LECTURES ON ETALE COHOMOLOGY

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Abstract. These are the notes for a course taught at the University of Michigan in W89 as Math 732 and in W98 as Math 776. They are available at www.math.lsa.umich.edu/~jmilne/

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Contents

Part I: Basic Theory

In comparison with my book, the emphasis is on heuristic arguments rather than formal proofs and on varieties rather than schemes. The notes also discuss the proof of the Weil conjectures (Grothendieck and Deligne).

Notations and Conventions. The conventions concerning varieties are the same as those in my notes on Algebraic Geometry. For example, an affine algebra over a field k is a finitely generated k-algebra A such that $A \otimes_k k^{\mathrm{al}}$ is has no nonzero nilpotents for one (hence every) algebraic closure k^{al} of k — this implies that A itself has no nilpotents. With such a k-algebra, we associate a ringed space $\mathrm{Specm}(A)$ (topological space endowed with a sheaf of k-algebras), and an affine variety over k is a ringed space isomorphic to one of this form. A variety over k is a ringed space $(X, \mathcal{O}_X)$ admitting a finite open covering $X = \cup U_i$ such that $(U_i, \mathcal{O}_X|U_i)$ is an affine variety for each i and which satisfies the separation axiom. We often use X to denote $(X, \mathcal{O}_X)$ as well as the underlying topological space. A regular map of varieties will sometimes be called a morphism of varieties.

For those who prefer schemes, a variety is a separated geometrically reduced scheme X of finite type over a field k with the nonclosed points omitted.

If X is a variety over k and $K \supset k$, then $X(K)$ is the set of points of X with coordinates in K and X_K or $X_{/K}$ is the variety[1] over K obtained from X. For example, if $X = \mathrm{Specm}(A)$, then $X(K) = \mathrm{Hom}_{k\text{-alg}}(A, K)$ and $X_K = \mathrm{Specm}(A \otimes_k K)$. In general, $X(K)$ is just a set, but I usually endow $X(\mathbb{C})$ with its natural complex topology. A separable closure k^{sep} of a field k is a field algebraic over k such that every separable polynomial with coefficients in k has a root in k^{sep}.

Our terminology concerning schemes is standard, except that I shall always assume that our rings are Noetherian and that our schemes are locally Noetherian.

Our terminology concerning rings is standard. In particular, a homomorphism $A \to B$ of rings maps 1 to 1. A homomorphism $A \to B$ of rings is finite, and B is a finite A-algebra, if B is finitely generated as an A-module. When A is a local ring, I often denote its (unique) maximal ideal by $\mathfrak{m}_A$. A local homomorphism of local rings is a homomorphism $f\colon A \to B$ such that $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$ (equivalently, $f(\mathfrak{m}_A) \subset \mathfrak{m}_B$).

Generally, when I am drawing motivation from the theory of sheaves on a topological space, I assume that the spaces are Hausdorff, i.e., not some weird spaces with points whose closure is the whole space.

I use the following notations:

$X \approx Y$	X and Y are isomorphic;
$X \cong Y$	X and Y are canonically isomorphic or there is a given or unique isomorphism;
$X \stackrel{\mathrm{df}}{=} Y$	X is defined to be Y, or equals Y by definition;
$X \subset Y$	X is a subset of Y (not necessarily proper).

References.

Homological algebra. I shall assume some familiarity with the language of abelian categories and derived functors. There is a summary of these topics in my Class Field Theory notes pp 69–76, and complete presentations in several books, for example, in Weibel, C.A., *An Introduction to Homological Algebra*[2], Cambridge U.P., 1994.

We shall not be able to avoid using spectral sequences — see pp 307–309 of my book on Etale Cohomology for a brief summary of spectral sequences and Chapter 5 of Weibel's book for a complete treatment.

Sheaf theory. Etale cohomology is modelled on the cohomology theory of sheaves in the usual topological sense. Much of the material in these notes parallels that in, for example,

Iverson, B., *Cohomology of Sheaves*, Springer, 1986.

[1]Our terminology follows that of Grothendieck. There is a conflicting terminology, based on Weil's Foundations and frequently used by workers in the theory of algebraic groups, that writes these the other way round.

[2]On p 9, Weibel defines $C[p]^n = C^{n-p}$. The correct original definition, universally used by algebraic and arithmetic geometers, is that $C[p]^n = C^{n+p}$(see Hartshorne, R., *Residues and Duality*, 1966, p26). Also, in Weibel, a "functor category" need not be a category.

Algebraic geometry. I shall assume familiarity with the theory of algebraic varieties, for example, as in my notes on Algebraic Geometry (Math. 631). Also, sometimes I will mention schemes, and so the reader should be familiar with the basic language of schemes as, for example, the first 3 sections of Chapter II of Hartshorne, *Algebraic Geometry*, Springer 1977, the first chapter of Eisenbud and Harris, *Schemes,* Wadsworth, 1992, or Chapter V of Shafarevich, *Basic Algebraic Geometry*, 2nd Edition, Springer, 1994.

For commutative algebra, I usually refer to

Atiyah, M., and MacDonald, I., *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

Etale cohomology. There are the following books:

Freitag, E., and Kiehl, R., *Etale Cohomology and the Weil Conjecture,* Springer, 1988.

Milne, J., *Etale Cohomology*, Princeton U.P. 1980 (cited as **EC**).

Tamme, *Introduction to Etale Cohomology*, Springer.

The original sources are:

Artin, M., *Grothendieck Topologies*, Lecture Notes, Harvard University Math. Dept. 1962.

Artin, M., *Théorèmes de Représentabilité pour les Espace Algébriques*, Presses de l'Université de Montréal, Montréal, 1973

Grothendieck, A., et al., Séminaire de Géométrie Algébrique.

SGA 4: (with Artin, M., and Verdier, J.-L.). *Théorie des topos et cohomologie étale des schémas* (1963–64). Springer Lecture Notes 1972–73.
SGA 4 1/2: (by Deligne, P., with Boutot, J.-F., Illusie, L., and Verdier, J.-L.) *Cohomologie étale*. Springer Lecture Notes 1977.
SGA 5: *Cohomologie l-adique et fonctions L* (1965–66). Springer Lecture Notes 1977.
SGA 7: (with Deligne, P., and Katz, N.) *Groupes de monodromie en géométrie algébriques* (1967–68). Springer Lecture Notes 1972–73.

Except for SGA $4\frac{1}{2}$, these are the famous seminars led by Grothendieck at I.H.E.S..

Whenever possible, I use my other course notes as references (because they are freely available to everyone).

GT: Group Theory (Math 594).
FT: Field Theory (Math 594).
AG: Algebraic Geometry (Math 631).
ANT: Algebraic Number Theory (Math 676).
AV: Abelian Varieties (Math 731).
CFT: Class Field Theory (Math 776).

Comment. The major theorems in étale cohomology are proved in SGA 4 and SGA 5, but often under unnecessarily restrictive hypotheses. Some of these hypotheses were removed later, but by proofs that used much of what is in those seminars. Thus the structure of the subject needs to be re-thought. Also, algebraic spaces should be more fully incorporated into the subject (see Artin 1973). It is likely that de Jong's resolution theorem (Smoothness, semi-stability and alterations. Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 51–93) will allow many improvements. Finally, such topics as intersection cohomology and Borel-Moore homology need to be added to the exposition. None of this will be attempted in these notes.

1. Introduction

For a variety X over the complex numbers, $X(\mathbb{C})$ acquires a topology from that on $\mathbb{C}$, and so one can apply the machinery of algebraic topology to its study. For example, one can define the Betti numbers $\beta^r(X)$ of X to be the dimensions of the vector spaces $H^r(X(\mathbb{C}), \mathbb{Q})$, and such theorems as the Lefschetz fixed point formula are available.

For a variety X over an arbitrary algebraically closed field k, there is only the Zariski topology, which is too coarse (i.e., has too few open subsets) for the methods of algebraic topology to be useful. For example, if X is irreducible, then the groups $H^r(X, \mathbb{Z})$, computed using the Zariski topology, are zero for all $r > 0$.

In the 1940s, Weil observed that some of his results on the numbers of points on certain varieties (curves, abelian varieties, diagonal hypersurfaces..) over finite fields would be explained by the existence of a cohomology theory giving vector spaces over a field of characteristic zero for which a Lefschetz fixed point formula holds. His results predicted a formula for the Betti numbers of a diagonal hypersurface in $\mathbb{P}^{d+1}$ over $\mathbb{C}$ which was later verified by Dolbeault.

About 1958, Grothendieck defined the étale "topology" of a scheme, and the theory of étale cohomology was worked out by him with the assistance of M. Artin and J.-L. Verdier. The whole theory is closely modelled on the usual theory of sheaves and their derived functor cohomology on a topological space. For a variety X over $\mathbb{C}$, the étale cohomology groups $H^r(X_{et}, \Lambda)$ coincide with the complex groups $H^r(X(\mathbb{C}), \Lambda)$ when Λ is finite, the ring of ℓ-adic integers $\mathbb{Z}_\ell$, or the field $\mathbb{Q}_\ell$ of ℓ-adic numbers (but not for $\Lambda = \mathbb{Z}$). When X is the spectrum of a field K, the étale cohomology theory for X coincides with the Galois cohomology theory of K. Thus étale cohomology bridges the gap between the first case, which is purely geometric, and the second case, which is purely arithmetic.

As we shall see in the course, étale cohomology does give the expected Betti numbers. Moreover, it satisfies analogues of the Eilenberg-Steenrod axioms, the Poincaré duality theorem, the Lefschetz fixed point formula, the Leray spectral sequence, etc.. The intersection cohomology of Goresky and MacPherson has an étale analogue, which provides a Poincaré duality theorem for *singular* varieties. Etale cohomology has been brilliantly successful in explaining Weil's observation.

Algebraic Topology. We briefly review the origins of the theory on which étale cohomology is modelled.

Algebraic topology had its origins in the late 19th century with the work of Riemann, Betti, and Poincaré on "homology numbers". After an observation of Emmy Noether, the focus shifted to "homology groups". By the 1950s there were several different methods of attaching (co)homology groups to a topological space, for example, there were the singular homology groups of Veblen, Alexander, and Lefschetz, the relative homology groups of Lefschetz, the Vietoris homology groups, the Čech homology groups, and the Alexander cohomology groups.

The situation was greatly clarified by Eilenberg and Steenrod 1953[3], which showed that for any "admissible" category of pairs of topological spaces, there is exactly one

[3]Foundations of Algebraic Topology, Princeton.

cohomology theory satisfying a certain short list of axioms. Consider, for example, the category whose objects are the pairs (X, Z) with X a locally compact topological space and Z a closed subset of X, and whose morphisms are the continuous maps of pairs. A cohomology theory on this category is a contravariant functor attaching to each pair a sequence of abelian groups and maps

$$\cdots \to H^{r-1}(U) \to H^r_Z(X) \to H^r(X) \to H^r(U) \to \cdots, \qquad U = X \setminus Z,$$

satisfying the following axioms:

(a) (exactness axiom) the above sequence is exact;
(b) (homotopy axiom) the map f^* depends only on the homotopy class of f;
(c) (excision) if V is open in X and its closure is disjoint from Z, then the inclusion map $(X \setminus V, Z) \to (X, Z)$ induces an isomorphisms $H^r_Z(X) \to H^r_Z(X \smallsetminus V)$;
(d) (dimension axiom) if X consists of a single point, then $H^r(P) = 0$ for $r \neq 0$.

The topologists usually write $H^r(X, U)$ for the group $H^r_Z(X)$. The axioms for a homology theory are similar to the above except that the directions of all the arrows are reversed. If $(X, Z) \mapsto H^r_Z(X)$ is a cohomology theory such that the $H^r_Z(X)$ are locally compact abelian groups (e.g., discrete or compact), then $(X, Z) \mapsto H^r_Z(X)^\vee$ (Pontryagin dual) is a homology theory. In this approach there is implicitly a single coefficient group.

In the 1940s, Leray attempted to understand the relation between the cohomology groups of two spaces X and Y for which a continuous map $Y \to X$ is given. This led him to the introduction of sheaves (local systems of coefficient groups), sheaf cohomology, and spectral sequences (about the same time as Roger Lyndon, who was trying to understand the relations between the cohomologies of a group G, a normal subgroup N, and the quotient group G/N).

Derived functors were used systematically in Cartan and Eilenberg 1956[4], and in his 1955 thesis a student of Eilenberg, Buchsbaum, defined the notion of an abelian category and extended the Cartan-Eilenberg theory of derived functors to such categories. (The name "abelian category" is due to Grothendieck).

Finally Grothendieck, in his famous 1957 Tohôku paper[5], showed that the category of sheaves of abelian groups on a topological space is an abelian category with enough injectives, and so one can define the cohomology groups of the sheaves on a space X as the right derived functors of the functor taking a sheaf to its abelian group of global sections. One recovers the cohomology of a fixed coefficient group Λ as the cohomology of the constant sheaf it defines. This is now the accepted definition of the cohomology groups, and it is the approach we follow to define the étale cohomology groups. Instead of fixing the coefficient group and having to consider all (admissible) pairs of topological spaces in order to characterize the cohomology groups, we fix the topological space but consider all sheaves on the space.

Brief review of sheaf cohomology. Let X be a topological space. We make the open subsets of X into a category with the inclusions as the only morphisms, and define a presheaf to be a contravariant functor from this category to the category *Ab* of abelian groups. Thus, such a presheaf $\mathcal{F}$ attaches to every open subset U of X an

[4]Homological Algebra, Princeton.
[5]Sur quelques points d'algèbre homologique, Tôhoku Math. J. 9, 119–221.

abelian group $\mathcal{F}(U)$ and to every inclusion $V \subset U$ a restriction map $\rho_V^U\colon \mathcal{F}(U) \to \mathcal{F}(V)$ in such way that $\rho_U^U = \mathrm{id}_{\mathcal{F}(U)}$ and, whenever $W \subset V \subset U$,

$$\rho_W^U = \rho_W^V \circ \rho_V^U.$$

For historical reasons, the elements of $\mathcal{F}(U)$ are called the sections of $\mathcal{F}$ over U, and the elements of $\mathcal{F}(X)$ the global sections of $\mathcal{F}$. Also, one sometimes writes $\Gamma(U,\mathcal{F})$ for $\mathcal{F}(U)$ and $s|V$ for $\rho_V^U(s)$.

A presheaf $\mathcal{F}$ is said to be a sheaf if

(a) a section $f \in \mathcal{F}(U)$ is determined by its restrictions $\rho_U^{U_i}(f)$ to the sets of an open covering $(U_i)_{i\in I}$ of U;
(b) a family of sections $f_i \in \mathcal{F}(U_i)$ for $(U_i)_{i\in I}$ an open covering of U arises by restriction from a section $f \in \mathcal{F}(U)$ if $f_i|U_i \cap U_j = f_j|U_i \cap U_j$ for all i and j.

In other words, $\mathcal{F}$ is a sheaf if, for every open covering $(U_i)_{i\in I}$ of an open subset U of X, the sequence

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j),$$

is exact — by definition, this means that the first arrow maps $\mathcal{F}(U)$ onto the subset of $\prod \mathcal{F}(U_i)$ on which the next two arrows agree. The first arrow sends $f \in \mathcal{F}(U)$ to the family $(f|U_i)_{i\in I}$, and the next two arrows send $(f_i)_{i\in I}$ to the families $(f_i|U_i\cap U_j)_{(i,j)\in I\times I}$ and $(f_j|U_i \cap U_j)_{(i,j)\in I\times I}$ respectively. Since we are considering only (pre)sheaves of abelian groups, we can restate the condition as: the sequence

$$\begin{array}{ccccccc} 0 & \to & \mathcal{F}(U) & \to & \prod_{i\in I} \mathcal{F}(U_i) & \to & \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \cap U_j) \\ & & f & \mapsto & (f|U_i), & & \\ & & & & (f_i) & \mapsto & (f_j|U_i \cap U_j - f_i|U_i \cap U_j) \end{array}$$

is exact. When applied to the empty covering of the empty set, the condition implies that $\mathcal{F}(\emptyset) = 0$.

For example, if Λ is a topological abelian group (e.g., $\mathbb{R}$ or $\mathbb{C}$), then we can define a sheaf on any topological space X by setting $\mathcal{F}(U)$ equal to the set of continuous maps $U \to \Lambda$ and taking the restriction maps to be the usual restriction of functions.

