k)$.

We need a notion of compatibility of PD-structures. Let $(A, I), (B, J)$ be two PD-rings, and let B be an A-algebra. Then we say that a PD-structure on J is compatible with I if

- On IB, there exist a PD-structure extending the PD-structure of I.
- The PD-structure on IB and on J coincide on the intersection $J \cap IB$.

Equivalently, there exist a PD-structure on $J + IB$ compatible with the PD-structure of J and of I.

DEFINITION 1.12. (The free PD-algebra $A < \underline{T} >$) Let A be a ring. We define the *free PD-algebra* $A < t_1, \ldots, t_r >$ as a direct sum $\oplus_{n \geq 0} \Gamma^n$, where a basis of Γ^n as an A-module is given by the symbols $t_1^{[k_1]} \cdot \cdots \cdot t_r^{[k_r]}$ with $k_1 + \cdots k_r = n$ and $k_i \in \mathbb{N}$. The algebra structure is defined by the relations $t_i^{[m]} t_i^{[n]} = \begin{pmatrix} m+n \\ n \end{pmatrix} t_i^{[m+n]}$. The ideal $I = A^+ < t_1, \ldots, t_r >= \oplus_{n \geq 1} \Gamma^n$ has a unique PD-structure such that $\gamma_n(t_i) = t_i^{[n]}$ for all $i \in \{1, \ldots r\}$ and $n \geq 1$.

The divided powers are the missing ingredient in many characteristic p recipes: e.g. an integrable connexion is equivalent to the data of a so-called *PD* stratification, adding divided powers yields the *PD* Poincaré Lemma which replaces the classical (or power series) Poincaré Lemma (see [**4**]), which fails in characteristic p (try integrating $x^{p-1}dx$); etc.

LEMMA 1.13. *(PD Poincaré Lemma) ([**1**, Lemme 2.12, p.296]) For any ring A and integer n, the de Rham complex of $A[t_1, \ldots, t_n]/A$ with coefficients in the PD algebra $A < t_1, \ldots, t_n >$, viewed as an $A[t_1, \ldots, t_n]$-module, and endowed with the integrable connexion D defined by $D(\gamma_k t_i) = \gamma_{k-1} t_i \otimes dt_i$, is a resolution of A, i.e. the sequence*

$$0 \longrightarrow A \longrightarrow A < t_1, \ldots, t_n > \xrightarrow{D} \oplus A < t_1, \ldots, t_n > dt_i \longrightarrow \oplus A < t_1, \cdots, t_n > dt_{i_1} \cdots dt_{i_r}$$

where the differential is the natural extension of D and the first map is the natural inclusion, is exact.

PROOF. If $n = 1$, the complex $A < t > \longrightarrow A < t >$ is given by:

$$\sum_{k \geq 0} a_k t^{[k]} \mapsto \sum_{k \geq 1} a_k t^{[k-1]} dt.$$

Thence the complex $0 \longrightarrow A \longrightarrow A < t > \longrightarrow A < t > dt \longrightarrow 0$ is acyclic, and the lemma follows for $n = 1$. Let us denote by $\omega^\bullet_{A<t_1, \ldots, t_n>}$ this complex and continue the induction: suppose that $A \longrightarrow \omega^\bullet_{A<t_1, \cdots, t_{n-1}>}$ is a quasi-isomorphism. Thence, $\omega^\bullet_{A<t_n>}$ is a complex whose terms are locally free modules on A, thus the morphism

$$A \otimes \omega^\bullet_{A<t_n>} \longrightarrow \omega^\bullet_{A<t_1, \cdots, t_{n-1}>} \otimes \omega^\bullet_{A<t_n>},$$

is again a quasi-isomorphism. Thus, the morphism of complexes $A \longrightarrow \omega^{\bullet}_{A<t_1,\dots,t_n>}$ is a quasi-isomorphism. □

REMARK 1.14. You have already encountered this proof before! The idea is to isolate the n-th variable: write a closed form $\omega = \omega_1 \wedge dt_n + \omega_2$. We write $d = d' + \frac{\delta}{\delta t_n} \wedge dt_n$. Thus $d'\omega_2 = 0$ and $d'\omega_1 + (-1)^{n-1}\frac{\delta}{\delta t_n}\omega_2 = 0$. By induction, $\omega_2 = d'\eta_2$ and $\omega_1 + (-1)^{n-1}\frac{\delta}{\delta t_n}\eta_2 = d\ \eta_1$. Thus we see that $\omega = d(\eta_1 \wedge dt_n + \eta_2)$.

Let (A, I) be a couple where A is a $W(k)$-algebra and I a PD-ideal of A compatible with the canonical PD-structure on $pW(k)$. The functor which associates to each pair (B, J) the set of $W(k)$-algebra morphisms $A \longrightarrow B$ sending I to J is represented by a pair $(D_{A/W(k)}(I), \overline{I})$, where $\overline{I}$ is a PD-ideal of $D_{A/W(k)}(I)$, called the *PD-envelope* of I relative to $W(k)$. It satisfies the universal property that for any morphism f : $(A, I) \longrightarrow (B, J)$ such that $IB \subset J$, there exists a unique PD-morphism $\overline{f}$ such that $(A, I) \longrightarrow (D_{A/W(k)}(I), \overline{I}) \xrightarrow{\overline{f}} (B, J)$ coincide with f. Clearly, $W(k)$ can be replaced in this discussion by any PD-ring, for example $W_n(k) := W(k)/p^nW(k)$ equipped with the canonical quotient PD-structure coming from $W(k)$.
A basic property is that $D_{A/W(k)}(I)/\overline{I} = A$. If the characteristic of A is zero or $I = pA$, $(D_{A/W(k)}(I), \overline{I}) = (A, I)$, and $D_{A/R}(I)$ is generated as an A-algebra by the $\gamma_n(x)$ for $x \in I$. This construction can be sheafified. In brief, if (S, I, γ) is a PD-scheme, X an S-scheme, and J a quasi-coherent ideal of $\mathcal{O}_X$, Y the closed subscheme defined by J, then the $\mathcal{O}_X$-PD-algebra $\widetilde{D_{\mathcal{O}_X}(I)}$ defined by $D_{\mathcal{O}_X}(I)$ is quasi-coherent, and we can look at the scheme $\mathrm{Spec}(\widetilde{D_{\mathcal{O}_X}(I)})$. This will be used later on.