When Λ has the discrete topology, every continuous map $f\colon U \to \Lambda$ is constant on each connected component of U, and hence factors through $\pi_0(U)$, the space of connected components of U. When this last space is discrete, $\mathcal{F}(U)$ is the set of all maps $\pi_0(U) \to \Lambda$, i.e., $\mathcal{F}(U) = \Lambda^{\pi_0(U)}$. In this case, we call $\mathcal{F}$ the constant sheaf defined by the abelian group Λ.

Grothendieck showed that, with the natural structures, the sheaves on X form an abelian category. Thus, we have the notion of an injective sheaf: it is a sheaf $\mathcal{I}$ such that for any subsheaf $\mathcal{F}'$ of a sheaf $\mathcal{F}$, every homomorphism $\mathcal{F}' \to \mathcal{I}$ extends to a homomorphism $\mathcal{F} \to \mathcal{I}$. Grothendieck showed that every sheaf can be embedded into an injective sheaf. The functor $\mathcal{F} \mapsto \mathcal{F}(X)$ from the category of sheaves on X to the category of abelian groups is left exact but not (in general) right exact. We define $H^r(X,\cdot)$ to be its r^{th} right derived functor. Thus, given a sheaf $\mathcal{F}$, we choose an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots$$

with each $\mathcal{I}^r$ injective, and we set $H^r(X,\mathcal{F})$ equal to the r^{th} cohomology group of the complex of abelian groups

$$\mathcal{I}^0(X) \to \mathcal{I}^1(X) \to \mathcal{I}^2(X) \to \cdots$$

While injective resolutions are useful for defining the cohomology groups, they are not convenient for computing it. Instead, one defines a sheaf $\mathcal{F}$ to be flabby if the restriction maps $\mathcal{F}(U) \to \mathcal{F}(V)$ are surjective for all open $U \supset V$, and shows that $H^r(X,\mathcal{F}) = 0$ if $\mathcal{F}$ is flabby. Thus, resolutions by flabby sheaves can be used to compute cohomology.

The inadequacy of the Zariski topology. As we noted above, for many purposes, the Zariski topology has too few open subsets. We list some situations where this is evident.

The cohomology groups are zero. Recall that a topological space X is said to be irreducible if any two nonempty open subsets of X have nonempty intersection, and that a variety (or scheme) is said to be irreducible if it is irreducible as a (Zariski) topological space.

THEOREM 1.1 (Grothendieck's Theorem). *If X is an irreducible topological space, then $H^r(X,\mathcal{F}) = 0$ for all constant sheaves and all $r > 0$.*

PROOF. Clearly, any open subset U of an irreducible topological space is connected. Hence, if $\mathcal{F}$ is the constant sheaf defined by the group Λ, then $\mathcal{F}(U) = \Lambda$ for every nonempty open U. In particular, $\mathcal{F}$ is flabby, and so $H^r(X,\mathcal{F}) = 0$ for $r > 0$. $\square$

REMARK 1.2. The Čech cohomology groups are also zero. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X. Then the Čech cohomology groups of the covering $\mathcal{U}$ are the cohomology groups of a complex whose rth group is

$$\prod_{(i_0,\dots,i_r)\in I^{r+1},\quad U_{i_0}\cap\ldots\cap U_{i_r}\neq\emptyset} \Lambda$$

with the obvious maps. For an irreducible space X, this complex is independent of the space X; in fact, it depends only on the cardinality of I (assuming the U_i are nonempty). It is easy to construct a contracting homotopy for the complex, and so deduce that the complex is exact.

The inverse mapping theorem fails. A C^∞ map $\varphi\colon N \to M$ of differentiable manifolds is said to be étale at $n \in N$ if the map on tangent spaces $d\varphi\colon \operatorname{Tgt}_n(N) \to \operatorname{Tgt}_{\varphi(n)}(M)$ is an isomorphism.

THEOREM 1.3 (Inverse Mapping Theorem). *A C^∞ map of differentiable manifolds is a local isomorphism at any point at which it is étale, i.e., if $\varphi\colon N \to M$ is étale at $n \in N$, then there exist open neighbourhoods V and U of n and $\varphi(n)$ respectively such that φ restricts to an isomorphism $V \to U$.*

Let X and Y be nonsingular algebraic varieties over an algebraically closed field k. A regular map $\varphi\colon Y \to X$ is said to *étale at $y \in Y$* if $d\varphi\colon \operatorname{Tgt}_y(Y) \to \operatorname{Tgt}_{\varphi(y)}(X)$ is an isomorphism.

For example, as we shall see shortly, $x \mapsto x^n\colon \mathbb{A}^1_k \to \mathbb{A}^1_k$ is étale except at the origin (provided n is not divisible by the characteristic of k). However, if $n > 1$ this map is not a local isomorphism at any point; in fact, there do not exist nonempty open subsets V and U of $\mathbb{A}^1_k$ such that map $x \mapsto x^n$ sends V isomorphically onto U. To see this, note that $x \mapsto x^n$ corresponds to the homomorphism of k-algebras $T \mapsto T^n\colon k[T] \to k[T]$. If $x \mapsto x^n$ sends V into U, then the corresponding map $k(U) \to k(V)$ on the function fields is $T \mapsto T^n\colon k(T) \to k(T)$. If $V \to U$ were an isomorphism, then so would be the map on the function fields, but it isn't.

Take $n = 2$ and $k = \mathbb{C}$, so that the map is $z \mapsto z^2\colon \mathbb{C} \to \mathbb{C}$. To prove that this is a local isomorphism at $z_0 \neq 0$, we have to construct an inverse function $z \mapsto \sqrt{z}$ to $z \mapsto z^2$ on some open set containing z_0^2. In order to do this complex analytically, we need to remove a curve in the complex plane from 0 to ∞. The complement of such curve is open for the complex topology but not the Zariski topology.

Fibre bundles aren't locally trivial. A topology on a set allows us to speak of something being true "near", rather than "at", a point and to speak of it being true "locally", i.e., in a neighbourhood of every point.

For example, suppose we are given a regular map of varieties $\varphi\colon Y \to X$ over an algebraically closed field k and the structure of a k-vector space on each fibre $\varphi^{-1}(x)$. The choice of a basis for $\varphi^{-1}(x)$ determines a k-linear isomorphism of algebraic varieties $\varphi^{-1}(x) \to \mathbb{A}^n_k$ for some n. The map $\varphi\colon Y \to X$ is said to be a vector bundle if it is locally trivial in the sense that every point $x \in X$ has an open neighbourhood U for which there is a commutative diagram

$$\begin{array}{ccc} \varphi^{-1}(U) & \to & U \times \mathbb{A}^n \\ \downarrow & & \downarrow \\ U & = & U \end{array}$$

with the top arrow a k-linear isomorphism of algebraic varieties.

For a smooth variety over $\mathbb{C}$, a regular map $Y \to X$ as above is locally trivial for the Zariski topology if it is locally trivial for complex topology. Thus the Zariski topology is fine enough for the study of vector bundles (see Weil, A., Fibre spaces in algebraic geometry, Notes of a course 1952, University of Chicago). However, it is not fine enough for the study of more exotic bundles. Consider for example a regular map $\pi\colon Y \to X$ endowed with an action of an algebraic group G on Y over X, i.e., a regular map $m\colon Y \times G \to Y$ satisfying the usual conditions for a group action and such that $\pi(yg) = \pi(y)$. In a talk in the Séminaire Chevalley in 1958, Serre called such a system locally isotrivial if, for each $x \in X$, there is a finite étale map $U \to U' \subset X$ with U' a Zariski open neighbourhood of x such that the pull-back of Y to U becomes isomorphic to the trivial system $U \times G \to U$.

The usefulness of this notion, together with Weil's observation, led Grothendieck to introduce the étale topology.

Etale cohomology. Let X and Y be smooth varieties over an algebraically closed field k. A regular map $\varphi\colon Y \to X$ is said to be étale if it is étale at all points $y \in Y$. An étale map is quasifinite (its fibres are finite) and open.

The étale "topology" on X is that for which the "open sets" are the étale morphisms $U \to X$. A family of étale morphisms $(U_i \to U)_{i\in I}$ over X is a covering of U if $U = \cup \varphi_i(U_i)$.

An étale neighbourhood of a point $x \in X$ is an étale map $U \to X$ together with a point $u \in U$ mapping to x.

Define Et/X to be the category whose objects are the étale maps $U \to X$ and whose arrows are the commutative diagrams

$$\begin{array}{ccc} V & \to & U \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

with the maps $V \to X$ and $U \to X$ étale (then $V \to U$ is also automatically étale).

A presheaf for the étale topology on X is a contravariant functor $\mathcal{F}\colon Et/X \to Ab$. It is a sheaf if the sequence

$$\mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j)$$

is exact for all étale coverings $(U_i \to U)$. One shows, much as in the classical case, that the category of sheaves is abelian, with enough injectives. Hence one can define étale cohomology groups $H^r(X_{et}, \mathcal{F})$ exactly as in the classical case, by using the derived functors of $\mathcal{F} \mapsto \mathcal{F}(X)$.

The rest of the course will be devoted to the study of these groups and of their applications.

Comparison with the complex topology. Let X be a nonsingular algebraic variety over $\mathbb{C}$. Which is finer, the étale topology on X or the complex topology on $X(\mathbb{C})$? Strictly speaking, this question doesn't make sense, because they are not the same types of objects, but the complex inverse mapping theorem shows that every étale neighbourhood of a point $x \in X$ "contains" a complex neighbourhood. More precisely, given an étale neighbourhood $(U, u) \to (X, x)$ of x, there exists a neighbourhood (V, x) of V for the complex topology, such that the inclusion $(V, x) \hookrightarrow (X, x)$ factors into

$$(V, x) \to (U, u) \to (X, x)$$

(the first of these maps is complex analytic and the second is algebraic).

Thus, any étale covering of X can be "refined" by a covering for the complex topology. From this one obtains canonical maps $H^r(X_{et}, \Lambda) \to H^r(X(\mathbb{C}), \Lambda)$ for any abelian group Λ.

Theorem 1.4 (Comparison Theorem). *For any finite abelian group Λ, the canonical maps $H^r(X_{et}, \Lambda) \to H^r(X(\mathbb{C}), \Lambda)$ are isomorphisms.*

By taking $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ and passing to the inverse limit over n, we obtain an isomorphism $H^r(X_{et}, \mathbb{Z}_\ell) \to H^r(X(\mathbb{C}), \mathbb{Z}_\ell)$. When tensored with $\mathbb{Q}_\ell$, this becomes an isomorphism $H^r(X_{et}, \mathbb{Q}_\ell) \to H^r(X(\mathbb{C}), \mathbb{Q}_\ell)$.

Applications of étale cohomology. Etale cohomology has become a prerequisite for arithmetic geometry, algebraic geometry over ground fields other than $\mathbb{C}$, parts of number theory, parts of K-theory, and the representation theory of finite and p-adic groups. I hope to explain some of these applications during the course. Here I briefly describe two.

Classical algebraic geometry. Given an algebraic variety X over a field k, and a homomorphism of fields $\sigma\colon k \to K$, we obtain an algebraic variety σX over K. For example, if X is the affine (or projective) variety defined by equations

$$\sum a_{i_1\cdots i_n} T_1^{i_1}\cdots T_n^{i_n} = 0,$$

then σX is the variety defined by the equations

$$\sum \sigma a_{i_1\cdots i_n} T_1^{i_1}\cdots T_n^{i_n} = 0.$$

Note that we also get a map $(t_1,\ldots,t_n) \mapsto (\sigma t_1,\ldots,\sigma t_n)\colon X(k) \to X(K)$.

QUESTION 1.5. Let X be a variety over $\mathbb{C}$, and let σ be an automorphism of $\mathbb{C}$ (as a field only, i.e., not necessarily continuous). What is the relation between the cohomology groups $H^r(X(\mathbb{C}),\mathbb{Q})$ and $H^r(\sigma X(\mathbb{C}),\mathbb{Q})$?

The map $(t_1,\ldots,t_n) \mapsto (\sigma t_1,\ldots,\sigma t_n)\colon X(\mathbb{C}) \to X(\mathbb{C})$ won't be a homeomorphism unless σ is the identity map or complex conjugation. In fact, Serre[6] constructed a nonsingular projective surface X over $\mathbb{C}$ such that $X(\mathbb{C})$ and $(\sigma X)(\mathbb{C})$ have nonisomorphic fundamental groups and so are not homeomorphic by any map. The theory of Shimura varieties gives other examples of varieties X such that $X(\mathbb{C})$ and $(\sigma X)(\mathbb{C})$ have nonisomorphic fundamental groups.

The answer is to (1.5) is: the groups $H^r(X(\mathbb{C}),\mathbb{Q})$ and $H^r(\sigma X(\mathbb{C}),\mathbb{Q})$ become canonically isomorphic after they have been tensored with $\mathbb{Q}_\ell$ (or $\mathbb{C}$).

To prove this, use the comparison theorem and the fact that $H^r(X_{et},\mathbb{Q}_\ell)$ is canonically isomorphic to $H^r((\sigma X)_{et},\mathbb{Q}_\ell)$ (because the isomorphism $\sigma\colon \mathbb{C}\to\mathbb{C}$ carries étale maps $U \to X$ to étale maps $\sigma U \to \sigma X$ etc.—it is an algebraic isomorphism, and the definition of the étale cohomology groups is purely algebraic).

This implies that X and σX have the same Betti numbers, and so $H^r(X(\mathbb{C}),\mathbb{Q})$ and $H^r((\sigma X)(\mathbb{C}),\mathbb{Q})$ are isomorphic (but *not* canonically isomorphic). For a smooth projective variety, this weaker statement was proved by Serre[7] using Dolbeault's theorem

$$\beta^r(X) = \sum_{p+q=r} \dim H^q(X(\mathbb{C}),\Omega^p)$$

and his own comparison theorem

$$H^r(X,\mathcal{F}) = H^r(X(\mathbb{C}),\mathcal{F}^h)$$

($\mathcal{F}$ is a coherent sheaf of $\mathcal{O}_X$-modules for the Zariski topology on X, and $\mathcal{F}^h$ is the associated sheaf of $\mathcal{O}_{X^h}$-modules on $X(\mathbb{C})$ for the complex topology). The case of a general variety was not known before Artin proved the Comparison Theorem.

[6] Exemples de variétés projectives conjuguées non homéomorphes. C. R. Acad. Sci. Paris 258 (1964), 4194–4196.

[7] Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, Grenoble 6 (1956), pp 1–42.

Now I can explain why one should expect the groups $H^r(X_{et}, \mathbb{Q})$ to be anomolous. Given a variety X over $\mathbb{C}$, there will exist a subfield k of $\mathbb{C}$ such that $\mathbb{C}$ is an infinite Galois extension of k and such that X can be defined by equations with coefficients in k. Then $\sigma X = X$ for all $\sigma \in \operatorname{Gal}(\mathbb{C}/k)$, and so $\operatorname{Gal}(\mathbb{C}/k)$ will act on the groups $H^r(X_{et}, \mathbb{Q}_\ell)$. It is expected (in fact, the Tate conjecture implies) that the image of $\operatorname{Gal}(\mathbb{C}/k)$ in $\operatorname{Aut}(H^r(X_{et}, \mathbb{Q}_\ell))$ will usually be infinite (and hence uncountable). Since $H^r(X_{et}, \mathbb{Q})$ is a finite dimensional vector space over $\mathbb{Q}$, $\operatorname{Aut}(H^r(X_{et}, \mathbb{Q}))$ is countable, and so the action of $\operatorname{Gal}(\mathbb{C}/k)$ on $H^r(X_{et}, \mathbb{Q}_\ell)$ can not arise from an action on $H^r(X_{et}, \mathbb{Q})$.

Representation theory of finite groups. Let G be a finite group and k a field of characteristic zero. The group algebra $k[G]$ is a semisimple k-algebra, and so there exists a finite set $\{V_1, \ldots, V_r\}$ of simple $k[G]$-modules such that every $k[G]$-module is isomorphic to a direct sum of copies of the V_i and every simple $k[G]$-module is isomorphic to exactly one of the V_i. In fact, one knows that r is the number of conjugacy classes in G.

For each i, define $\chi_i(g)$ for $g \in G$ to be the trace of g acting on V_i. Then $\chi_i(g)$ depends only on the conjugacy class of g. The character table of G lists the values of $\chi_1, \ldots, \chi_r$ on each of the conjugacy classes. One would like to compute the character table for each simple finite group.

For the commutative simple groups, this is trivial. The representation theory of A_n is closely related to that of S_n, and was worked out by Frobenius and Young between 1900 and 1930. The character tables of the sporadic groups are known—there are, after all, only 26 of them. That leaves those related to algebraic groups (by far the largest class).

The representation theory of $\mathrm{GL}_n(\mathbb{F}_q)$ was worked out by J.A. Green and that of $\mathrm{Sp}_4(\mathbb{F}_q)$ by B. Srinivasan.

Let G be an algebraic group over $\mathbb{F}_q$. Then $G(\mathbb{F}_q)$ is the set of fixed points of the Frobenius map $t \mapsto t^q$ acting on $G(\mathbb{F}_q^{\mathrm{al}})$:

$$G(\mathbb{F}_q) = G(\mathbb{F}_q^{\mathrm{al}})^F.$$

In order to get all the simple finite groups, one needs to consider reductive groups G over $\mathbb{F}_p^{\mathrm{al}}$ and a map $F\colon G(\mathbb{F}_p^{\mathrm{al}}) \to G(\mathbb{F}_p^{\mathrm{al}})$ such that some power F^m of F is the Frobenius map of a model of G over $\mathbb{F}_q$ for some q; the finite group is again the set of fixed points of F on $G(\mathbb{F}_p^{\mathrm{al}})$.

Deligne and Lusztig 1976, and many subsequent papers of Lusztig, very successfully apply étale cohomology to work out the representation theory of these groups. They construct a variety X over $\mathbb{F}$ on which G acts. Then G acts on the étale cohomology groups of X, and the Lefschetz fixed point formula can be applied to compute the traces of these representations.

2. Etale Morphisms

An étale morphism is the analogue in algebraic geometry of a local isomorphism of manifolds in differential geometry, a covering of Riemann surfaces with no branch points in complex analysis, and an unramified extension in algebraic number theory.

For varieties, it is possible to characterize étale morphisms geometrically; for arbitrary schemes, there is only the commutative algebra.

Etale morphisms of nonsingular algebraic varieties. Throughout this subsection, all varieties will be defined over an algebraically closed field k. Let W and V be nonsingular algebraic varieties over k. As in the introduction, a regular map $\varphi\colon W \to V$ is said to be *étale* at $Q \in W$ if the the map $d\varphi\colon \operatorname{Tgt}_Q(W) \to \operatorname{Tgt}_{\varphi(Q)}(V)$ on tangent spaces is an isomorphism, and φ is said to be *étale* if it is étale at every point of W.

Proposition 2.1. *Let $V = \operatorname{Specm}(A)$ be a nonsingular affine variety over k, and let W be the subvariety of $V \times \mathbb{A}^n$ defined by the equations*

$$g_i(Y_1, \dots, Y_n) = 0, \quad g_i \in A[Y_1, \dots, Y_n], \quad i = 1, \dots, n.$$

The projection map $Y \to \mathbb{A}^n$ is étale at a point $(P; b_1, \dots, b_n)$ of Y if and only if the Jacobian matrix $\left(\frac{\partial g_i}{\partial Y_j}\right)$ is nonsingular at $(P; b_1, \dots, b_n)$.

Proof. Set

$$A = k[X_1, \dots, X_m]/(F_1, \dots, F_r)$$

and

$$P = (a_1, \dots, a_m).$$

The tangent space to V at P is the solution space of the system of linear equations

$$(dF_i)_P \stackrel{\text{df}}{=} \sum_{j=1}^{m} \left.\frac{\partial F_i}{\partial X_j}\right|_P (dX_j)_P = 0, \quad i = 1, \dots r,$$

in the variables $(dX_i)_P$ (see AG §4).

By definition, $W = \operatorname{Specm} A[Y_1, \dots, Y_n]/(g_1, \dots, g_n)$. For each i, choose a $G_i \in k[X_1, \dots, X_m, Y_1, \dots, Y_n]$ mapping to $g_i \in A[Y_1, \dots, Y_n]$. Then

$$A[Y_1, \dots, Y_n]/(g_1, \dots, g_n) = k[X_1, \dots, X_m, Y_1, \dots, Y_n]/(F_1, \dots, F_r, G_1, \dots, G_n),$$

and so the tangent space to W at $(P; b_1, \dots, b_n)$ is the solution space of the system of linear equations

$$\begin{aligned} \sum_{j=1}^{m} \left.\frac{\partial F_i}{\partial X_j}\right|_P (dX_j)_{(P;\mathbf{b})} &= 0, \quad i = 1, \dots r, \\ \sum_{j=1}^{m} \left.\frac{\partial G_i}{\partial X_j}\right|_{(P;\mathbf{b})} (dX_j)_P + \sum_{j=1}^{n} \left.\frac{\partial G_i}{\partial Y_j}\right|_{(P;\mathbf{b})} (dY_j)_{(P;\mathbf{b})} &= 0, \quad i = 1, \dots n, \end{aligned}$$

($r+n$ equations in $m+n$ variables). The map $\operatorname{Tgt}_{(P;\mathbf{b})}(W) \to \operatorname{Tgt}_P(V)$ is the obvious projection map. Thus φ is étale at $(P; \mathbf{b})$ if and only if every solution of the first system of equations extends uniquely to a solution of the second system. This will be so if and only if the $n \times n$ matrix $\left(\frac{\partial G_i}{\partial Y_j}\Big|_{(P;\mathbf{b})}\right)$ is nonsingular. □

COROLLARY 2.2. *Let $\varphi\colon U \to V$ be a regular map, where U and V both equal $\mathbb{A}^m$. Then φ is étale at $(a_1, \dots, a_m)$ if and only if the Jacobian matrix $\left(\frac{\partial (X_i \circ \varphi)}{\partial Y_j}\Big|_{(a_1,\dots,a_m)}\right)$ is nonsingular. (Here X_i is the i^{th} coordinate function on V and Y_j is the j^{th} coordinate function on U.)*

PROOF. Each $X_i \circ \varphi$ is a regular function on $U = \mathbb{A}^m$, and hence is a polynomial $G_i(Y_1, \dots, Y_m)$ in the coordinate functions Y_j on U. The graph Γ_φ of φ is the subvariety of $\mathbb{A}^m \times \mathbb{A}^m$ defined by the equations

$$X_i = G_i(Y_1, \dots, Y_m), \quad i = 1, \dots, m.$$

The map φ is the composite of the isomorphism $P \mapsto (P, \varphi(P))\colon V \to \Gamma_\varphi$ (see AG 3.25) and the projection $\Gamma_\varphi \to \mathbb{A}^m$, and so is étale at $P_0 = (a_1, \dots, a_m)$ if and only if the projection map is étale at $(P_0, \varphi(P_0))$. The statement now follows from the theorem. □

REMARK 2.3. Note that the condition

the Jacobian matrix $\left(\frac{\partial (X_i \circ \varphi)}{\partial Y_j}\Big|_{(a_1,\dots,a_m)}\right)$ is nonsingular

is precisely the hypothesis of the Inverse Mapping Theorem in advanced calculus.

EXERCISE 2.4. (a) Prove the proposition using the definition of the tangent space in terms of dual numbers (see AG 4.34).

(b) Prove the corollary with U replaced by an open subset of $\mathbb{A}^n$.

EXAMPLE 2.5. Let $V = \mathrm{Specm}(A)$ be a nonsingular affine variety over k, and let

$$f(T) = a_0 T^m + \cdots + a_m$$

be a polynomial with coefficients in A. Thus, each a_i is a regular function on V, and we can regard $f(T)$ as a continuous family $f(P;T)$ of polynomials parametrized by $P \in V$. Let $W = \mathrm{Specm}(A[T]/(f(T))$ (assuming the ring to be reduced). Then W is the subvariety of $V \times \mathbb{A}^1$ defined by the equation

$$f(P;T) = 0,$$

and the inclusion $A \hookrightarrow A[T]/(f(T))$ corresponds to the projection map $\pi\colon W \to V$, $(P;c) \mapsto P$. For each $P_0 \in V$, $\pi^{-1}(P_0)$ is the set of roots of $f(P_0;T)$ where

$$f(P_0;T) = a_0(P_0)T^m + \cdots + a_m(P_0) \in k[T].$$

We have:

(a) the fibre $\pi^{-1}(P_0)$ is finite if and only if P_0 is not a common zero of the a_i; thus π is quasi-finite (AG p101) if and only if the ideal generated by $a_0, a_1, \dots, a_m$ is A;

(b) the map π is finite if and only if a_0 is a unit in A (cf. AG 6.1).

(c) the map π is étale at $(P_0;c)$ if and only if c is a simple root of $f(P_0;T)$.

To prove (c), note that c is a simple root of $f(P_0;T)$ if and only if c is not a root of $\frac{df(P_0;T)}{dT}$. We can now apply the Proposition 2.1.

EXAMPLE 2.6. Consider the map $x \mapsto x^n \colon \mathbb{A}^1 \to \mathbb{A}^1$. Since $\frac{dX^n}{dX} = nX^{n-1}$, we see from (2.2) that the map is étale at no point of $\mathbb{A}^1$ if the characteristic of k divides n, and that otherwise it is étale at all $x \neq 0$.

Etale morphisms of arbitrary varieties. The tangent space at a singular point of a variety says little about the point. Instead, one must use the tangent cone (AG §4).

Recall that for an affine variety $V = \operatorname{Specm} k[X_1, \ldots, X_n]/\mathfrak{a}$ over an algebraically closed field k, the tangent cone at the origin P is defined by the ideal $\mathfrak{a}_* = \{f_* \mid f \in \mathfrak{a}\}$ where, for $f \in k[X_1, \ldots, X_n]$, f_* is the homogeneous part of f of lowest degree. The geometric tangent cone at P is the zero set of $\mathfrak{a}_*$. However, $k[X_1, \ldots, X_n]/\mathfrak{a}_*$ may have nilpotents, and so (as in (2.7b) below) the geometric tangent cone may not determine this ring. Instead, we define the tangent cone $C_P(V)$ to be the k-algebra $k[X_1, \ldots, X_n]/\mathfrak{a}_*$ (or, equivalently but better, the affine k-scheme $\operatorname{Spec} k[X_1, \ldots, X_n]/\mathfrak{a}_*$).

Let $\varphi \colon W \to V$ be a regular map of varieties over an algebraically closed field k. Then φ is said to be *étale* at $Q \in W$ if it induces an isomorphism $C_{\varphi(Q)}(V) \to C_Q(W)$ of tangent cones (as k-algebras). For nonsingular varieties, this agrees with the definition in the last subsection.

EXAMPLE 2.7. (a) The tangent cone at the origin to the curve

$$V: \quad Y^2 = X^3 + X^2$$

is defined by the equation

$$Y^2 = X^2.$$

The map $t \mapsto (t^2 - 1, t(t^2 - 1)) \colon \mathbb{A}^1 \to V$ is not étale at the origin because the map

$$X \mapsto T^2 - 1, \quad Y \mapsto T(T^2 - 1) \colon k[X, Y]/(Y^2 - X^2) \to k[T]$$

it defines on the tangent cones is not an isomorphism.

(b) The tangent cone at the origin to the curve

$$V: \quad Y^2 = X^3$$

is defined by the equation

$$Y^2 = 0.$$

Thus it is the line $Y = 0$ with multiplicity 2. The map $t \mapsto (t^2, t^3) \colon \mathbb{A}^1 \to V$ is not étale at the origin because the map $k[Y]/(Y^2) \to k[T]$ it defines on the tangent cones is not an isomorphism.

The tangent cone can be defined more canonically. Recall (Atiyah and MacDonald 1969, p111) that to any local ring A, one attaches a graded ring

$$gr(A) \stackrel{\text{df}}{=} \bigoplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1}, \quad \mathfrak{m} = \mathfrak{m}_A.$$

The multiplication on $gr(A)$ is induced by the multiplication

$$a, b \mapsto ab \colon \mathfrak{m}^i \times \mathfrak{m}^j \to \mathfrak{m}^{i+j}.$$

A local homomorphism $A \to B$ defines a homomorphism of graded rings $gr(A) \to gr(B)$. It is easily shown that if $gr(A) \to gr(B)$ is an isomorphism, so also is the map induced on the completions $\hat{A} \to \hat{B}$ (ibid. 10.23). The converse is also true, because $gr(A) = gr(\hat{A})$ (ibid. 10.22).

For any point P on a variety V over an algebraically closed field k, $C_P(V) = gr(\mathcal{O}_P)$ (AG 4.36). Thus we arrive at the following criterion:

> a regular map $\varphi\colon W \to V$ of varieties over an algebraically closed field k is étale at $Q \in W$ if and only if the map $\hat{\mathcal{O}}_{V,\varphi(Q)} \to \hat{\mathcal{O}}_{W,Q}$ induced by φ is an isomorphism.

In particular, if Q maps to P, then a necessary condition for φ to be étale at Q, is that V must have the same type of singularity at P as W has at Q.

Etale morphisms of varieties over arbitrary fields. Let $\varphi\colon W \to V$ be a regular map of varieties over a field k. We say that φ is *étale* at $w \in W$ if, for some algebraic closure k^{al} of k, $\varphi_{k^{\mathrm{al}}}\colon W_{k^{\mathrm{al}}} \to V_{k^{\mathrm{al}}}$ is étale at the points of $W_{k^{\mathrm{al}}}$ mapping to w.

EXERCISE 2.8. Make this condition explicit in terms of the tangent spaces to W and V (nonsingular case) or the tangent cones (general case). Prove (or disprove) that $\varphi\colon W \to V$ is étale at w if and only if the map $gr(\mathcal{O}_{V,v}) \otimes_{\kappa(v)} \kappa(w) \to gr(\mathcal{O}_{W,w})$ is an isomorphism.

Etale morphisms of schemes. For proofs of the statements in this subsection, see EC I.1–I.3.

Flat morphisms. Recall that a homomorphism of rings $A \to B$ is *flat* if the functor $M \mapsto B \otimes_A M$ from A-modules to B-modules is exact. One also says that B is a *flat A-algebra.* To check that $f\colon A \to B$ is flat, it suffices to check that the local homomorphism $A_{f^{-1}(\mathfrak{m})} \to B_{\mathfrak{m}}$ is flat for every maximal ideal $\mathfrak{m}$ in B.

If A is an integral domain, then $x \mapsto ax\colon A \to A$ is injective for all nonzero a. Therefore, so also is $x \mapsto ax\colon B \to B$ for any flat A-algebra B, and it follows that $A \to B$ is injective. For a Dedekind domain A the converse holds: a homomorphism $A \to B$ is flat if (and only if) it is injective.

A morphism $\varphi\colon Y \to X$ of schemes (or varieties) is *flat* if the local homomorphisms $\mathcal{O}_{X,\varphi(y)} \to \mathcal{O}_{Y,y}$ are flat for all $y \in Y$. The remark following the definition of flatness shows that it suffices to check this for the closed points $y \in Y$.

A flat morphism $\varphi\colon Y \to X$ of varieties is the analogue in algebraic geometry of a continuous family of manifolds $Y_x \overset{\text{df}}{=} \varphi^{-1}(x)$ parametrized by the points of X. If φ is flat, then

$$\dim Y_x = \dim Y - \dim X$$

for all (closed) $x \in X$ for which $Y_x \neq \emptyset$; the converse is true if X and Y are nonsingular. Moreover, a finite morphism $\varphi\colon Y \to X$ of varieties is flat if and only if each fibre $\varphi^{-1}(x)$ has the same number of points counting multiplicities. (If φ is the map of affine varieties defined by the homomorphism $A \to B$ of affine k-algebras, then the condition means that $\dim_{A/\mathfrak{m}} B/\mathfrak{m}B$ is independent of the maximal ideal $\mathfrak{m} \subset A$).

Let Z be a closed subscheme of X. Then the inclusion $Z \hookrightarrow X$ will be flat if and only if Z is also open in X (and so is a connected component of X).

Unramified morphisms. A local homomorphism $f\colon A \to B$ of local rings is *unramified* if $B/f(\mathfrak{m}_A)B$ is a finite separable field extension of $A/\mathfrak{m}_A$, or, equivalently, if

(a) $f(\mathfrak{m}_A)B = \mathfrak{m}_B$, and
(b) the field $B/\mathfrak{m}_B$ is finite and separable over $A/\mathfrak{m}_A$.

This agrees with the definition in algebraic number theory where one only considers discrete valuation rings.

A morphism $\varphi\colon Y \to X$ of schemes is *unramified* if it is of finite type and if the maps $\mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ are unramified for all $y \in Y$. It suffices to check the condition for the closed points y of Y.