1.3. The crystalline Grothendieck topology.

1.3.1. *Smoothness and nil-immersions.* We gather the various results around nilimmersions, discuss the relationship between smoothness and liftability, and try to motivate the geometric ideas entering the definition of the crystalline site: i.e. why does this all actually work ?

An ideal I in a ring A is a nil-ideal if there exists an integer $m > 0$ such that $x^m = 0$ for all $x \in I$. Observe that a nilpotent ideal is necessarily a nil-ideal, but not reciprocally. If A is a ring killed by p (more generally, any integer $n \geq 2$), then any PD-ideal I in A is a nil-ideal (it consists of nilpotent elements), as follows from the equality: $x^p = p!\gamma_n(x) = 0$ for all $x \in I$.

DEFINITION 1.15. A *nilimmersion* is a closed immersion of schemes whose corresponding quasi-coherent sheaf of ideals is a locally a nil-ideal.

EXAMPLE 1.16. Let A be a commutative ring, I a nilpotent ideal. The homomorphism $A \twoheadrightarrow A/I$ corresponds to a map of schemes $\mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}(A)$ which is a closed immersion. A point we want to stress is that $\mathrm{Spec}(A)$ and $Spec(A/I)$ have the same underlying topological space (this remains valid for I a nil-ideal). The reader should keep in minde the intuitive picture suggested by the terminology of (nilpotent) thickenings.

REMARK 1.17. If $f : X \longrightarrow Y$ is a morphism of k-schemes, a thickening of Y does not necessarily yield a thickening of X. This is why the category of sheaves on the cristalline site is more flexible than the site itself: using sheaves allows to consider the sheaf on X obtained by *inverse image* of a thickening of Y, while this operation is not well-behaved on thickenings themselves.

Let X be a scheme over k. Let U,T be Zariski open subsets of X, and let $i : X \hookrightarrow Z$ be a closed immersion of X in a smooth $W_n(k)$-scheme Z. Recall that the universal PD-thickening (corresponding to the PD-envelope)

$$\bar{i} : X \longrightarrow X_Z \to Z$$

consists in adding the PD-structure in an universal way to the ideal of i: it is universal among the PD-thickenings $\widetilde{X_Z} \longrightarrow Z$ such that $X \hookrightarrow \widetilde{X_Z} \longrightarrow Z$ coincides with $i : X \longrightarrow Z$. If X is smooth and Z is a smooth lifting of X (N.B. there is a natural closed immersion of the special fiber in the lifting), $X_Z = Z$. Note that X_Z and X have the same underlying topological space. The following, I believe, is the main geometric property which justifies the definition of the crystalline site: It is a property used in the proof of the isomorphism between the crystalline cohomology and the de Rham cohomology to show that the PD thickening X_Z covers the *final object* $*$, the sheaf associated to the presheaf whose sections over a non-empty open is the set $\{0\}$ with one element. In the case of the topos of a topological space, this sheaf is represented by the open X, but in the cristalline case, this sheaf is *not* representable in general: a section $s \in \mathrm{Hom}(*, F)$ is just a compatible collection $s_T \in F(T)$ for all objects T. Let us explain in geometric terms what a covering of the final object amounts to: if $U \subset T$ is a thickening whose corresponding ideal has a PD-structure, a morphism $U \longrightarrow X_Z$ extends locally over T to a morphism $T \longrightarrow X_Z$: from the definition of X_Z, it suffices to prove that the closed immersion $U \longrightarrow Z$ extends locally over T to an immersion $T \longrightarrow Z$, which follows from using the fact that $U \longrightarrow T$ is a nilimmersion and Z is smooth. This follows, with a little algebraic nonsense (that can be found in [**3**]), from the definition of *smoothness*:

DEFINITION 1.18. A morphism $\pi : X \longrightarrow S$ is *smooth* if it is locally of finite presentation and *formally smooth*, i.e. for all affine scheme S', any closed subscheme $S_0^{'}$ of S' defined by a nilpotent ideal I of $\mathcal{O}_{S'}$, and any morphism $S' \longrightarrow S$, the application:

$$\mathrm{Hom}_S(S', X) \longrightarrow \mathrm{Hom}_S(S_0^{'}, X),$$

deduced from the canonical injection $S_0^{'} \longrightarrow S'$ is surjective; in other words, there exists a lifting.

N.B. An *étale* morphism can be characterized in the same way, by replacing "surjective" in the definition of a smooth morphism with "bijective"; in other words, there exists a *unique* lifting.

We state some basic properties of smooth (resp. étale) morphisms:

PROPOSITION 1.19. *(*[**12**, Proposition 2.4, 2.6]*)*

- *If $X \longrightarrow Y$ is smooth, the $\mathcal{O}_X$-module $\Omega^1_{X/Y}$ is locally free of finite type.*
- *Suppose $X \xrightarrow{f} Y \xrightarrow{g} S$ are S-morphisms. Then if f is smooth, the exact sequence:*

$$0 \longrightarrow f^*\Omega^1_{X/S} \longrightarrow \Omega^1_{X/S} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

is locally split. If f is étale, then $\Omega^1_{X/Y} = 0$ and $f^*\Omega^1_{Y/S} \cong \Omega^1_{X/S}$.
- *Suppose the morphism gf is smooth. Then if the above exact sequence is locally split, then f is smooth. If the canonical homomorphism $f^*\Omega^1_{Y/S} \longrightarrow \Omega^1_{X/S}$ is an isomorphism, then f is étale.*

PROPOSITION 1.20. *(*[**12**, Proposition 2.10]*) Let $f : X \longrightarrow Y$ be a morphism locally of finite presentation. The following are equivalent:*

- *f is smooth*
- *f is flat and the geometric fibers of f are regular schemes (i.e. all local rings $\mathcal{O}_{X,x}$ are regular).*