Let $\varphi\colon Y \to X$ be a morphism of finite type. Then φ is unramified if and only if the sheaf $\Omega^1_{Y/X}$ is zero (see EC I.3.5).

Etale morphisms. A morphism $\varphi\colon Y \to X$ of schemes is *étale* if it is flat and unramified. In particular, this means that φ is of finite type.[8]

A homomorphism of rings $f\colon A \to B$ is étale if $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale. Equivalently, it is étale if

(a) B is a finitely generated A-algebra;
(b) B is a flat A-algebra;
(c) for all maximal ideals $\mathfrak{n}$ of B, $B_\mathfrak{n}/f(\mathfrak{p})B_\mathfrak{n}$ is a finite separable field extension of $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$, where $\mathfrak{p} = f^{-1}(\mathfrak{n})$.

Let $X = \operatorname{Spec} A$ where A is an integral domain. For any proper ideal $\mathfrak{a} \subset A$, the map $Z \hookrightarrow X$ corresponding to the homomorphism $A \to A/\mathfrak{a}$ is unramified, but not flat, and hence not étale. This agrees with intuition of "étale" meaning "local isomorphism": the inclusion of a proper closed submanifold into a connected manifold is not a local isomorphism.

PROPOSITION 2.9. *For a regular map $\varphi\colon Y \to X$ of varieties over an algebraically closed field, the definition of "étale" in this subsection agrees with that in the previous subsection.*

PROOF. Note first that every regular map of varieties is of finite type. Thus, it suffices to show that, for any local homomorphism of local k-algebras $f\colon A \to B$ arising from a regular map of k-varieties, the homomorphism $\hat{f}\colon \hat{A} \to \hat{B}$ is an isomorphism if and only if f is flat and unramified.

Certainly, if $\hat{A} \to \hat{B}$ is an isomorphism, then it is both flat and unramified. Thus for the necessity it suffices to show that if $\hat{A} \to \hat{B}$ flat or unramified, then $A \to B$ has the same property. If $\hat{A} \to \hat{B}$ is flat, then an exact sequence of A-modules $M' \to M \to M''$ gives rise to an exact sequence of $\hat{B}$-modules

$$\hat{B} \otimes_A M' \to \hat{B} \otimes_A M \to \hat{B} \otimes_A M''$$

(because $\hat{A}$ is a flat A-algebra — Atiyah and MacDonald 1969, 10.14), and this implies that

$$B \otimes_A M' \to B \otimes_A M \to B \otimes_A M''$$

[8] Grothendieck requires it to be only locally of finite type, or, for arbitrary (not necessarily Noetherian) schemes that it be locally of finite presentation.

is exact (because $\hat{B}$ is a *faithfully* flat B-algebra, ibid. 10, Ex. 7). If $\hat{A} \to \hat{B}$ is unramified, then the ideals $\mathfrak{m}_B$ and $\mathfrak{m}_A B$ both generate the maximal ideal in $\hat{B}$, which implies that they are equal (because[9] $\mathfrak{a}S \cap R = \mathfrak{a}$ for any ideal $\mathfrak{a}$ in a ring R and faithfully flat homomorphism $R \to S$.

The converse is less elementary. Zariski's Main Theorem (EC I 1.8) allows us to assume that B is the localization of a finite A-algebra C at a maximal ideal, and, in fact, that $C = A[T]/(f(T))$ for some monic polynomial $f(T)$. Now $\hat{B}$ is the completion of $\hat{A}[T]/(f(T))$ at some maximal ideal lying over $\mathfrak{m}_{\hat{A}}$. Because of Hensel's lemma (Atiyah and MacDonald 1969, 10, Ex. 9) a factorization $\bar{f} = g_1^{e_1} \cdots g_n^{e_n}$ in $k[T]$, $k = A/\mathfrak{m}_A$, with the g_i the distinct irreducible factors of $\bar{f}$ lifts to a factorization $f = f_1 \cdots f_n$ in $\hat{A}[T]$. Correspondingly, $\hat{A}[T]/(f(T)) = \prod_i \hat{A}[T]/(f_i(T))$, and so $\hat{B} = \hat{A}[T]/(f_i(T))$ for some i. Thus, $\hat{A} \to \hat{B}$ is finite, and so we can apply the next lemma. □

LEMMA 2.10. *Let $\varphi\colon A \to B$ be a local homomorphism of local rings. If*

(a) *φ is injective,*
(b) *the map on residue fields $A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ is an isomorphism,*
(c) *φ is unramified, and*
(d) *B is a finite A-algebra,*

then φ is an isomorphism.

PROOF. Because of (a), we only have to show that φ is surjective. From (b) and (c) we see that

$$B = \varphi(A) + \mathfrak{m}_B = \varphi(A) + \mathfrak{m}_A B.$$

Condition (d) allows us to apply Nakayama's Lemma to the A-module B, which gives that $B = \varphi(A)$. □

Examples.

The Jacobian criterion. Proposition 2.1 holds with V an affine scheme.

Fields. Let k be a field. A local k-algebra A is unramified if and only if it is a finite separable field extension of k. Let A be an étale k-algebra. Because A is unramified over k, no maximal ideal of A contains a proper prime ideal, and so A has dimension 0. It is therefore an Artin ring (Atiyah and MacDonald 1969, 8.5), and so is a finite product of finite separable field extensions of k (ibid. 8.7). Conversely, a finite product of finite separable field extensions of k is an étale k-algebra.

Dedekind domains. Let A be a Dedekind domain, and let L be a finite separable extension of its field of fractions K. Let B be the integral closure of A in L, and let $\mathfrak{P}$ be a prime ideal of B. Then $\mathfrak{p} \overset{\text{df}}{=} \mathfrak{P} \cap A$ is a prime ideal of A, and $\mathfrak{P}$ is said to be unramified in B/A if

(a) in the factorization of $\mathfrak{p}B$ into a product of prime ideals, $\mathfrak{P}$ occurs with exponent one;
(b) the residue field extension $B/\mathfrak{P} \supset A/\mathfrak{p}$ is separable.

[9] Matsumura, H., Commutative Algebra, Benjamin, 1970, 4.C.

This is equivalent to the map $A_{\mathfrak{p}} \to B_{\mathfrak{P}}$ being unramified. Let b be an element of B that is contained in all the ramified prime ideals of B. Then $B_b \stackrel{\text{df}}{=} B[b^{-1}]$ is an étale A-algebra, and every étale A-algebra is a finite product of algebras of this type.

Standard étale morphisms. Let A be a ring, and let $f(T)$ be a monic polynomial with coefficients in A. Because $f(T)$ is monic, $A[T]/(f(T))$ is a free A-module of finite rank, and hence flat. For any $b \in A[T]/(f(T))$ such that $f'(T)$ is invertible in $(A[T]/(f(T))_b$, the homomorphism $A \to (A[T]/(f(T))_b$ étale. An étale morphism $\varphi\colon V \to U$ is said to be *standard* if it is isomorphic to the Spec of such a homomorphism, i.e., if there exists a ring A, a monic polynomial $f(T) \in A[T]$, and a $b \in A[T]/(f(T))$ satisfying the above condition for which there is a commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\approx} & \operatorname{Spec}(A[T]/(f(T)))_b \\ \downarrow{\scriptstyle\varphi} & & \downarrow \\ U & \xrightarrow{\approx} & \operatorname{Spec} A. \end{array}$$

It is an important fact that locally *every* étale morphism is standard, i.e., for any étale morphism $\varphi\colon Y \to X$ and $y \in Y$, there exist open affine neighbourhoods V of y and U of $\varphi(y)$ such that $\varphi(V) \subset U$ and $\varphi|V\colon V \to U$ is standard (EC I 3.14).

Normal schemes. Let X be a connected normal scheme (or variety); if X is affine, this means that X is the spectrum of an integrally closed integral domain A. Let K be the field of rational functions on X, and let L be a finite separable field extension of K. Let Y be the normalization of X in L: if $X = \operatorname{Spec} A$, then $Y = \operatorname{Spec} B$ where B is the integral closure of A in L. Then $Y \to X$ is finite (Atiyah and MacDonald 1969, 5.17), and hence of finite type. Let U be an open subset of Y excluding any closed points of y where φ is ramified. Then $U \to X$ is flat, and hence étale; moreover, every étale X-scheme is a disjoint union of étale X-schemes of this type (EC I 3.20, 3.21).

Schemes with nilpotents. Let X be a scheme, and let X_0 be the subscheme of X defined by a nilpotent ideal (so X_0 and X have the same underlying topological space); then the map $U \mapsto U_0 \stackrel{\text{df}}{=} U \times_X X_0$ is an equivalence from the category of étale X-schemes to the category of étale X_0-schemes.

Properties of étale morphisms. Roughly speaking, étale morphisms have all the properties suggested by the analogy with local isomorphisms. Here we confine ourselves to listing the most important properties with a brief discussion of their proofs.

PROPOSITION 2.11. (a) *Any open immersion is étale.*
(b) *The composite of two étale morphisms is étale.*
(c) *Any base change of an étale morphism is étale.*
(d) *If $\varphi \circ \psi$ and φ are étale, then so also is ψ.*

Statement (a) says that if U is an open subscheme (or variety) of X, then the inclusion $U \hookrightarrow X$ — this is obvious from any definition.

Statement (b) is also obvious from any definition.

For the notion of fibre product of varieties and the base change of a regular map, see (AG §6). Given a point $(u, y) \in U \times_X Y$, one shows easily (for example, by using dual numbers, ibid. 4.34) that

$$\mathrm{Tgt}_{(u,y)}(U \times_X Y) = \mathrm{Tgt}_u(U) \times_{\mathrm{Tgt}_x(X)} \mathrm{Tgt}_y(Y)$$

where x is the common image of u and y in X. Thus, (c) is obvious for nonsingular varieties.

Finally, (d) is obvious for varieties using the definition in terms of tangent cones, for example. For schemes, see (EC I 3.6).

PROPOSITION 2.12. *Let $\varphi\colon Y \to X$ be an étale morphism.*

(a) *For all $y \in Y$, $\mathcal{O}_{Y,y}$ and $\mathcal{O}_{X,x}$ have the same Krull dimension.*
(b) *The morphism φ is quasi-finite.*
(c) *The morphism φ is open.*
(d) *If X is reduced, then so also is Y.*
(e) *If X is normal, then so also is Y.*
(f) *If X is regular, then so also is Y.*

When X and Y are connected varieties, (a) says that they have the same dimension — this is obvious, because the tangent cone has the same dimension as the variety.

Statement (b) says that the fibres of φ are all finite.

Statement (c) follows from the more general fact that flat morphisms of finite type are open (EC I 2.12).

Statement (e) is implied by our description of the étale morphisms to a normal scheme.

For varieties, (f) implies that if X is nonsingular and $Y \to X$ is étale, then Y is also nonsingular. This is obvious, because a point P is nonsingular if and only if the tangent cone at P is a polynomial ring in $\dim X$ variables.

EXERCISE 2.13. Find a simple direct proof that a quasi-finite regular map $Y \to X$ of nonsingular varieties is open. (Such a map is flat, hence open; the problem is to find a direct proof using only, for example, facts proved in AG.)

PROPOSITION 2.14. *Let $\varphi\colon Y \to X$ be a morphism of finite type. The set of points $U \subset Y$ where φ is étale is open in Y.*

Of course, U may be empty. If X is normal, then U is the complement of the support of the $\mathcal{O}_Y$-sheaf $\Omega^1_{Y/X}$.

PROPOSITION 2.15. *Let $\varphi\colon Y \to X$ be an étale morphism of varieties. If X is connected, then any section to φ is an isomorphism of X onto a connected component of Y.*

PROOF. For simplicity, we assume that the ground field is algebraically closed. Let $s\colon X \to Y$ be a section to φ, i.e., a regular map such that $\varphi \circ s = \mathrm{id}_X$. The graph $\Gamma_s \overset{\mathrm{df}}{=} \{(x, s(x)) \mid x \in X\}$ is closed in $X \times Y$ (see AG 3.25), and its inverse image under the regular map $y \mapsto (\varphi(y), y)\colon Y \to X \times Y$ is $s(X)$. Therefore, $s(X)$ is closed in Y. It is also open, because s is étale (2.12c), and so is a connected component of Y. Thus $\varphi|s(X)$ and s are inverse isomorphisms. $\square$

COROLLARY 2.16. *Let φ, φ' be étale morphisms $Y \to X$ where X and Y are varieties over an algebraically closed field and Y is connected. If φ and φ' agree at a single point of Y, then they are equal on the whole of Y.*

PROOF. The maps $(1, \varphi)$, $(1, \varphi')\colon Y \to Y \times X$ are sections to the projection map $(y, x) \mapsto y\colon Y \times X \to Y$, and the hypothesis says that they agree at some point $y_0 \in Y$. Therefore they are equal. Since φ and φ' are their composites with the projection map $X \times Y \to X$, φ and φ' are also equal. $\square$

REMARK 2.17. Proposition 2.15 holds for schemes provided φ is assumed to be separated. The corollary also holds for schemes provided that one assumes, not only that φ and φ' agree at a point, but also that they induce the same map on the residue field at the point. See (EC I 3.12, I 3.13).

3. The Etale Fundamental Group

The étale fundamental group classifies the finite étale coverings of a variety (or scheme) in the same way that the usual fundamental group classifies the covering spaces of a topological space. This section is only a survey of the topic—see the references at the end of the section for full accounts.

We begin by reviewing the classical theory from a functorial point of view.

The topological fundamental group. Let X be a connected topological space. In order to have a good theory, we assume that X is pathwise connected and semi-locally simply connected (i.e., every $P \in X$ has a neighbourhood U such that every loop in U based at P can be shrunk in X to P).

Fix an $x \in X$. The *fundamental group* $\pi_1(X, x)$ is defined to be the group of homotopy classes of loops in X based at x.

A continuous map $\pi\colon Y \to X$ is a *covering space* of X if every $P \in X$ has an open neighbourhood U such that $\pi^{-1}(U)$ is a disjoint union of open sets U_i each of which is mapped homeomorphically onto U by π. A *map of covering spaces* $(Y, \pi) \to (Y', \pi')$ is a continuous map $\alpha\colon Y \to Y'$ such that $\pi' \circ \alpha = \pi$. Under our hypotheses, there exists a simply connected covering space $\tilde{\pi}\colon \tilde{X} \to X$. Fix an $\tilde{x} \in \tilde{X}$ mapping to $x \in X$. Then $(\tilde{X}, \tilde{x})$ has the following universal property: for any covering space $Y \to X$ and $y \in Y$ mapping to $x \in X$, there is a unique covering space map $\tilde{X} \to Y$ sending $\tilde{x}$ to y. In particular, the pair $(\tilde{X}, \tilde{x})$ is unique up to a unique isomorphism; it is called the *universal covering space* of (X, x).

Let $\mathrm{Aut}_X(\tilde{X})$ denote the group of covering space maps $\tilde{X} \to \tilde{X}$, and let $\alpha \in \mathrm{Aut}_X(\tilde{X})$. Because α is a covering space map, $\alpha\tilde{x}$ also maps to $x \in X$. Therefore, a path from $\tilde{x}$ to $\alpha\tilde{x}$ is mapped by $\tilde{\pi}$ to a loop in X based at x. Because $\tilde{X}$ is simply connected, the homotopy class of the loop doesn't depend on the choice of the path, and so, in this way, we obtain a map $\mathrm{Aut}_X(\tilde{X}) \to \pi_1(X, x)$. This map an isomorphism.

For proofs of the above statements, see Part I of Greenberg, M., *Lectures on Algebraic Topology*, Benjamin, 1966.

Let $Cov(X)$ be the category of covering spaces of X with only finitely many connected components — the morphisms are the covering space maps. We define $F\colon Cov(X) \to Sets$ to be the functor sending a covering space $\pi\colon Y \to X$ to the set $\pi^{-1}(x)$. This functor is representable by $\tilde{X}$, i.e.,

$$F(Y) \cong \mathrm{Hom}_X(\tilde{X}, Y) \quad \text{functorially in } Y.$$

Indeed, we noted above that to give a covering space map $\tilde{X} \to Y$ is the same as to give a point $y \in \pi^{-1}(x)$.

If we let $\mathrm{Aut}_X(\tilde{X})$ act on $\tilde{X}$ on the right, then it acts on $\mathrm{Hom}_X(\tilde{X}, Y)$ on the left:

$$\alpha f \overset{\text{df}}{=} f \circ \alpha, \quad \alpha \in \mathrm{Aut}_X(\tilde{X}), \quad f\colon \tilde{X} \to Y.$$

Thus, we see that F can be regarded as a functor from $Cov(X)$ to the category of $\mathrm{Aut}_X(\tilde{X})$ (or $\pi_1(X, x)$) sets.

That $\pi_1(X,x)$ classifies the covering spaces of X is beautifully summarized by the following statement: the functor F defines an equivalence from $Cov(X)$ to the category of $\pi_1(X,x)$-sets with only finitely many orbits.

The étale fundamental group. Let X be a connected variety (or scheme). We choose a geometric point $\bar{x} \to X$, i.e., a point of X with coordinates in a separably algebraically closed field. When X is a variety over an algebraically closed field k, we can take $\bar{x}$ to be an element of $X(k)$. For a scheme X, choosing $\bar{x}$ amounts to choosing a point $x \in X$ together with a separably algebraically closed field containing the residue field $\kappa(x)$ at x.

We can no longer define the fundamental group to consist of the homotopy classes of loops. For example, if X is a smooth curve over $\mathbb{C}$, then $X(\mathbb{C})$ is a Riemann surface and a loop on $X(\mathbb{C})$ has complex (or algebraic dimension) $\frac{1}{2}$ — not being physicists, we don't allow such objects. Instead, we mimic the characterization of the fundamental group as the group of covering transformations of a universal covering space.

Recall that a finite étale map $\pi\colon Y \to X$ is open (because étale — see 2.12) and closed (because finite — see AG 6.7) and so it is surjective (provided $Y \neq \emptyset$). If X is a variety over an algebraically closed field and $\pi\colon Y \to X$ is finite and étale, then each fibre of π has exactly the same number of points. Moreover, each $P \in X$ has an étale neighbourhood $(U,u) \to (X,x)$ such that $Y \times_X U$ is a disjoint union of open subvarieties (or subschemes) U_i each of which is mapped isomorphically onto U by $\pi \times 1$ (see later). Thus, a finite étale map is the natural analogue of a finite covering space.

We define FEt/X to be the category whose objects are the finite étale maps $\pi\colon Y \to X$ (sometimes referred to as finite étale coverings of X) and whose arrows are the X-morphisms.

Define $F\colon FEt/X \to Sets$ to be the functor sending (Y,π) to the set of $\bar{x}$-valued points of Y lying over x, so $F(Y) = \mathrm{Hom}_X(\bar{x}, Y)$. If X is a variety over an algebraically closed field and $\bar{x} \in X(k)$, then $F(Y) = \pi^{-1}(\bar{x})$.

We would like to define the universal covering space of X to be the object representing F, but unfortunately, there is (usually) no such object. For example, let $\mathbb{A}^1$ be the affine line over an algebraically closed field k of characteristic zero. Then the finite étale coverings of $\mathbb{A}^1 \setminus \{0\}$ are the maps

$$t \mapsto t^n\colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$$

(see Example 2.6). Among these coverings, there is no "biggest" one—in the topological case, with $k = \mathbb{C}$, the universal covering is

$$\mathbb{C} \xrightarrow{\exp} \mathbb{C} \setminus \{0\},$$

which has no algebraic counterpart.

However, the functor F is pro-representable. This means that there is a projective system $\tilde{X} = (X_i)_{i \in I}$ of finite étale coverings of X indexed by a directed set I such that

$$F(Y) = \mathrm{Hom}(\tilde{X}, Y) \stackrel{\mathrm{df}}{=} \varinjlim_{i \in I} \mathrm{Hom}(X_i, Y) \quad \text{functorially in } Y.$$

In the example considered in the last paragraph, $\tilde{X}$ is the family $t \mapsto t^n \colon \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1 \setminus \{0\}$ indexed by the positive integers partially ordered by division.

We call $\tilde{X}$ "the" universal covering space of X. It is possible to choose $\tilde{X}$ so that each X_i is *Galois over* X, i.e., has degree over X equal to the order of $\mathrm{Aut}_X(X_i)$. A map $X_j \to X_i$, $i \le j$, induces a homomorphism $\mathrm{Aut}_X(X_j) \to \mathrm{Aut}_X(X_i)$, and we define

$$\pi_1(X, \bar{x}) = \mathrm{Aut}_X(\tilde{X}) \overset{\mathrm{df}}{=} \varprojlim{}_i \, \mathrm{Aut}_X(X_i),$$

endowed with its natural topology as a projective limit of finite discrete groups. If $X_n \to \mathbb{A}^1 \setminus \{0\}$ denotes the covering in the last paragraph, then

$$\mathrm{Aut}_X(X_n) = \mu_n(k) \text{ (group of } n\text{th roots of 1 in } k)$$

with $\zeta \in \mu_n(k)$ acting by $x \mapsto \zeta x$. Thus

$$\pi_1(\mathbb{A}^1 \setminus 0, \bar{x}) = \varprojlim{}_n \mu_n(k) \approx \hat{\mathbb{Z}}.$$

Here $\hat{\mathbb{Z}} \cong \prod_\ell \mathbb{Z}_\ell$ is the completion of $\mathbb{Z}$ for the topology defined by the subgroups of finite index. The isomorphism is defined by choosing a compatible system of isomorphisms $\mathbb{Z}/n\mathbb{Z} \to \mu_n(k)$ or, equivalently, by choosing primitive nth roots ζ_n of 1 for each n such that $\zeta_{mn}^m = \zeta_n$ for all $m, n > 0$.

The action of $\pi_1(X, x)$ on $\tilde{X}$ (on the right) defines a left action of $\pi_1(X, x)$ on $F(Y)$ for each finite étale covering Y of X. This action is continuous when $F(Y)$ is given the discrete topology—this simply means that it factors through a finite quotient $\mathrm{Aut}_X(X_i)$ for some $i \in I$.

THEOREM 3.1. *The functor $Y \mapsto F(Y)$ defines an equivalence from the category of finite étale coverings of X to the category of finite discrete $\pi_1(X, \bar{x})$-sets.*

Thus $\pi_1(X, \bar{x})$ classifies the finite étale coverings of X in the same way that the topological fundamental group classifies the covering spaces of a topological space.

EXAMPLE 3.2. Let $\mathbb{P}^1$ denote the projective line over an algebraically closed field. The differential $\omega = dz$ on $\mathbb{P}^1$ has no poles except at ∞, where it has a double pole. For any finite étale covering $\pi \colon Y \to X$ with Y connected, $\pi^{-1}(\omega)$ will have double poles at each of the $\deg(\pi)$-points lying over ∞, and no other poles. Thus the divisor of $\pi^{-1}(\omega)$ has degree $-2\deg(\pi)$. Since this equals $2\mathrm{genus}(Y) - 2$, $\deg(\pi)$ must equal 1, and π is an isomorphism. Therefore $\pi_1(\mathbb{P}^1, \bar{x}) = 1$, as expected.

The same argument shows that $\pi_1(\mathbb{A}^1, \bar{x}) = 0$ when the ground field has characteristic zero. More generally, there is no connected curve Y and finite map $Y \to \mathbb{P}^1$ of degree > 1 that is unramified over $\mathbb{A}^1$ and tamely ramified over ∞.

REMARK 3.3. (a) If $\bar{\bar{x}}$ is a second geometric point of X, then there is an isomorphism $\pi_1(X, \bar{x}) \to \pi_1(X, \bar{\bar{x}})$, well-defined up to conjugation.

(b) The fundamental group $\pi_1(X, \bar{x})$ is a covariant functor of $(X, \bar{x})$.

(c) Let $k \subset \Omega$ be algebraically closed fields of characteristic zero. For any variety X over k, the functor $Y \mapsto Y_\Omega \colon FEt/X \to FEt/X_\Omega$ is an equivalence of categories, and so defines an isomorphism $\pi_1(X, \bar{x}) \cong \pi_1(X_\Omega, \bar{x})$ for any geometric point $\bar{x}$ of X_Ω. This statement is false for p-coverings in characteristic p; for example, for each $\alpha \in k[T]$ we have an Artin-Schreier finite étale covering of $\mathbb{A}^1$ defined by the equation

$$Y^p - Y + \alpha,$$

and one gets more of these when k is replaced by a larger field.

Varieties over $\mathbb{C}$. Let X be a variety over $\mathbb{C}$, and let $Y \to X$ be a finite étale covering of X. If X is nonsingular, so also is Y and, when $X(\mathbb{C})$ and $Y(\mathbb{C})$ are endowed with their complex topologies, $Y(\mathbb{C}) \to X(\mathbb{C})$ becomes a finite covering space of $X(\mathbb{C})$.

The key result that allows us to relate the classical and étale fundamental groups is:

Theorem 3.4 (Riemann Existence Theorem). *Let X be a nonsingular variety over $\mathbb{C}$. The functor sending a finite étale covering (Y, π) of X to the finite covering space $(Y(\mathbb{C}), \pi)$ of $X(\mathbb{C})$ is an equivalence of categories.*

The difficult part of the proof is showing that the functor is essentially surjective, i.e., that every finite (topological) covering space of $X(\mathbb{C})$ has a natural algebraic structure. For proofs, see: Grauert and Remmert, Math. Ann. (1958), 245–318; SGA 4, XI.4.3; and SGA 1, XII. Section 21 below contains a sketch of the last proof.

It follows from the theorem that the étale universal covering space $\tilde{X} = (X_i)_{i \in I}$ of X has the property that any finite topological covering space of $X(\mathbb{C})$ is a quotient of some $X_i(\mathbb{C})$. Let $x = \bar{x}$ be any element of $X(\mathbb{C})$. Then

$$\pi_1(X, x) \stackrel{\text{df}}{=} \varprojlim{}_i \operatorname{Aut}_X(X_i) = \varprojlim{}_i \operatorname{Aut}_{X(\mathbb{C})}(X_i(\mathbb{C})) = \pi_1(X(\mathbb{C}), x)\hat{}$$

where $\pi_1(X(\mathbb{C}), x)\hat{}$ is the completion of $\pi_1(X(\mathbb{C}), x)$ for the topology defined by the subgroups of finite index.

Since $\pi_1(\mathbb{C} \setminus \{0\}, x) \approx \mathbb{Z}$, we recover the fact that $\pi_1(\mathbb{A}^1 \setminus \{0\}, \bar{x}) \approx \hat{\mathbb{Z}}$.

Remark 3.5. Let X be a nonsingular variety over an algebraically closed field k of characteristic zero, and let σ and τ be embeddings of k into $\mathbb{C}$. Then, for the étale fundamental groups,

$$\pi_1(X, \bar{x}) \cong \pi_1(\sigma X, \sigma\bar{x}) \cong \pi_1(\tau X, \tau\bar{x})$$

(by 3.3c). Hence

$$\pi_1((\sigma X)(\mathbb{C}), \sigma\bar{x})\hat{} \cong \pi_1((\tau X)(\mathbb{C}), \tau\bar{x})\hat{}.$$

As we noted in the introduction, there are examples where

$$\pi_1((\sigma X)(\mathbb{C}), \sigma\bar{x}) \not\approx \pi_1((\tau X)(\mathbb{C}), \tau\bar{x}).$$

The statements in this section (appropriately modified) hold for any connected scheme X of finite type over $\mathbb{C}$; in particular, $\pi_1(X, \bar{x}) \cong \pi_1(X(\mathbb{C}), x)\hat{}$ for such a scheme.

Examples.

The spectrum of a field. For $X = \operatorname{Spec} k$, k a field, the étale morphisms $Y \to X$ are the spectra of étale k-algebras A, and each is finite. Thus, rather than working with FEt/X, we work with the opposite category Et/k of étale k-algebras.

The choice of a geometric point for X amounts to the choice of a separably algebraically closed field Ω containing k. Define $F\colon Et/k \to Sets$ by

$$F(A) = \operatorname{Hom}_k(A, \Omega).$$

Let $\tilde{k} = (k_i)_{i\in I}$ be the projective system system consisting of all finite Galois extensions of k contained in Ω. Then $\tilde{k}$ ind-represents F, i.e.,

$$F(A) \cong \operatorname{Hom}_k(A, \tilde{k}) \overset{\text{df}}{=} \varinjlim_{i\in I} \operatorname{Hom}_k(A, k_i) \qquad \text{functorially in } A,$$

—this is obvious. Define

$$\operatorname{Aut}_k(\tilde{k}) = \varprojlim_{i\in I} \operatorname{Aut}_{k\text{-alg}}(k_i).$$

Thus

$$\operatorname{Aut}_k(\tilde{k}) = \varprojlim_i \operatorname{Gal}(k_i/k) = \operatorname{Gal}(k^{\text{sep}}/k)$$

where k^{sep} is the separable algebraic closure of k in Ω. Moreover, F defines an equivalence of categories from Et/k to the category of finite discrete $\operatorname{Gal}(k^{\text{sep}}/k)$-sets. This statement summarizes, and is easily deduced from, the usual Galois theory of fields.

Normal varieties (or schemes). For a connected normal variety (or scheme) X, it is most natural to take the geometric point $\bar{x}$ to lie over the generic point x of X. Of course, strictly speaking, we can't do this if X is a variety because varieties don't have generic points, but what it amounts to is choosing a separably algebraically closed field Ω containing the field $k(X)$ of rational functions on X. We let L be the union of all the finite separable field extensions K of $k(X)$ in Ω such that the normalization of X in K is étale over X; then

$$\pi_1(X, \bar{x}) = \operatorname{Gal}(L/k(X))$$

with the Krull topology.

Albanese variety. [To be added.]

Computing the étale fundamental group. For a connected variety over $\mathbb{C}$, we can exploit the relation to the classical fundamental group to compute the étale fundamental group.

For a connected variety over an arbitrary algebraically closed field of characteristic zero, we can exploit the fact that the fundamental group doesn't change when we make a base change from one algebraically closed field to a second such field.

For a connected variety X_0 over an algebraically closed field k of characteristic $p \neq 0$ one can sometimes obtain information on the "non-p" part of $\pi_1(X_0, x)$ by lifting X_0 to characteristic zero. Suppose that there exists a complete discrete valuation ring R whose residue field is k and whose field of fractions K is of characteristic zero, and a scheme X smooth and proper over R whose special fibre is X_0. There is a canonical surjective homomorphism $\pi_1(X_{K^{\text{al}}}, \bar{x}) \to \pi_1(X_0, \bar{x}_0)$ whose kernel is contained in the kernel of every homomorphism of $\pi_1(X_{K^{\text{al}}}, \bar{x})$ into a finite group of order prime to

p; in particular, the two groups have the same finite quotients of order prime to p. Since every smooth projective curve X_0 lifted to characteristic zero, one obtains a description of $\pi_1(X_0, \bar{x}_0)$ (ignoring p-quotients) as the completion of a free group on $2\text{genus}(X_0)$ generators subject to the standard relation.[10]

Let X be a connected algebraic variety over a field k, and assume that $X_{k^{\text{sep}}}$ is also connected. Then there is an exact sequence

$$1 \to \pi_1(X_{k^{\text{sep}}}, \bar{x}) \to \pi_1(X, \bar{x}) \to \text{Gal}(k^{\text{sep}}/k) \to 1.$$

The maps come from the functoriality of the fundamental group.

The most interesting object in mathematics. Arguably, it is

$$\pi_1(X, \bar{x}),$$

where X is the projective line over $\mathbb{Q}$ (not $\mathbb{C}$!) with the three points $0, 1, \infty$ removed. The above exact sequence becomes in this case:

$$1 \to \pi_1(X_{\mathbb{Q}^{\text{al}}}, \bar{x}) \to \pi_1(X, x) \to \text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q}) \to 1.$$

Here $\pi_1(X_{\mathbb{Q}^{\text{al}}}, \bar{x})$ is canonically isomorphic to the completion of the fundamental group of the Riemann sphere with three points removed, which is generated by loops γ_0, γ_1, γ_∞ around the three deleted points with the single relation $\gamma_0\gamma_1\gamma_\infty = 1$. The group $\text{Gal}(\mathbb{Q}^{\text{al}}/\mathbb{Q})$ is already very mysterious, and understanding this extension of it involves mixed Tate motives, the K-groups of number fields, the special values of zeta functions, etc.. See Deligne's article in *Galois Groups over* $\mathbb{Q}$, Springer, 1989, pp 79–297.

Apparently Grothendieck was the first to emphasize the significance of this group—see *Geometric Galois Actions, 1. Around Grothendieck's Esquisse d'un Programme,* (Schneps and Lochak, Eds), Cambridge University Press, 1997. There are also several papers by Y. Ihara, G. Anderson, and others.