1.3.2. *Crystalline and de Rham cohomologies.* The basic idea of Grothendieck for crystalline cohomology was to lift varieties to characteristic zero and take the de Rham cohomology. The work of Monsky-Washnitzer proving that such a procedure works in the affine case was certainly decisive. Explicitly, let k be a perfect field in characteristic $p > 0$ and let $W(k)$ be the Witt vectors (e.g. for $k = \mathbb{F}_p, W(\mathbb{F}_p) = \mathbb{Z}_p$). A lift of smooth, proper scheme X/k is a smooth, proper scheme $Z/W(k)$ such that $X = Z \times_{\mathrm{Spec}(W(k))} \mathrm{Spec}(k)$. The algebraic de Rham cohomology of Z over $W(k)$ is thus defined and we can take its hypercohomology $H^i_{dR}(Z/W) := \mathbb{H}^i(Z, \Omega^\bullet_{Z/W})$, as explained in [**4**]. We give an heuristic explanation based on the Gauss-Manin connexion of why this makes good sense. Let $f : \mathcal{X} \longrightarrow S = SpfW[[t_1, \dots, t_r]]$ be a universal deformation of X/k, that is any scheme Z/W lifting X corresponds to a point $x \in S(W(k))$., i.e. a continous morphism $W(k)[[T]] \longrightarrow W(k)$ such that

$$Z \cong \mathcal{X} \times_S W(k) = \mathcal{X} \otimes_{W[[t]]} (W(k), x).$$

In terms of Schlessinger's theory, I believe the obstruction to the existence of a universal deformation lies in the cohomology group $H^2(X, \mathcal{T}_X), \mathcal{T}_X$ the tangent sheaf of X. Suppose that the relative de Rham cohomology groups $\mathcal{H}^i := R^i f_* \Omega^\bullet_{\mathcal{X}/S}$ are free for all i and compatible to base change (this is true in the complex algebraic category). Then $H^i_{dR}(Z/W(k)) = x^* \mathcal{H}^i = \mathcal{H}^i \otimes_{W[[t]]} (W(k), x)$. As we explain in more detail below, in the algebraic category, the $\mathcal{H}^i$ are also equipped with a Gauss-Manin connexion:

$$\nabla_{GM} : \mathcal{H}^i \longrightarrow \mathcal{H}^i \otimes \Omega^1_{S/W(k)}.$$

If Z_1, Z_2 are two lifts of X/k, they correspond to two morphisms $x, y : W(k)[[t]] \longrightarrow W(k)$ such that $x = y \mod p$. The Gauss-Manin connexion allows to define an isomorphism $\chi(x, y) : \mathcal{H}^i \otimes (W(k), x) \longrightarrow \mathcal{H}^i \otimes (W(k), y)$ by the explicit formula (found in [**16**]):

$$\chi(x,y)(s) = \sum_{|\underline{n}| \geq 0} \frac{(x(t) - y(t))^{\underline{n}}}{\underline{n}!} \Big(\nabla_{GM}\Big(\frac{\delta}{\delta t}\Big)^{\underline{n}} s\Big)(y).$$

The convergence of this series is the so-called *nilpotence* of the Gauss-Manin connexion. Moreover, $\chi(x, x)$ is the identity and $\chi(x, y)\chi(y, z) = \chi(x, z)$.
The second step is to try to design an algebraic cohomology theory which will give the same cohomology group without having to assume the existence of a lift (which does not exist in general: there are non-liftable, smooth, proper varieties in characteristic p), i.e. that could be computed more generally for $X \hookrightarrow Z$, where $\hookrightarrow$ is a closed immersion, and Z is smooth (this is possible e.g. for X affine or projective). It is worthwhile to look at what happens in characteristic zero. Let X be a variety over $\mathbb{C}$. If $i : X \longrightarrow Y$ is a closed immersion of X into a smooth variety, let $\Omega^\bullet_Y$ be the (algebraic) de Rham complex of Y. Denote by $\widehat{(\Omega^\bullet_Y)}_{/X}$ its formal completion along X, which can be viewed as a complex of (Zariski) sheaves on X. Then it is a theorem of Deligne (see [**10**] for a proof) that up to canonical isomorphism, the hypercohomology $H^*(X, \widehat{(\Omega^\bullet_Y)}_{/X})$ does not depend on the immersion i. The proof of this result relies on the *formal* Poincaré Lemma, which says that the continuous de Rham complex of $\mathbb{C}[[t_1, \dots, t_n]]$ is a resolution of $\mathbb{C}$

A similar result holds in characteristic p with the use of PD-envelopes. This works roughly as follows: Let $X \longrightarrow X_Z \longrightarrow Z$ be the PD-envelope of a closed immersion

$X \hookrightarrow Z$, Z a smooth $W_n(k)$-scheme. The scheme X_Z is affine over Z, and $\mathcal{O}_{X_Z}$, viewed as an $\mathcal{O}_Z$-module, has a natural integrable connexion ∇ with respect to $W_n(k)$, satisfying $\nabla x^{[m]} = x^{[m-1]} \otimes dx$, for $x \in I_{X \subset X_Z}$. This allows to consider the de Rham complex $\mathcal{O}_{X_Z} \otimes \Omega^\bullet_{Z/W_n(k)}$ of $Z/W_n(k)$ with coefficients in $\mathcal{O}_{X_Z}$:

$$\mathcal{O}_{X_Z} \longrightarrow \mathcal{O}_{X_Z} \otimes_{\mathcal{O}_Z} \Omega^1_{Z/W_n(k)} \longrightarrow \cdots ,$$

a complex of $\mathcal{O}_{X_Z}$-modules with differential given by differential operators of order 1. By the PD-Poincaré Lemma, the hypercohomology of X_Z with values in $\mathcal{O}_{X_Z} \otimes \Omega^\bullet_{Z/W_n(k)}$ do not depend on the immersion $i : X \longrightarrow Z$ and it computes $H^*(X/W(k))$ by taking the inverse limit over n. In the next section, we obtain these cohomology groups as the cohomology of a site intrinsically attached to X, which is then only assumed to be of finite type.