References. The original source is Grothendieck's seminar, SGA 1. The most useful account is in:

Murre, J.P., *Lectures on an Introduction to Grothendieck's Theory of the Fundamental Group*, Tata Institute, 1967.

[10]The classification by Grothendieck of the finite étale coverings of a curve in characteristic p of degree prime to p was one of his first major successes, illustrating the importance of his approach to algebraic geometry. As far as I know, there is as yet no purely algebraic proof of the result.

4. The Local Ring for the Etale Topology

Let X be a topological space endowed with a sheaf $\mathcal{O}_X$ of functions. A *germ of a function* at x is represented by a pair (U, f) consisting of an open neighbourhood U of x and a function f on U; two pairs (U, f) and (U', f') represent the same germ if $f|V = f'|V$ for some open neighbourhood V of x contained in $U \cap U'$. In other words, a germ of a function at x is an element of the ring $\mathcal{O}_{X,x} \stackrel{\text{df}}{=} \varinjlim \Gamma(U, \mathcal{O}_X)$ where U runs over the open neighbourhoods of x.

For example, if $X = \mathbb{C}$ endowed with the sheaf of holomorphic functions, then a germ of a function at $c \in \mathbb{C}$ defines a power series $\sum_{n \geq 0} a_n(z-c)^n$ that converges in some open neighbourhood of c, and every such power series arises from a unique germ; thus $\mathcal{O}_{X,c}$ is the ring of such power series. If X is an algebraic variety over an algebraically closed field, then, for any open affine subvariety U containing x, $\mathcal{O}_{X,x} = A_{\mathfrak{m}}$ where $A = \Gamma(U, \mathcal{O}_X)$ and $\mathfrak{m}$ is the maximal ideal in A corresponding to x—see AG 2.7.

More generally, for any ringed space $(X, \mathcal{O}_X)$ (so $\mathcal{O}_X$ is not necessarily a sheaf of functions on X) and $x \in X$, we define $\mathcal{O}_{X,x} = \varinjlim_U \Gamma(U, \mathcal{O}_X)$ where U runs over the open neighbourhoods of x. For example, if X is a scheme, then, for any open affine U containing x, $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ where $A = \Gamma(U, \mathcal{O}_X)$ and $\mathfrak{p}$ is the prime ideal in A corresponding to x.

A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_{X,x}$ is a local ring for all $x \in X$—then $\mathcal{O}_{X,x}$ is called the *local ring* at x (for the given ringed space structure). We shall study the analogous ring for the étale topology.

The case of varieties. Throughout this subsection, X will be a variety over an algebraically closed field k.

Recall that an étale neighbourhood of a point $x \in X$ is an étale map $U \to X$ together with a point $u \in U$ mapping to x. A morphism (or map) of étale neighbourhoods $(V, v) \to (U, u)$ is a regular map $V \to U$ over X sending v to u. It follows from Corollary 2.16 that there is at most one map from a connected étale neighbourhood to a second étale neighbourhood. The connected affine étale neighbourhoods form a directed set[11] with the definition,

$$(U, u) \leq (U', u') \text{ if there exists a map } (U', u') \to (U, u),$$

and we define the *local ring at x for the étale topology* to be

$$\mathcal{O}_{X,\bar{x}} = \varinjlim_{(U,u)} \Gamma(U, \mathcal{O}_U).$$

Since every open Zariski neighbourhood of x is also an étale neighbourhood of x, we get a homomorphism $\mathcal{O}_{X,x} \to \mathcal{O}_{X,\bar{x}}$. Similarly, we get a homomorphism $\mathcal{O}_{U,u} \to \mathcal{O}_{X,\bar{x}}$ for any étale neighbourhood (U, u) of x, and clearly

$$\mathcal{O}_{X,\bar{x}} = \varinjlim_{(U,u)} \mathcal{O}_{U,u}.$$

[11] Actually, they don't form a set — only a class. However, it is possible to replace the class with a set. For example, if V is affine (which we may assume), say, $V = \operatorname{Specm}(A)$, then we need only consider étale maps $\operatorname{Specm} B \to \operatorname{Specm} A$ with B a ring of fractions of a quotient of an A-algebra of the form $A[T_1, \ldots, T_n]$ (some fixed set of symbols $T_1, T_2, \ldots$). Similar remarks apply elsewhere, and will generally be omitted.

The transition maps $\mathcal{O}_{U,u} \to \mathcal{O}_{V,v}$ in the direct system are all flat (hence injective) unramified local homomorphisms of local rings with Krull dimension $\dim X$.

PROPOSITION 4.1. *The ring $\mathcal{O}_{X,\bar{x}}$ is a local Noetherian ring with Krull dimension $\dim X$.*

PROOF. The direct limit of a system of local homomorphisms of local rings is local (with maximal ideal the limit of the maximal ideals); hence $\mathcal{O}_{X,\bar{x}}$ is local. The maps on the completions $\hat{\mathcal{O}}_{U,u} \to \hat{\mathcal{O}}_{V,v}$ are all isomorphisms — see the proof of (2.9) — and it follows that $\hat{\mathcal{O}}_{X,\bar{x}} = \hat{\mathcal{O}}_{X,x}$. An argument of Nagata (Artin 1962, 4.2) now shows that $\mathcal{O}_{X,\bar{x}}$ is Noetherian, and hence has Krull dimension $\dim X$. □

The ring $\mathcal{O}_{X,\bar{x}}$ is Henselian. The most striking property of $\mathcal{O}_{X,\bar{x}}$ is that it satisfies the conclusion of Hensel's lemma — in fact (see the next subsection), it is characterized by being the "smallest" local ring containing $\mathcal{O}_{X,x}$ having this property.

DEFINITION 4.2. A local ring A is said to be *Henselian* if it satisfies the following condition:

let $f(T)$ be a monic polynomial in $A[T]$, and let $\bar{f}(T)$ denote its image in $k[T]$, $k = A/\mathfrak{m}_A$; then any factorization of $\bar{f}$ into a product of two monic relatively prime polynomials lifts to a factorization of f into the product of two monic polynomials, i.e., if $\bar{f} = g_0 h_0$ with g_0 and h_0 monic and relatively prime in $k[T]$, then $f = gh$ with g and h monic, $\bar{g} = g_0$, and $\bar{h} = h_0$.

REMARK 4.3. (a) The g and h in the above factorization are strictly coprime, i.e., $A[T] = (g, h)$. To see this, note that $M \stackrel{\text{df}}{=} A[T]/(g,h)$ is finitely generated as an A-module (because $g(T)$ is monic), and $M = \mathfrak{m}_A M$ (because $(g_0, h_0) = k[T]$). Now Nakayama's Lemma implies that $M = 0$.

(b) The g and h in the above factorization are unique, for let $f = gh = g'h'$ with g, h, g', h' all monic and $\bar{g} = \bar{g}'$, $\bar{h} = \bar{h}'$. The argument in (a) shows that g and h' are strictly coprime, and so there exist $r, s \in A[T]$ such that $gr + h's = 1$. Now

$$g' = g'gr + g'h's = g'gr + ghs,$$

and so g divides g'. As they have the same degree and are monic, they must be equal.

THEOREM 4.4. *Every complete local ring is Henselian.*

PROOF. The standard proof of Hensel's Lemma in number theory works for any complete local ring—see Atiyah and MacDonald 1969, p115. □

THEOREM 4.5. *For any point x in X, $\mathcal{O}_{X,\bar{x}}$ is Henselian.*

This follows from the next two lemmas.

Consider the following condition on a local ring A (again the quotient map $A[T] \to k[T]$, $k = A/\mathfrak{m}_A$, is denoted by $f \mapsto \bar{f}$):

(∗) let $f_1, \ldots, f_n \in A[T_1, \ldots, T_n]$; every common zero $\mathbf{a}_0$ in k^n of the f_i for which $\mathrm{Jac}(f_1, \ldots, f_n)(\mathbf{a}_0)$ is nonzero lifts to a common zero of the f_i in A^n.

LEMMA 4.6. *For any point x in X, $\mathcal{O}_{X,\bar{x}}$ satisfies (∗).*

PROOF. Let $f_1, f_2, \ldots, f_n \in \mathcal{O}_{X,\bar{x}}[T_1, \ldots, T_n]$ and $\mathbf{a}_0 \in k^n$ be as in $(*)$. By definition

$$\mathcal{O}_{X,\bar{x}} = \varinjlim k[U], \quad \mathfrak{m}_{\bar{x}} = \varinjlim \mathfrak{m}_u$$

(direct limit over the étale neighbourhoods (U,u) of x; $k[U] = \Gamma(U, \mathcal{O}_U)$ the ring of regular functions on U; $\mathfrak{m}_{\bar{x}}$ is the maximal ideal of $\mathcal{O}_{X,\bar{x}}$; $\mathfrak{m}_u$ is the ideal of regular functions on U zero at u). Because there are only finitely many f_i's, and each has only finitely many coefficients, there exists a (U,u) such that the $f_i \in B[T_1, \ldots, T_n]$, $B = k[U]$.

Let $C = B[T_1, \ldots, T_n]/(f_1, \ldots, f_n)$, and let $V = \operatorname{Specm}(C)$. The homomorphism $B \to C$ defines a map $V \to U$. From $\mathbf{a}_0$, we obtain a point $v \in V$ lying over u, and the condition $\operatorname{Jac}(f_1, \ldots, f_n)(\mathbf{a}_0) \neq 0$ implies that $V \to U$ is étale at v (see 2.1). Therefore (2.14), there exists a $g \in C$, $g \notin \mathfrak{m}_v$ such that $D(g) \to U$ is[12] étale. Hence there is a homomorphism $C_g \to \mathcal{O}_{X,\bar{x}}$ such that the inverse image of the maximal ideal of $\mathcal{O}_{X,\bar{x}}$ is $\mathfrak{m}_v$. But to give such a homomorphism is to give a common zero of the f_i in $\mathcal{O}_{X,\bar{x}}$ lifting $\mathbf{a}_0$. □

LEMMA 4.7. *Let A be a local ring. If A satisfies $(*)$, then it is Henselian.*

PROOF. Let

$$f(T) = T^n + a_1 T^{n-1} + \cdots + a_n,$$

and suppose $\bar{f}(T) = g_0(T) h_0(T)$ with g_0 and h_0 monic of degrees r and s respectively and relatively prime. We seek a factorization

$$f(T) = (T^r + x_1 T^{r-1} + \cdots + x_r)(T^s + y_1 T^{s-1} + \cdots + y_s)$$

in $A[T]$ lifting the given factorization of $\bar{f}(T)$. To obtain a factorization, we must solve the equations

$$\begin{aligned} x_1 + y_1 &= a_1 \\ x_2 + x_1 y_1 + y_2 &= a_2 \\ x_3 + x_2 y_1 + x_1 y_2 + y_3 &= a_3 \\ &\cdots \\ x_r y_s &= a_n \end{aligned}$$

(n equations in n unknowns). The factorization $\bar{f} = g_0 h_0$ provides a solution of the system modulo $\mathfrak{m}_A$. The Jacobian matrix of the system of equations is

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & \ldots & 0 \\ y_1 & 1 & 0 & \ldots & 0 & x_1 & 1 & 0 & \ldots & 0 \\ y_2 & y_1 & 1 & \ldots & 0 & x_2 & x_1 & 1 & \ldots & 0 \\ & & & \ldots & & & & & \ldots & \\ & & & \ldots & & & & & \ldots & \\ y_s & y_{s-1} & y_{s-2} & \ldots & 1 & & & & \ldots & \\ 0 & y_s & y_{s-1} & \ldots & y_1 & & & & \ldots & \\ 0 & 0 & y_s & \ldots & y_2 & & & & \ldots & \\ & & & \ldots & & & & & \ldots & \\ & & & \ldots & & & & & \ldots & \end{pmatrix}$$

[12] Notations as in AG: $D(g)$ is the open set where g is nonzero.

The determinant of the transpose of this matrix is the resultant of $T^r+y_1T^{r-1}+\cdots+y_r$ and $T^s+x_1T^{s-1}+\cdots+x_s$. Because g_0 and h_0 are monic and relatively prime, $\mathrm{Res}(g_0,h_0)\neq 0$ (AG 5.34). Therefore the Jacobian matrix of the system of equations is nonzero at the solution in k provided by the factorization $\bar{f}=g_0h_0$. Now $(*)$ shows the factorization lifts to A. $\square$

The local ring at a nonsingular point. The local ring at a point on a differential manifold depends only on the dimension of the manifold and whether the manifold is real or complex. The next two results show that, similarly, the local ring for the étale topology at a nonsingular point of a variety over an algebraically closed field depends only on the dimension of the variety and the field.

PROPOSITION 4.8. *If $\varphi: Y \to X$ is étale at y, then the map $\mathcal{O}_{X,\overline{\varphi(y)}} \to \mathcal{O}_{Y,\bar{y}}$ induced by φ is an isomorphism.*

PROOF. After replacing Y by an open neighbourhood of y, we may suppose that it is étale over X (see 2.14). Then any étale neighbourhood of y can be regarded also as an étale neighbourhood of x, and such neighbourhoods are cofinal in the set of all étale neighbourhoods of x. Therefore, the two direct limits are equal. $\square$

PROPOSITION 4.9. *Let P be a nonsingular point on a variety X, and let $d=\dim X$. Then there is a regular map $\varphi\colon U\to\mathbb{A}^d$ étale at P.*

PROOF. Choose any set of regular function $f_1,\ldots,f_d$ defined in a neighbourhood U of P for which $(df_1)_P,\ldots,(df_d)_P$ is a basis of the dual space to

$\mathrm{Tgt}_P(U)$ (equivalently, that generate the maximal ideal in $\mathcal{O}_{X,P}$). Then

$$\varphi\colon U\to\mathbb{A}^d,\quad \varphi(Q)=(f_1(Q),\ldots,f_d(Q)),$$

is étale at P because the map dual to $(d\varphi)_P$ is $(dX_i)_{\text{origin}}\mapsto(df_i)_P$. (For more details, see AG 4.29.) $\square$

Note that the map sends P to the origin in $\mathbb{A}^d$.

PROPOSITION 4.10. *The local ring for the étale topology at the origin in $\mathbb{A}^d$ is*

$$k[[T_1,\ldots,T_d]]\cap k(T_1,\ldots,T_d)^{al},$$

i.e., it consists of the elements of the complete ring $k[[T_1,\ldots,T_d]]$ that are roots of a polynomial (not necessarily monic) in $k[T_1,\ldots,T_d]$.

We discuss the proof below.

Interlude on Henselian rings.

PROPOSITION 4.11. *Let A be a local ring, and let $k=A/\mathfrak{m}_A$. The following statements are equivalent:*

(a) *A is Henselian;*
(b) *let $f\in A[T]$; if $\bar{f}=g_0h_0$ with g_0 monic and g_0 and h_0 relatively prime, then then $f=gh$ with g monic and $\bar{g}=g_0$ and $\bar{h}=h_0$.*
(c) *A satisfies condition $(*)$ above;*
(d) *let B be an étale A-algebra; a decomposition $B/\mathfrak{m}_AB=k\times\bar{B}'$ lifts to a decomposition $B=A\times B'$.*

PROOF. (d) $\Rightarrow$ (c). Let $f_1, \ldots, f_n$ and $\mathbf{a}_0$ be as in $(*)$. Let $B = (A[T_1, \ldots, T_n]/(f_1, \ldots, f_n))_b$, where b has been chosen so that $b(\mathbf{a}_0) \neq 0$ and $\mathrm{Jac}(f_1, \ldots, f_n)$ is a unit in B. Then $A \to B$ is étale. From $\mathbf{a}_0$, we obtain a decomposition $B/\mathfrak{m}_A B = k \times \bar{B}'$, which (d) shows lifts to a decomposition $B = A \times B'$. The image of the n-tuple $(T_1, \ldots, T_n)$ under the projection map $B \to A$ is a lifting of $\mathbf{a}_0$ to A.

(c) $\Rightarrow$ (b). The proof of Lemma 4.7 can be modified to show that $(*)$ implies (b).

(b) $\Rightarrow$ (a). Trivial.

(a) $\Rightarrow$ (d). We may suppose that $A \to B$ is a standard étale homomorphism (see §2), i.e., that $B = (A/(f(T))_b$ with $f(T)$ a monic polynomial such that $f'(T)$ is invertible in B. A decomposition $B/\mathfrak{m}_A B = k \times \bar{B}'$ arises from a simple root a_0 of $\bar{f}(T)$, which (a) implies lifts to a root of $f(T)$ in A. Now the map $B \to A$, $T \mapsto a$, yields a decomposition $B = A \times B'$. □

DEFINITION 4.12. Let A be a local ring. A local homomorphism $A \to A^h$ of local rings with A^h Henselian is called the *Henselization* of A if any other local homomorphism $A \to B$ with B Henselian factors uniquely into $A \to A^h \to B$.