1.3.3. *The crystalline site.* In this section we describe the crystalline site as a Grothendieck topology.

Example 1.21. Keep in mind the following basic example: $S = \mathrm{Spec}(W_n(k))$, $I = (p)$ with the canonical PD structure γ coming from $W(k)$, and X a scheme over $\mathrm{Spec}(k)$.

Remark 1.22. Why are the Witt vectors lurking in the background of p-adic cohomology theories ? The coefficients cannot always be $\mathbb{Q}_p$, by Serre's argument, as explain in [**4**, p.9]: a supersingular elliptic curve has a 4-dimensional endomorphism algebra, the quaternion algebra ramified at p and ∞, so it is impossible to obtain a 2-dimension H^1 over $\mathbb{Q}_p$. The Witt vectors $W(k)$ are the most natural choice after $\mathbb{Z}_p$, being *the* complete discrete valuation ring with residue field k whose maximal ideal is generated by p.

Let $S = (S, I, \gamma)$ be a PD-scheme, i.e. I is a quasi-coherent ideal of $\mathcal{O}_S$ and γ is a divided powers structure on I. Let X be an S-scheme (on which γ extends).

The crystalline site $Cris(X/S, I, \gamma)$ is defined as follows:

- Objects

  The objects are triples $(U \subset U', \gamma_{U'})$, where U is a Zariski open subset of X, U' is a nilpotent thickening of U), and $\gamma_{U'}$ is a divided powers structure on the ideal of definition J in $\mathcal{O}_{U'}$ which defines U in U' and which is *compatible* with the structure γ on I. This notion of compatibility is local.

  Example 1.23. The triple $(U \xrightarrow{id_U} U, 0)$, for any Zariski open $U \subset X$ is an object of the cristalline site $Cris(X/S, I, \gamma)$.

- Morphisms

  A morphism of $(U \subset U', \gamma_{U'})$ into $(V \subset V', \gamma_{V'})$ is an inclusion $U \subset V$ and a morphism of schemes $U' \longrightarrow V'$ which makes the obvious diagram commute and is such that $\gamma_{U'}$ and $\gamma_{V'}$ are compatible.

- Coverings

  The coverings are families of morphisms $(U_i, V_i, \gamma_i) \longrightarrow (U, V, \gamma)$ such that $V_i \longrightarrow V$ is an open immersion and $V = \cup_i V_i$, i.e. the family of open immersion is surjective.

We describe sheaves (of sets) over the cristalline site. The general definition applies: a sheaf is a presheaf which satisfies the sheaf exact sequence

$$0 \longrightarrow F(T) \longrightarrow \prod_i F(T_i) \rightrightarrows \prod_{i,j} F(T_i \cap T_j),$$

for any covering $\{T_i \longrightarrow T\}$. Let $\mathcal{F}$ be a sheaf and (U,V) an object in $Cris(X/S)$. If we associate to an open W of V the sections of $\mathcal{F}$ over $(U \cap W, W)$, we define a sheaf over V for the Zariski topology. A morphism $g : (U,V) \longrightarrow (U',V')$ in $Cris(X/S)$ gives a morphism $g^*_{\mathcal{F}} : g^{-1}\mathcal{F}_{(U',V')} \longrightarrow F_{(U,V)}$ satisfying the two properties: transitivity condition for morphisms $(U,V) \longrightarrow (U',V') \longrightarrow (U'',V'')$; if $V \xrightarrow{g} V'$ is an open immersion $U = U' \times_{V'} V$, then $g^*_{\mathcal{F}}$ is an isomorphism. Reciprocally, any Zariski sheaf equipped with transition morphisms satisfying the two properties defines a cristalline sheaf.

Example 1.24. **Important !** The (co-)functor $V \mapsto \mathcal{O}_V$ defines a sheaf of rings on $Cris(X/S)$, which we call the "structure sheaf" and denoted by $\mathcal{O}_{X/S}$.

The cristalline cohomology is, by definition, the cohomology of the structure sheaf:

$$H^i_{cris}(X/S) := H^i(X/S, \mathcal{O}_{X/S}).$$

Remark 1.25. **Points galore.** The Zariski interpretation of the sheaves shows that isomorphisms of sheaves can be detected at the level of stalks, since the crystalline topology has "enough" points. It is stated in SGA 4, IV, p.389 that Deligne has a theorem which implies that all topoi "encountered in algebraic geometry" have enough points.

How do we get $W(k)$-modules in characteristic p ? Recall our typical sitation with $S = \mathrm{Spec}(W_n(k)), I = (p)$. We apply our general definition to this setting: for example, the objects are commutative diagrams $U/\mathrm{Spec}(k) \hookrightarrow V/Spec(W_n(k))$, where $U \subset X$ is a Zariski open, $i : U \hookrightarrow V$ is a PD-thickening of U, a closed immersion of $W_n(k)$-schemes such that $\mathrm{Ker}(\mathcal{O}_V \longrightarrow \mathcal{O}_U)$ is equipped with a PD-structure compatible with the canonical structure induced on $W_n(k)$, etc. Then $H^i(X/W_n(k)) := H^i((X/W_n(k))_{cris}, \mathcal{O}_{X/W_n(k)})$ and $H^i(X/W(k))$ is defined as the inverse limit of the $H^i(X/W_n(k))$.

2. Examples: Crystals

"Un cristal possède deux propriétés caractéristiques: la rigidité, et la faculté de croître, dans un voisinage approprié. Il y a des cristaux de toute espéce de substance: des cristaux de soude, de soufre, de modules, d'anneaux, de schémas relatifs, etc..." quoted (reputedly) from ([**9**]).

Crystals in modules generalize modules equipped with an integrable, quasi-nilpotent connexion in characteristic p. We explain these terms in a utilitarian manner, sending the interested reader to [**16**] for more details. Sheaves of quasi-coherent $\mathcal{O}_S$-modules with a connexion form a nice category: they are stable under internal Hom (when defined) and under tensor product, behave well under smooth base change, and have enough injectives, i.e. shows good signs of being a good category of coefficients for a cohomology theory. More concretely, $H^i_{cris}(X/S)$ are crystals in modules.