Clearly the Henselization of a local ring is unique (if it exists).

PROPOSITION 4.13. *For a local ring A, $A^h = \varinjlim_{(B,\mathfrak{q})} B$, where the limit is over all pairs $(B, \mathfrak{q})$ consisting of an étale A-algebra B and a prime ideal $\mathfrak{q}$ such that $\mathfrak{q} \cap A = \mathfrak{m}_A$ and the induced map $A/\mathfrak{m}_A \to B/\mathfrak{q}$ is an isomorphism.*

We leave the proof as an exercise.

COROLLARY 4.14. *Let X be a variety over an algebraically closed field k. For any $x \in X$, $\mathcal{O}_{X,\bar{x}}$ is the Henselization of $\mathcal{O}_{X,x}$.*

PROPOSITION 4.15. *Let A be a local ring, and let B be the intersection of all local Henselian rings H with*

$$A \subset H \subset \hat{A}, \quad \mathfrak{m}_A \subset \mathfrak{m}_H \subset \hat{\mathfrak{m}}_A.$$

Then B is Henselian, and $A \to B$ is the Henselization of A.

PROOF. Let $f(T)$ be a monic polynomial in $B[T]$ such that $\bar{f} = g_0 h_0$ with g_0 and h_0 monic and relatively prime. For each H, the factorization lifts uniquely to $f = gh$ with $g, h \in H[T]$. Because of the uniqueness, g and h have coefficients in $\cap H = B$. Therefore, B is Henselian.

Let $A \to A^h$ be the Henselization of A. Because B is Henselian, there is a unique local A-homomorphism $A^h \to B$. The image of the homomorphism is again Henselian, and hence equals B (because B, by definition, contains no proper Henselian local subring). □

M. Artin has many theorems (and conjectures) stating that things that are known to happen over the completion of a local ring happen already over the Henselization. We state one of these.

THEOREM 4.16. *Let k be a field, and let $f_1, \ldots, f_m \in k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$. Let $a_1, \ldots, a_r \in k[[X_1, \ldots, X_n]]$ be formal power series such that*

$$f_i(X_1, \ldots, X_n, a_1, \ldots, a_r) = 0, \quad i = 1, \ldots, m.$$

Let $N \in \mathbb{N}$. Then there exist elements $b_1, \ldots, b_r$ in the Henselization of $k[X_1, \ldots, X_n]_{(X_1,\ldots,X_n)}$ such that

$$\begin{aligned} b_j &\equiv a_j \mod (X_1, \ldots, X_n)^N, \quad j = 1, \ldots r; \\ f_i(X_1, \ldots, X_n, b_1, \ldots, b_r) &= 0, \quad i = 1, \ldots, m. \end{aligned}$$

PROOF. See Artin 1973 , p48. □

COROLLARY 4.17. *The Henselization of $A \overset{df}{=} k[T_1, \ldots, T_d]_{(T_1,\ldots,T_d)}$ is*

$$B \overset{df}{=} k[[T_1, \ldots, T_d]] \cap k(T_1, \ldots, T_d)^{al}.$$

PROOF. Certainly, by construction, every element of A^h is algebraic over A, and so $A^h \subset B$. Conversely, let $\alpha \in k[[T_1, \ldots, T_d]]$ be a root of a polynomial $f \in k[T_1, \ldots, T_d, X]$. For every $N \in \mathbb{N}$, there exists an $a \in A^h$ such that $f(a) = 0$ and $a \equiv \alpha \mod \mathfrak{m}_A^N$. But $f \in A[X]$ has only finitely many roots. Thus if $a \equiv \alpha \mod \mathfrak{m}_A^N$ for a large enough N, then $a = \alpha$. □

A local ring A is said to be *strictly Henselian* if it is Henselian and its residue field is separably algebraically closed or, equivalently, if every monic polynomial $f(T) \in A[T]$ such that $\bar{f}(T)$ is separable splits into factors of degree 1 in $A[T]$.

DEFINITION 4.18. Let A be a local ring. A local homomorphism $A \to A^{sh}$ from A into a strictly Henselian ring A^{sh} is a *strict Henselization* of A if any other local homomorphism from A into a strictly Henselian ring H extends to A^{sh}, and, moreover, the extension is uniquely determined once the map $A^{sh}/\mathfrak{m}^{sh} \to H/\mathfrak{m}_H$ on residue fields has been specified.

For example, for a field k (regarded as a local ring), the Henselization is k itself, and any separable algebraic closure of k is a strict Henselization of k.

Schemes. A *geometric point* of a scheme X is a morphism $\bar{x}\colon \operatorname{Spec}\Omega \to X$ with Ω a separably closed field. An *étale neighbourhood* of such a point $\bar{x}$ is an étale map $U \to X$ together with a geometric point $\bar{u}\colon \operatorname{Spec}\Omega \to U$ lying over $\bar{x}$. The *local ring at $\bar{x}$ for the étale topology* is

$$\mathcal{O}_{X,\bar{x}} \overset{\text{df}}{=} \varinjlim_{(U,\bar{u})} \Gamma(U, \mathcal{O}_U)$$

where the limit is over the connected affine étale neighbourhoods $(U, \bar{u})$ of $\bar{x}$.

When X is a variety and $x = \bar{x}$ is a closed point of X, this agrees with the previous definition.

Most of the results for varieties over algebraically closed fields extend *mutatis mutandis* to schemes. For example, $\mathcal{O}_{X,\bar{x}}$ is a strict Henselization of $\mathcal{O}_{X,x}$.

In the remainder of these notes, the local ring for the étale topology at a geometric point $\bar{x}$ of a scheme X (or variety) will be called the *strictly local ring* at $\bar{x}$, $\mathcal{O}_{X,\bar{x}}$.

The strictly local ring at a nonclosed point of an algebraic scheme is used in Section 15: Cohomological Dimension.

5. Sites

In order to develop a sheaf theory, and a cohomology theory for sheaves, as for example in Grothendieck's 1957 Tohôku paper, it is not necessary to have a topological space in the conventional sense. In fact, it suffices to have a category $\mathbf{C}$ together with, for each object U of $\mathbf{C}$, a distinguished set of families of maps $(U_i \to U)_{i \in I}$, called the *coverings* of U, satisfying the following axioms:

(a) for any covering $(U_i \to U)_{i\in I}$ and any morphism $V \to U$ in $\mathbf{C}$, the fibre products[13] $U_i \times_U V$ exist, and $(U_i \times_U V \to V)_{i\in I}$ is a covering of V;
(b) if $(U_i \to U)_{i\in I}$ is a covering of U, and if for each $i \in I$, $(V_{ij} \to U_i)_{j\in J_i}$ is a covering of U_i, then the family $(V_{ij} \to U)_{i,j}$ is a covering of U;
(c) for any U in $\mathbf{C}$, the family $(U \xrightarrow{\mathrm{id}} U)$ consisting of a single map is a covering of U.

In (b), the map $V_{ij} \to U$ is the composite of $V_{ij} \to U_i \to U$.

The system of coverings is then called a *(Grothendieck) topology*, and $\mathbf{C}$ together with the topology is called a *site.* If $\mathbf{T}$ is a site, then $Cat(\mathbf{T})$ denotes the underlying category.

For example, let X be a topological space, and let $\mathbf{C}$ be the category whose objects are the open subsets of X and whose morphisms are the inclusion maps. Then the families $(U_i \to U)_{i\in I}$ such that $(U_i)_{i\in I}$ is an open covering of U is a Grothendieck topology on $\mathbf{C}$. For open subsets U and U' of V, $U \times_V U' = U \cap U'$.

A *presheaf of sets* on a site $\mathbf{T}$ is a contravariant functor $\mathcal{F}\colon Cat(\mathbf{T}) \to Sets$. Thus, to each object U in $Cat(\mathbf{T})$, $\mathcal{F}$ attaches a set $\mathcal{F}(U)$, and to each morphism $\varphi\colon U \to V$ in $Cat(\mathbf{T})$, a map $\mathcal{F}(\varphi)\colon \mathcal{F}(V) \to \mathcal{F}(U)$ in such a way that $\mathcal{F}(\psi \circ \varphi) = \mathcal{F}(\varphi) \circ \mathcal{F}(\psi)$ and $\mathcal{F}(\mathrm{id}_U) = \mathrm{id}_{\mathcal{F}(U)}$. Note that the notion of a presheaf on $\mathbf{T}$ does not depend on the coverings. We sometimes denote $\mathcal{F}(\varphi)\colon \mathcal{F}(V) \to \mathcal{F}(U)$ by $a \mapsto a|U$, although this is can be confusing because there may be more than one morphism $V \to U$.

Similarly, a presheaf of (abelian) groups or rings on $\mathbf{T}$ is a contravariant functor from $Cat(\mathbf{T})$ to the category of (abelian) groups or rings.

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[13] For X an object of a category $\mathbf{C}$, $\mathbf{C}/X$ denotes the category whose objects are the morphisms $U \to X$ in $\mathbf{C}$ and whose arrows are the commutative diagrams

$$\begin{array}{ccc} U & \to & U' \\ & \searrow \quad \swarrow & \\ & X. & \end{array}$$

A morphism $U \to U'$ making this diagram commute is also referred to as an X-*morphism.* The *fibre product* $U_1 \times_X U_2$ of morphisms $\varphi_1\colon U_1 \to X$, $\varphi_2\colon U_2 \to X$ in $\mathbf{C}$ is their product in the category $\mathbf{C}/X$. Thus, there is a commutative diagram

$$\begin{array}{ccc} U_1 \times_X U_2 & \xrightarrow{\psi_2} & U_2 \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \varphi_2} \\ U_1 & \xrightarrow[\varphi_1]{} & X \end{array}$$

having the obvious universal property.

A *sheaf* on $\mathbf{T}$ is a presheaf $\mathcal{F}$ that satisfies the sheaf condition:

$$\text{(S)}: \qquad \mathcal{F}(U) \to \prod_{i\in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j)\in I\times I} \mathcal{F}(U_i \times_U U_j)$$

is exact for every covering $(U_i \to U)$. Thus $\mathcal{F}$ is a sheaf if the map

$$f \mapsto (f|U_i)\colon \mathcal{F}(U) \to \prod \mathcal{F}(U_i)$$

identifies $\mathcal{F}(U)$ with the subset of the product consisting of families (f_i) such that

$$f_i|U_i \times_U U_j = f_j|U_i \times_U U_j$$

for all $i, j \in I \times I$.

When $\mathbf{T}$ is the site arising from a topological space, these definitions coincide with the usual definitions.

A *morphism of presheaves* is simply a morphism of functors (alias, natural transformation) and a *morphism of sheaves* is a morphism of presheaves.

We now list some sites. In the remainder of this section, X will be a variety or a scheme. A family of regular maps $(\varphi_i\colon U_i \to U)$ of varieties will be said to be *surjective* if $U = \cup\varphi_i(U_i)$, and similarly for a family of morphisms of schemes.

The Zariski site on X. The site X_{zar} is that associated (as above) with X regarded as a topological space for the Zariski topology.

The étale site on X. The site X_{et} has as underlying category Et/X, whose objects are the étale morphisms $U \to X$ and whose arrows are the X-morphisms $\varphi\colon U \to V$. The coverings are the surjective families of étale morphisms $(U_i \to U)$ in Et/X.

Variants of the étale site on X. Apparently, in Grothendieck's original definition, the coverings were taken to be surjective families of maps $(U_i \to U)_{i\in I}$ each of which is a finite étale map onto a Zariski open subset of U. In other words, the definition was chosen so that being "trivial" for this topology coincides with being "isotrivial" in Serre's terminology (see the Introduction). This is the *finite-étale* topology, and the corresponding site is denoted X_{fet}.

Another variant, which was suggested by Nisnevich, and which has proved useful in K-theory and in the study of the cohomology of group schemes takes as its coverings the surjective families of étale maps $(U_i \to U)_{i\in I}$ with the following property: for each $u \in U$ there exists an $i \in I$ and $u_i \in U$ such that map $\kappa(u) \to \kappa(u_i)$ on the residue fields is an isomorphism. The condition is imposed for all $u \in U$, including the nonclosed points, and so even for a variety over an algebraically closed field, this differs from the étale topology. It is called the *completely decomposed topology*, and the corresponding site is denoted X_{cd}. The local ring at a point x of X for this topology is the Henselization (rather than the strict Henselization) of $\mathcal{O}_{X,x}$.

In the next two examples, we take X to be a scheme.

The flat site on X. The site X_{Fl} has as underlying category Sch/X, the category of all X-schemes. The coverings are the surjective families of X-morphisms $(U_i \xrightarrow{\varphi_i} U)$ with each φ_i flat and of finite-type.

The big étale site on X. The site X_{Et} has as underlying category Sch/X. The coverings are the surjective families of étale X-morphisms $(U_i \to U)_{i\in I}$.

The site $\mathbf{T}_G$. To each profinite group G, there corresponds a site $\mathbf{T}_G$ whose underlying category is that of all finite discrete G-sets. A covering is any surjective family of G -maps.

REMARK 5.1. (a) When $U \to X$ is a member of a covering of X, we can think of U as being an "open subset" of X for the "topology". Note however that for the flat site, $U \to X$ and $U' \to X$ may both be "open subsets" of X, but an X-morphism $U \to U'$ may not realize U as an "open subset" of U, i.e., $U \to X$ and $U' \to X$ can be flat without $U \to U'$ being flat.

(b) If all the schemes in the underlying category are quasicompact (e.g., varieties) then one can take the coverings to be *finite* surjective families.

DEFINITION 5.2. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ be two sites. A functor $Cat(\mathbf{T}_2) \to Cat(\mathbf{T}_1)$ transforming coverings into coverings is called a continuous map $\mathbf{T}_1 \to \mathbf{T}_2$.

For example, a map of topological spaces $Y \to X$ defines a continuous map of the corresponding sites if and only if it is continuous in the usual sense.

There are obvious continuous maps of sites

$$X_{Fl} \to X_{Et} \to X_{et} \to X_{cd} \to X_{zar}.$$

If $X = \operatorname{Spec} k$ with k a field, and k^{sep} is an algebraic closure of k, then there are essentially-inverse continuous maps

$$\mathbf{T}_G \to X_{et}, \quad X_{et} \to \mathbf{T}_G$$

where $G = \mathrm{Gal}(k^{\mathrm{al}}/k)$ — see §3.

REMARK 5.3. Note that a continuous map $\mathbf{T}_1 \to \mathbf{T}_2$ is actually a functor in the opposite direction. This seemed so illogical to the participants in SGA 4 that they defined a continuous map $\mathbf{T}_1 \to \mathbf{T}_2$ to be a functor $Cat(\mathbf{T}_1) \to Cat(\mathbf{T}_2)$ preserving coverings. However, this conflicts with our intuition from topological spaces and was abandoned in the published version.

Dictionary. What we have called a topology is called a pre-topology in SGA 4. There they define a *sieve* (crible in French) to be a full[14] subcategory containing with any object V, all objects of the category admitting a morphism to V. To give a *topology* on a category $\mathbf{C}$ is to give a set of sieves in $\mathbf{C}/U$ for each object U of $\mathbf{C}$ satisfying certain natural axioms (ib. II.1).

Let $\mathbf{T}$ be a site according to the definition at the start of this section. To any covering $\mathcal{U} = (U_i \to U)_{i\in I}$ one attaches the sieve $s(\mathcal{U})$ whose objects are those in $Cat(\mathbf{T})$ admitting a morphism to one of the U_i. The sieves arising from coverings define a topology in the sense of SGA 4.

A *topos* is any category equivalent to the category of sheaves on a site. In SGA 4, p299, it is argued that toposes are the natural objects of study rather than sites. However, I shall not use the word again in these notes.

[14]This means that the Hom sets in the subcategory equal those in the ambient category.

6. Sheaves for the Etale Topology

Let X be a variety (or scheme). According to the general definition, a sheaf $\mathcal{F}$ on X_{et} is a contravariant functor $Et/X \to Sets$ (or Ab, or ...) such that

$$\text{(S):} \qquad \mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I \times I} \mathcal{F}(U_i \times_U U_j)$$

is exact for every $U \to X$ étale and every étale covering $(U_i \to U)$.