Definition 2.1. A sheaf F of $\mathcal{O}_{X/S}$-modules such that all $u^\sharp : u^*F_{(U,U')} \longrightarrow F_{(V,V')}$ are isomorphisms for all maps between objects of the cristalline site $u : (V,V') \longrightarrow (U,U')$ is called a crystal in modules. Here u^* denotes the module pullback, i.e. u^{-1} tensored with the structure sheaf.

Example 2.2. The sheaf $\mathcal{O}_{X/S}$ is a crystal in modules.

A crystal in modules F yields canonical isomorphisms:

$$pr_1^*(F_{X \subset X}) \cong F_{X \subset \Delta^1 X} \cong pr_2^*(F_{X \subset X}),$$

and thus can be equipped with a *connexion*. The infinitesimal descent data aspect of connexions is expounded in [**1**]; we are content with giving the descent-theoretic definition of a connexion. Let S be a scheme, I a quasi-coherent ideal of $\mathcal{O}_S$ and γ a PD-system on I satisfying Berthelot's condition $(n-1)!I^{[r]} = 0$ for $n >> 0$. We denote by $\Delta^1(X)$ the first infinitesimal neighborhood of X in $X \times_S X$. The ideal $I := \mathcal{I}/\mathcal{I}^2$ (where $\mathcal{I}$ is the ideal of Δ) defining X in $\Delta^1(X)$ is nilpotent of order 2, and we define a PD-structure by putting $\gamma_1(x) = x$, and $\gamma_n(x) = 0$ for $n \geq 2$. This yields an object $X \subset \Delta^1(X)$ of the cristalline site. Denote by pr_1, pr_2 the two projections from $\Delta^1(X)$ to X.

Definition 2.3. A *connexion* on an $\mathcal{O}_X$-module M is given by an infinitesimal descent data :

$$\phi : pr_1^*(M) \cong pr_2^*(M),$$

satisfying the usual cocycle condition ($p_{12}^* \circ p_{23}^* = p_{13}^*$) for the diagram:

$$\Delta_2^1(X) \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} \Delta^1(X) \begin{array}{c}\longrightarrow\\ \longrightarrow\end{array} X,$$

where Δ_2^1 is the first infinitesimal neighborhood of X in $X \times_S X \times_S X$.

As shown in [**1**], this is equivalent to giving an $\mathcal{O}_S$-linear morphism:

$$\nabla : M \longrightarrow \Omega^1_{X/S} \otimes_{\mathcal{O}_X} M,$$

satisfying $\nabla(am) = da \otimes m + a\nabla(m)$, for all local sections a on $\mathcal{O}_X$ and m on M. Briefly put, the notion of a connexion is equivalent to first order descent data, which is equivalent to the data of a sheaf on a site made up of the first order thickenings of open subsets of X. A stratification, or n-connexion, is the suitable generalization for higher orders. We shall not explore this notion in detail, except for a few quick remarks.

Theorem 2.4. *Let X/S be smooth and S a scheme over* $\mathrm{Spec}(\mathbb{Q})$. *Then a connexion C extends to a stratification iff C is (so-called) integrable.*

In characteristic p, an integrable connexion is not necessarily a stratification. The prime example is the Gauss-Manin connexion. Again, adding PD structures saves the day: an integrable connexion is equivalent to the data of a so-called PD stratification (see [**1**]). We now define (!) and study integrable connexions. A connexion can be extended naturally to maps $\rho_i : \Omega^i_{X/S} \otimes M \longrightarrow \Omega^{i+1}_{X/S} \otimes M$. For our purposes, we only need $\rho_1 := \omega \otimes \mapsto dw \otimes f - w \otimes \nabla f$.

Definition 2.5. The *curvature* K is the $\mathcal{O}_S$-linear map:

$$K = \rho_1 \otimes \rho : M \longrightarrow M \otimes_{\mathcal{O}_X} \Omega^2_{X/S}.$$

Definition 2.6. A connexion is integrable (i.e. has zero curvature) if the map:

$$M \longrightarrow M \otimes_{\mathcal{O}_X} \Omega^2_{X/S}$$

is zero.

Remark 2.7. It can be shown that in the X/S smooth case, crystals in modules are the same as quasi-coherent modules equipped with an integrable connexion. Moreover, in characteristic p, the connexion is so-called quasi-nilpotent (see [**8**]).

The algebraic de Rham complex $\mathcal{H}^*(X/S)$ associated to a smooth morphism $f : X \longrightarrow S$ is a *crystal*, and it is therefore equipped with a canonical *integrable* connexion, called the Gauss-Manin connexion ∇_{GM} ([**18**]). We used it above to explain why the invariance of the de Rham cohomology of a lift sitting in a universal deformation. We follow the crisp presentation of [**15**].

Since $\pi : X \longrightarrow S$ is smooth, the sequence:

$$0 \longrightarrow \pi^*(\omega^1_{S/k}) \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/S} \longrightarrow 0$$

is exact. The complex $\Omega^\bullet_{X/k}$ admits a canonical filtration

$$\Omega^\bullet_{X/k} = F^0(\Omega^\bullet_{X/k}) \supset F^1(\Omega^\bullet_{X/k}) \supset F^2(\Omega^\bullet_{X/k}) \supset \cdots ,$$

where $F^i = F^(\Omega^\bullet_{X/k})$ is the image of the map $\Omega^{\bullet,-i}_{X/k} \otimes_{\mathcal{O}_X} \pi^*(\Omega^i_{S/k}) \longrightarrow \Omega^\bullet_{X/k}$. All sheaves $\Omega^i_{*/**}$ are locally free over their respective schemes, we can describe the associated graded objects of this filtration as:

$$gr^i(\Omega^\bullet_{X/k}) = F^i/F^{i+1} = \pi^*(\Omega^i_{X/k}) \otimes_{\mathcal{O}_X} \Omega^{\bullet,-i}_{X/k}.$$

Let $\mathbb{R}^0\pi_*$ denote the functor of complexes of abelian sheaves on X to the category of abelian sheaves on S. The derived functors of $\mathbb{R}^0\pi_*$ are $\mathbb{R}^q\pi_*$. Applying the spectral sequence for finitely filtered objects as in [**7**, 0_{III}, 13.6.4], we obtain a spectral sequence abutting to the associated graded object with respect to the filtration of $\mathbb{R}^q\pi_*(\Omega^\bullet_{X/k})$, while the E_1-term is $E_1^{p,q} = \Omega^p_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S)$. Using the canonical flasque resolution à la Godement (generalizing the procedure for constructing the cup product): we can equipp the spectral sequence with a product structure, explicitly, for each p, q, p', q' and r:

$$E_r^{p,q} \times E_r^{p',q'} \longrightarrow E_r^{p+p',q+q'}$$

sending (e, e') to $e \cdot e'$ where e, e' are sections of $E_r^{p,q}$ and $E_r^{p',q'}$ respectively over an open subset of S. This pairing satisfies the usual rules:

$$e \cdot e' = (-1)^{(p+q)(p'+q')} e' \cdot e,$$

and

$$d_r(e \cdot e') = d_r(e) \cdot e' + (-1)^{p+q} e \cdot d_r(e').$$

At the E_1-term, since d_1 has bidegree $(1, 0)$, we obtain, for any q, the complex $E_1^{\cdot,q}$, which is explicitly:

$$0 \longrightarrow \mathcal{H}^q_{DR}(X/S) \xrightarrow{d_1^{0,q}} \Omega^1_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S) \xrightarrow{d_1^{1,q}} \Omega^2_{S/k} \otimes_{\mathcal{O}_S} \mathcal{H}^q_{DR}(X/S) \longrightarrow \cdots$$

and we define the Gauss-Manin connexion to be $d_1^{0,q}$. The integrability follows from the observation that the connecting homomorphism defining the Gauss-Manin connexion is the differential between E_1-terms of a spectral sequences, that is the curvature is $d_1^{1,q} \cdot d_1^{0,q} = 0$. Thence, we can form the hypercohomology groups $\mathbb{H}^q(\mathcal{H}^p(X/S), \nabla_{GM})$ of the de Rham complex. We obtain the associated hypercohomology spectral sequence:

THEOREM 2.8. *(Leray spectral sequence)*

$$\mathbb{H}^q(\mathcal{H}^p(X/S), \nabla_{GM}) \implies H^{p+q}_{dR}(X).$$

The Gauss-Manin connexion has many interesting properties related to its behaviour with respect to spectral sequences: it is compatible with the Zariski filtration, but *not* compatible with the Hodge filtration: it respects it with a shift of one,

$$\nabla_{GM} F^i_{Hodge} \subset F^{i-1}_{Hodge}.$$

This phenomena is called Griffiths transversality, and arises naturally when studying families of varieties varying algebraically e.g. Shimura varieties. When the Hodge-to-de-Rham spectral sequence degenerates ($E_1 = E_\infty$) (for exemple, for $X \longrightarrow S$ proper, smooth, $\mathcal{E} = \mathcal{O}_X$ and in zero characteristic), then the Gauss-Manin connexion is given by the cup-product with the Kodaira-Spencer map

$$KS_{X/S} : Der_k(\mathcal{O}_S, \mathcal{O}_S) \longrightarrow H^1(X, Der(X/S)).$$

Remark 2.9. From the introduction of [**16**]. Let $S/\mathrm{Spec}(\mathbb{C})$ be a projective, connected, non-singular curve over with a finite number of points $\{x_1, \ldots, x_m\}$ removed. Let $\pi : X \longrightarrow S$ be a proper, smooth morphism. The space X can be views as a locally trivial C^∞-fibre space. Fix $i \geq 0$. When we let $s \in S$ vary, the complex cohomology groups $H^i(X_s, \mathbb{C})$ form a *local system* on $S^{analytic}$. The Gauss-Manin connexion enables us to construct this local system in an algebraic manner. The algebraic de Rham sheaves $\mathcal{H}^i_{dR}(X/S)$ are locally free coherent algebraic sheaves on S, with fiber at s $H^i(X_s, \mathbb{C})$, equipped with the Gauss-Manin connexion ∇_{GM}. The local system of $H^i(X_s, \mathbb{C})$ is retrieved as the sheaf of germs of horizontal sections of $H^i_{dR}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_S^{analytic}$. Differential equations give a geometric interpretation of these structures: $H^i_{dR}(X/S)$ can be viewed as a system of algebraic differential equations on S (called the Pichard-Fuchs equations), and the local system of $H^i(X_s, \mathbb{C})$ is the local system of germs of solutions of the Picard-Fuchs equations.

3. Crystalline cohomology is a Weil cohomology

The participants of the Seminar are familiar with the introduction of [**3**], which discusses the notion of a Weil cohomology. Berthelot's pavé ([**1**]) proves that for X smooth and proper over a field, crystalline cohomology is a Weil cohomology. Katz and Messing ([**14**] is in a missing *Inventiones* volume at Rosenthall Library) have shown that *if* H^* is any cohomology theory for projective smooth varieties over finite fields (with coefficients in a field of characteristic zero) that satisfies:

(1) Poincaré duality;
(2) the weak Lefschetz theorem;
(3) the Lefschetz trace formula giving the zeta function as the alternating product of the characteristic polynomials of the Frobenius acting as endomorphism on the cohomology H^*),

then the Riemann hypothesis (!) and the hard Lefschetz theorem are true for H^* as well. Bloch, Deligne and Illusie have shown that one can associate a complex $W\Omega^\bullet_X$ to any smooth, proper scheme over a perfect field of characteristic p, called the de Rham-Witt complex. It is a complex of sheaves whose hypercohomology (computed on the Zariski site) is isomorphic to the crystalline cohomology $H^*_{cris}(X/W(k))$. This complex has many virtues: its construction is independent of the construction of the crystalline site (no divided powers, no topos theory), it is elementary and the proof that crystalline cohomology is a cohomology theory à la Weil (among other properties, checking the above three conditions) is simpler than Berthelot's original treatment. To be fair, Berthelot's treatment is very thorough: he proved in his thesis the finiteness theorems for the H^i_{cris}, the isomorphism with de Rham cohomology in the smooth case, smooth base change, Künneth formula, the Lefschetz trace formula, Poincaré duality, the existence of a Gysin morphism and the existence of a cycle class map under certain conditions (Gillet and Messing later did it in general). He also proved the weak Lefschetz formula in a later paper ([**2**]). It is impossible to treat the exciting properties of the

Rham-Witt complex (and the associated spectral sequences) at any depth in just a few lines, and altogether, it is best to learn more about F-crystals and Cartier-Dieudonné modules before undertaking its study. Let us add a few (possibly unconvincing) remarks: this complex helps us understanding better the non-degeneracy of the Hodge-to-de-Rham spectral sequence briefly discussed by Pete (cf. [**11**]); it enables many comparison theorems with other cohomology theories. Ekedahl's book gives substance to the general principle that a complex is a more refined object to study than its hypercohomology, in particular it may have moduli (Ekedahl gives as an example that the hypercohomology of the de Rham-Witt complex of supersingular K3 surfaces with Artin invariant σ only depend on σ, while the whole complex has $\sigma - 1$ moduli).

3.1. ***F*-crystals.** Let X be a smooth projective variety defined over a field k and let $Alb(X)$ be its Albanese variety over k. Suppose that X has a k-rational point. Then we have the following properties:

- There is a map:
$$alb : X \longrightarrow Alb(X)$$
such that $alb(X)$ generates $Alb(X)$.
- (Universality) Given any map $f : X \longrightarrow A$ from X to an abelian variety A, there exists a map $g : Alb(X) \longrightarrow A$ such that $f = g \circ alb$.

The map $X \longrightarrow Alb(X)$ induces an *isomorphism* $H^1_{cris}(Alb(X)/W(k)) \longrightarrow H^1_{cris}(X/W(k))$. Moreover, Oda showed that $H^1_{cris}(Alb(X)/W(k))$ is the Dieudonné module of the p-divisible group $\varinjlim Alb(X)[p^n]$. If $g = \dim Alb(X)/k$, it is a free $W(k)$-module of rank $2g$ equipped with two semilinear (or σ-linear) endomorphisms F and V such that $FV = VP = p$, induces by the Frobenius and Verschiebung maps on $Alb(X)$, and where σ is the map induced from $x \mapsto x^p$, say. The module $H^1(X/W(k))$ is an example of an F-crystal:

Definition 3.1.
- An F-crystal M over $W(k)$ is a free module of finite rank over $W(k)$ with σ-linear injective map $F : M \longrightarrow M$, such that M/FM is of finite length. The homomorphisms are $W(k)$-module homomorphisms such that for
$$f \in \mathrm{Hom}((M_1, F_1), (M_2, F_2)), f \circ F_1 = F_2 \circ f.$$
- An F-isocrystal (V, F) over $B(k)$ ($B(k)$ being the field of fractions of $W(k)$) is a finite dimensional vector space over $B(k)$ with a σ-linear bijective map $F : V \longrightarrow V$. The homomorphisms are vector space maps respecting the F-structure.

When X is smooth, proper over k, $H^m_{cris}(X/W(k))/torsion$ equipped with F, the action of Frobenius, is an F-cristal. Poincaré duality insures that the map F is injective.

Theorem 3.2. (Dieudonné-Manin) *Let k be an algebraically closed field. The category of F-isocrystals over $B(k)$ is semisimple with simple objects parameterized by $\mathbb{Q}$. To a slope $\lambda \in \mathbb{Q}$ corresponds the object E_λ defined as follows. If $\lambda = \frac{r}{s}$, with $s, r \in \mathbb{Z}$, $s > 0, (r, s) = 1$, then*
$$E_\lambda = B(k)[F]/(F^s - T^r).$$

This result allows to define the *slopes* of an F-cristal as the set of λ. Poincaré duality allows to show that the slopes of $H^m_{cris}(X/W(k))/torsion$ are in $[0, d]$. The weak Lefschetz theorem imply that the slopes are in $[0, m]$, if $0 \leq m \leq d$ and in $[m - d, d]$ if $d \leq m \leq 2d$.

For a proof of a generalization of the Dieudonné-Manin theorem to Cohen rings, see [**17**, Theorem 5.6, p. 63]. Grothendieck, Mazur and Messing ([**19**]) and later de Jong, Kedlaya and others generalized classical results about F-crystals to crystals in modules over increasingly general base schemes, i.e. varying in families. For example, to give a flavour of this generalization, an F-crystal over $W(k)[[T]]$ is a finite, locally free $W(k)[[T]]$-module equipped with a $W(k)$-linear connection $\nabla : M \longrightarrow M \otimes dt$ for $r \in W(k)[[T]], v \in M$ and a map $F : \sigma^* M \longrightarrow M$ of $W(k)[[T]]$-modules with connection, called its “Frobenius”, whose kernel and cokernel are killed by some power of p. Note that F *induces* a σ-linear map $M \longrightarrow M$.

It is not too surprising that the first crystalline group is well-behaved, but it is rather striking that the definition also work give sensible $H^i_{cris}, i > 1$. We have seen that $H^1_{cris}(X/W(k))$ determines the p-divisible group of $Alb(X)$, another way to phrase it is to say that $H^1_{cris}(X/W(k))$ uniquely determines the completion of the Picard variety. Under certain hypotheses, there is an interpretation of $H^2_{cris}(X/W(k))$ in terms of a smooth formal group related to the Brauer group of X. I don't know of any further precise interpretation for $i > 2$.

4. Appendix: Witt vectors and a minuscule dose of semilinear algebra

We try explaining and motivating the use of Witt vectors.

Definition 4.1. Let k be a field of characteristic p. A *Cohen ring* is a complete discrete valuation ring W of unequal characteristic with residue field k whose maximal ideal is generated by p.

A Cohen ring W is unique up to non-canonical isomorphism. If k is perfect ($x \mapsto x^p$ is bijective), then W is unique up to canonical isomorphism, and we call it the ring of Witt vectors $W(k)$. The quotient $W_n(k) := W(k)/p^n W(k)$ is called the truncated Witt vectors of length n. For example, $W_2(k) \cong k \oplus k$ as k-varieties. $W_n(k)$ is flat over $\mathbb{Z}/p^n\mathbb{Z}$, and $W_n(k)/pW_n(k) \cong k$, and these two properties characterize $W_n(k)$.

Why are the Witt vectors such an important algebraic concept ? We throw in a couple of remarks:

- $W(k)$ is not too hairy: it is a discrete valuation ring, hence has (Krull) dimension one: the next best thing after a field; and it has characteristic zero, with a unique PD-structure on its maximal ideal;
- $W(\bar{k})$ is the maximal unramified extension of $W(k)$: this, I believe, was Witt's great breakthrough, as giving a systematic way of constructing unramified extensions;
- The Witt vectors are equipped with two semi-linear operators: Frobenius (for Frobenius) and Verschiebung (for translation) such that $FV = p = VF$;

N.B. These notions are useful and powerful (cf. [**17**]). A difference between semi-linear algebra and linear algebra is that the former depends on the (residue) field, but it is still quite rigid:

Lemma 4.2. (“Fitting's Lemma” for F-crystals) *We can decompose an F-crystal (M, F) uniquely in a direct sum*

$$M = M_{etale} \oplus M_{local}$$

of its étale part M_{etale} (on which Frobenius is an isomorphism) and its local part M_{local} (on which Frobenius is topologically nilpotent, i.e. $\cap_{i \geq 0} F^i M_{local} = 0$).

See what happens over an algebraically closed field:

EXERCISE 4.3. (Fitting) Let k be an algebraically closed field of characteristic p, $\mathcal{V}/k$ a k-vector space of dimension n. Let $q = p^a$, for some $a \in \mathbb{Z}\backslash 0$ and $\phi : \mathcal{V} \longrightarrow \mathcal{V}$ an additive bijection such that for all $\lambda \in k$, for all $v \in \mathcal{V}$, we have $\phi(\lambda v) = \lambda^q v$. Then there exists a base $(e_1, \ldots, e_n)$ of $\mathcal{V}$ such that $\phi(e_i) = e_i$ for all i.

Moreover, the presence of a Frobenius map built-in the Witt vectors is an interesting feature for us motivated by characteristic p geometry, where the map $x \mapsto x^p$ is so crucially used. As a matter of fact, it follows from the functoriality of the cohomology theory that the absolute Frobenius endomorphism $F_{abs} : X \longrightarrow X$ induces a semi-linear map: $F_{abs}^* : H^*(X/W(k)) \longrightarrow H^*(X/W(k))$. This, modulo torsion, gives an F-crystal structure to $H^i(X/W(k), F_{abs}^*)$.

Bibliography

[1] Berthelot, Pierre, *Cohomologie cristalline des schémas de caractéristique* $p > 0$. Lecture Notes in Mathematics, Vol. **407.** Springer-Verlag, Berlin-New York, 1974. 604 pp.

[2] Berthelot, Pierre Sur le "théorème de Lefschetz faible" en cohomologie cristalline. *C. R. Acad. Sci. Paris* Sér. A-B **277** (1973), A955–A958.

[3] Berthelot, Pierre; Ogus, Arthur, *Notes on crystalline cohomology.* Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978. vi+243 pp.

[4] Pete L. Clark, On Hodge Theory and DeRham Cohomology of Varieties, Notes for this seminar, 4th version, October 29th, 2003.

[5] Deligne, Pierre; Pappas, Georgios, Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant. *Compositio Math.* **90** (1994), no. 1, 59–79.

[6] Ekedahl, Torsten, *Diagonal complexes and F-gauge structures.* Travaux en Cours. Hermann, Paris, 1986. xii+122 pp.

[7] Dieudonné, J., Grothendieck, A. Éléments de géométrie algébrique III. Étude cohomologique des faisceaux cohérents. I. Inst. Hautes Études Sci. Publ. Math. No. **11**, 1961 167 pp.

[8] Grothendieck, Alexandre —em Groupes de Barsotti-Tate et cristaux de Dieudonné. Séminaire de Mathématiques Supérieures, No. **45**, 1970. Les Presses de l'Université de Montrtéal, Montréal, Qué., 1974. 155 pp.

[9] Grothendieck, A., Letter to John Tate.

[10] Hartshorne, Robin On the De Rham cohomology of algebraic varieties. *Inst. Hautes Études Sci. Publ. Math.* No. **45** (1975), 5–99.

[11] Illusie, Luc, Complexe de de Rham-Witt et cohomologie cristalline. *Ann. Sci. École Norm. Sup.* (4) **12** (1979), no. 4, 501–661.

[12] Bertin, José; Demailly, Jean-Pierre; Illusie, Luc; Peters, Chris, *Introduction à la théorie de Hodge.* Panoramas et Synthèses, 3. Société Mathématique de France, Paris, 1996. vi+273 pp.

[13] Katz, Nicholas, Travaux de Dwork. *Séminaire Bourbaki*, 24ème année (1971/1972), Exp. No. **409**, pp. 167–200. Lecture Notes in Math., Vol. 317, Springer, Berlin, 1973.

[14] Katz, Nicholas M.; Messing, William, Some consequences of the Riemann hypothesis for varieties over finite fields. *Invent. Math.* **23** (1974), 73–77.

[15] Katz, Nicholas M.; Oda, Tadao, On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.* **8**, 1968, 199–213.

[16] Katz, Nicholas M., Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin. *Inst. Hautes Études Sci. Publ. Math.* No. **39**, (1970), 175–232.

[17] Kedlaya, Kiran, A p-adic local monodromy theorem, arXiv:math.AG/0110124, to appear in *Annals of Mathematics.*

[18] Manin, Ju. I. Algebraic curves over fields with differentiation.*Izv. Akad. Nauk SSSR. Ser. Mat.* **22** 1958 737–756.

[19] Mazur, B.; Messing, William, *Universal extensions and one dimensional crystalline cohomology.* Lecture Notes in Mathematics, Vol. **370**. Springer-Verlag, Berlin-New York, 1974. vii+134 pp.