Note that a sheaf $\mathcal{F}$ on X_{et} defines by restriction a sheaf on U_{zar} for every $U \to X$ étale. In particular, if $U = \coprod U_i$, then $\mathcal{F} \xrightarrow{\approx} \Pi\mathcal{F}(U_i)$. On applying the sheaf condition in the case that I is the empty set, one finds that $\mathcal{F}(\emptyset)$ is a set with one element (or the zero group if $\mathcal{F}$ is a sheaf of groups). Here $\emptyset$ is the empty variety (or scheme).

For a Zariski covering, the sheaf condition (S) is the familiar one. We next examine what it means for Galois coverings.

Galois coverings. Let $\varphi\colon Y \to X$ be a morphism, and let G be a finite group. A right action of G on Y over X is a map $\alpha\colon G \to \mathrm{Aut}_X(Y)$ such that $\alpha(gh) = \alpha(h) \circ \alpha(g)$. If X and Y are affine, so that φ corresponds to a map of rings $A \to B$, then to give an right action of G on Y over X is the same as to give a left action of G on the A-algebra B.

Definition 6.1. Let $Y \to X$ be a faithfully flat map, and let G be a finite group acting on Y over X on the right. Then $Y \to X$ is called a *Galois covering of X with group G* if the morphism $Y \times G \to Y \times_X Y$, $(y, g) \mapsto (y, yg)$ is an isomorphism.

Here $Y \times G$ denotes a disjoint union of copies of Y indexed by the elements of G:

$$Y \times G = \amalg_g Y_g, \quad Y_g = Y.$$

The maps $\mathrm{id}\colon Y_g \to Y$ and $g\colon Y_g \to Y$ define a map $Y_g \to Y \times_X Y$ and the condition in the definition requires that these maps induce an isomorphism $\amalg Y_g \to Y \times_X Y$.

If $\varphi\colon Y \to X$ is a Galois covering, then φ is surjective, finite, and étale of degree equal to the order of G (because it becomes so after a flat base change). Conversely, it is possible to prove that, if $Y \to X$ is surjective, finite, and étale of degree equal to the order of $\mathrm{Aut}_X(Y)$, then it is Galois with group $\mathrm{Aut}_X(Y)$.

Remark 6.2. A morphism $\varphi\colon Y \to X$ of irreducible varieties is said to be *generically Galois* if $k(Y)$ is Galois over $k(X)$. Some miscreants drop the "generically" and call such a morphism Galois.

An A-algebra B is said to be *Galois* if there is a group G acting on B (by A-algebra automorphisms) in such a way that $\operatorname{Spec} B \to \operatorname{Spec} A$ is Galois with group G. Explicitly, this means that $A \to B$ is faithfully flat and that the homomorphism

$$B \otimes_A B \to \prod_{g \in G} B, \quad b \otimes b' \mapsto (\cdots, b \cdot g(b'), \cdots),$$

is an isomorphism. Here $\prod_{g \in G} B$ denotes a product of copies of B indexed by the elements of G.

EXAMPLE 6.3. Let $K = k[T]/(f(T))$ where k is a field and $f(T)$ is a monic irreducible polynomial in $k[T]$. If

$$f(T) = f_1(T)^{e_1} \cdots f_r(T)^{e_r}$$

is the factorization of $f(T)$ into powers of distinct irreducible polynomials in $K[T]$, then

$$K \otimes_k K \cong K[T]/(f(T)) \cong \prod K[T]/(f_i(T)^{e_i})$$

by the Chinese Remainder Theorem. It follows that K is a Galois k-algebra if and only if $f(T)$ splits into distinct linear factors in $K[T]$, i.e., $f(T)$ is separable and K is its splitting field.

Let $Y \to X$ be Galois with group G. Then G acts on Y on the right, and hence, for any presheaf $\mathcal{P}$, it acts on $\mathcal{P}(Y)$ on the left because $\mathcal{P}$ is a contravariant functor.

PROPOSITION 6.4. *Let $Y \to X$ be Galois with group G, and let $\mathcal{F}$ be a presheaf on X_{et} that takes disjoint unions to products. Then $\mathcal{F}$ satisfies the sheaf condition (S) for the covering $Y \to X$ if and only if the restriction map $\mathcal{F}(X) \to \mathcal{F}(Y)$ identifies $\mathcal{F}(X)$ with the subset $\mathcal{F}(Y)^G$ of elements of $\mathcal{F}(Y)$ fixed by G.*

PROOF. There is a commutative diagram

$$\begin{array}{ccccc} X & \leftarrow & Y & \leftleftarrows & Y \times_X Y \\ \| & & \| & & \uparrow\approx \\ X & \leftarrow & Y & \leftleftarrows & Y \times G. \end{array}$$

in which the projection maps $(y, y') \mapsto y$ and $(y, y') \mapsto y'$ on the top row correspond respectively to the maps $(y, g) \mapsto y$ and $(y, g) \mapsto yg$ on the bottom row. On applying $\mathcal{F}$ to the diagram, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{F}(X) & \to & \mathcal{F}(Y) & \rightrightarrows & \mathcal{F}(Y \times_X Y) \\ \| & & \| & & \downarrow\approx \\ \mathcal{F}(X) & \to & \mathcal{F}(Y) & \rightrightarrows & \Pi_{g\in G}\mathcal{F}(Y) \end{array}$$

Here $\prod_{g\in G} \mathcal{F}(Y)$ is a product of copies of $\mathcal{F}(Y)$ indexed by the elements of G, and the maps $\mathcal{F}(Y) \to \prod_{g\in G} \mathcal{F}(Y)$ are

$$s \mapsto (s, \dots, s, \dots, s), \quad s \mapsto (1s, \dots, gs, \dots)$$

respectively. These maps agree on $s \in \mathcal{F}(Y)$ if and only if $gs = s$ for all $g \in G$. $\square$

EXERCISE 6.5. (a) Let $\mathcal{F}$ be a sheaf of abelian groups on X_{et}, and let $Y \to X$ be a Galois covering with group G. Show that the complex

$$\mathcal{F}(X) \to \mathcal{F}(Y) \to \mathcal{F}(Y \times_X Y) \to \mathcal{F}(Y \times_X Y \times_X Y) \to \cdots$$

is isomorphic to the complex of inhomogeneous cochains for G acting on $\mathcal{F}(Y)$ (see CFT p43). Each map in the complex is the alternating sum of the maps given by the various projection maps.

(b) Show that 6.3 remains true with k replaced by a local Henselian ring.

A criterion to be a sheaf. The next result makes it easier to check that a presheaf is a sheaf.

PROPOSITION 6.6. *In order to verify that a presheaf $\mathcal{F}$ on X_{et} is a sheaf, it suffices to check that $\mathcal{F}$ satisfies the sheaf condition (S) for Zariski open coverings and for étale coverings $V \to U$ (consisting of a single map) with V and U both affine.*

PROOF. If $\mathcal{F}$ satisfies the sheaf condition for Zariski open coverings, then $\mathcal{F}(\coprod U_i) = \prod \mathcal{F}(U_i)$. From this it follows that the sequence (S) for a covering $(U_i \to U)_{i\in I}$ in which the indexing set I is finite is isomorphic to the sequence (S) arising from a single morphism $(\coprod U_i \to U)$ because

$$(\coprod U_i) \times_U (\coprod U_i) = \coprod_{(i,j)\in I\times I} (U_i \times_U U_j).$$

Since a finite disjoint union of affine varieties (or schemes) is again affine, the second condition in the statement of the proposition implies that (S) is exact for coverings $(U_i \to U)_{i\in I}$ in which the indexing set is finite and U and the U_i are affine.

For the rest of the proof, which involves only easy diagram chasing, see EC II 1.5. □

REMARK 6.7. It is possible to show that any finite étale map $V \to U$ is dominated by a Galois map $V' \to U$, i.e., there exists a Galois covering $V' \to U$ factoring through $V \to U$ and such that $V' \to V$ is surjective. It follows from this and the proposition that to verify that a presheaf $\mathcal{F}$ on X_{fet} (finite-étale topology — see §5) is a sheaf, it suffices to check that $\mathcal{F}$ satisfies the sheaf condition (S) for Zariski open coverings and for Galois coverings $V \to U$. These statements are not true for the étale topology.

Examples of sheaves on X_{et}. Let $A \to B$ be the homomorphism of rings corresponding to a surjective étale morphism $V \to U$ of affine varieties (or schemes). In checking the second condition in (6.6), we shall usually make use only of the fact that $A \to B$ is faithfully flat [15] (i.e., we shall not need to use that it is unramified).

The structure sheaf on X_{et}. For any $U \to X$ étale, define $\mathcal{O}_{X_{et}}(U) = \Gamma(U, \mathcal{O}_U)$. Certainly, its restriction to U_{zar} is a sheaf for any U étale over X. That it is a sheaf on X_{et} follows from Proposition 6.6 and the next proposition.

PROPOSITION 6.8. *For any faithfully flat homomorphism $A \to B$, the sequence*

$$0 \to A \to B \xrightarrow{b\mapsto 1\otimes b - b\otimes 1} B \otimes_A B$$

is exact.

[15] Recall that a flat homomorphism $A \to B$ is faithfully flat if it satisfies one of following equivalent conditions:

(a) if an A-module M is nonzero, then $B \otimes_A M$ is nonzero;
(b) if a sequence of A-modules $M' \to M \to M''$ is not exact, then neither is $B \otimes_A M' \to B \otimes_A M \to B \otimes_A M''$;
(c) the map $\operatorname{Spec} B \to \operatorname{Spec} A$ is surjective (for affine k-algebras, this is equivalent to $\operatorname{Specm} B \to \operatorname{Specm} A$ being surjective).

PROOF. *Step 1:* The statement is true if $f\colon A \to B$ admits a section, i.e., a homomorphism $s\colon B \to A$ such that $s \circ f = \mathrm{id}$.

To prove this, let $k\colon B \otimes_A B \to B$ send $b \otimes b' \mapsto b \cdot fs(b')$. Then

$$k(1 \otimes b - b \otimes 1) = fs(b) - b.$$

Thus, if $1 \otimes b - b \otimes 1 = 0$, then $b = fs(b) \in f(A)$.

Step 2: If the statement is true for $a' \mapsto a' \otimes 1\colon A' \to A' \otimes_A B$, where $A \to A'$ is some faithfully flat homomorphism, then it is true for $A \to B$.

The sequence for $A' \to A' \otimes B$ is obtained from that for $A \to B$ by tensoring with A'.

Step 3: The homomorphism $b \mapsto b \otimes 1\colon B \to B \otimes_A B$ has a section, namely, the map $b \otimes b' \mapsto bb'$.

Since, by assumption, $A \to B$ is faithfully flat, this completes the proof. □

REMARK 6.9. Let $K \to L$ be a map of fields that is Galois with group G in the above sense. Then $L^G = K$ (by Propositions 6.4 and 6.8), and so L is Galois over K in the usual sense (FT 3.23).

The sheaf defined by a scheme Z. An X-scheme Z defines a contravariant functor:

$$\mathcal{F}\colon Et/X \to Sets, \quad \mathcal{F}(U) = \mathrm{Hom}_X(U, Z).$$

I claim that this is a sheaf of sets. It is easy to see that $\mathcal{F}$ satisfies the sheaf criterion for open Zariski coverings. Thus it suffices to show that

$$Z(A) \to Z(B) \rightrightarrows Z(B \otimes_A B)$$

is exact for any faithfully flat map $A \to B$. If Z is affine, defined say by the ring C, then the sequence becomes

$$\mathrm{Hom}_{A\text{-alg}}(C, A) \to \mathrm{Hom}_{A\text{-alg}}(C, B) \rightrightarrows \mathrm{Hom}_{A-\mathrm{alg}}(C, B \otimes_A B)$$

The exactness of this follows immediately from Proposition 6.8. We leave the case of a nonaffine Z to the reader.

If Z is has a group structure, then $\mathcal{F}_Z$ is a sheaf of groups.

EXAMPLE 6.10. (a) Let μ_n be the variety (or scheme) defined by the single equation

$$T^n - 1 = 0.$$

Then $\mu_n(U)$ is the group of nth roots of 1 in $\Gamma(U, \mathcal{O}_U)$.

(b) Let $\mathbb{G}_a$ be the affine line regarded as group under addition. Then $\mathbb{G}_a(U) = \Gamma(U, \mathcal{O}_U)$ regarded as an abelian group.

(c) Let $\mathbb{G}_m$ be the affine line with the origin omitted, regarded as a group under multiplication. Then $\mathbb{G}_m(U) = \Gamma(U, \mathcal{O}_U)^\times$.

(d) Let GL_n be the variety (or scheme) defined by the single equation

$$T \cdot \det(T_{ij}) = 1$$

in the variables n^2+1 variables $T, T_{11}, \ldots, T_{nn}$. Then $\mathrm{GL}_n(U) = \mathrm{GL}_n(\Gamma(U, \mathcal{O}_U))$, the group of invertible $n \times n$-matrices with entries from the ring $\Gamma(U, \mathcal{O}_U)$. For example, $\mathrm{GL}_1 = \mathbb{G}_m$.

Constant sheaves. Let X be a variety or a quasi-compact scheme. For any set Λ, define

$$\mathcal{F}_\Lambda(U) = \Lambda^{\pi_0(U)}$$

— product of copies of Λ indexed by the set $\pi_0(U)$ of connected components of U. With the obvious restriction maps, this is a sheaf, called the *constant sheaf* on X_{et} defined by Λ. If Λ is finite, then it is the sheaf defined by scheme $X \times \Lambda$ (disjoint union of copies of X indexed by Λ). When Λ is a group, then $\mathcal{F}_\Lambda$ is a sheaf of groups.

The sheaf defined by a coherent $\mathcal{O}_X$-module. Let $\mathcal{M}$ be a sheaf of coherent $\mathcal{O}_X$-modules on X_{zar} in the usual sense of algebraic geometry. For any étale map $\varphi\colon U \to X$, we obtain a coherent $\mathcal{O}_U$-module $\varphi^*\mathcal{M}$ on U_{zar}. For example, if U and X are affine, corresponding to rings B and A, then $\mathcal{M}$ is defined by a finitely generated A-module M and $\varphi^*\mathcal{M}$ corresponds to the B-module $B \otimes_A M$. There is a presheaf $U \mapsto \Gamma(U, \varphi^*\mathcal{M})$ on X_{et}, which we denote $\mathcal{M}^{et}$. For example, $(\mathcal{O}_{X_{zar}})^{et} = \mathcal{O}_{X_{et}}$. To verify that $\mathcal{M}^{et}$ is a sheaf it suffices (thanks to 6.6) to show that the sequence

$$0 \to M \to B \otimes_A M \rightrightarrows B \otimes_A B \otimes_A M$$

is exact whenever $A \to B$ is faithfully flat. This can be proved exactly as in the case $M = A$: again one can assume that $A \to B$ has a section, which allows one to construct a contracting homotopy.

EXAMPLE 6.11. Let X be variety over a field k. For any mophism $\varphi\colon U \to X$, there is an exact sequence

$$\varphi^*\Omega^1_{X/k} \to \Omega^1_{U/k} \to \Omega^1_{U/X} \to 0$$

(Hartshorne 1977, II.8.11). If φ is étale, then $\Omega^1_{U/X} = 0$, and so $\varphi^*\Omega^1_{X/k} \to \Omega^1_{U/k}$ is surjective—since they are locally free sheaves of the same rank, this implies[16] that the map is an isomorphism. Thus, the restriction of $(\Omega^1_{X/k})^{et}$ to U_{zar} is $\Omega^1_{U/k}$.

EXERCISE 6.12. Show that for any faithfully flat homorphism $A \to B$ and A-module M, the sequence

$$0 \to M \to B \otimes_A M \to B \otimes_A B \otimes_A M \to B \otimes_A B \otimes_A B \otimes_A M \to \cdots$$

is exact. The map $B^{\otimes i} \otimes_A M \to B^{\otimes i+1} \otimes_A M$ is $d^i \otimes 1$ where $d^i\colon B^{\otimes i} \to B^{\otimes i+1}$ is the alternating sum of the maps obtained by inserting a 1 in the various positions.

The sheaves on $\mathrm{Spec}(k)$. Let k be a field. A presheaf $\mathcal{F}$ of abelian groups on $(\mathrm{Spec}\, k)_{et}$ can be regarded as a covariant functor $Et/k \to Ab$ (recall Et/k is the category of étale k-algebras). Such a functor will be a sheaf if and only if $\mathcal{F}(\Pi A_i) = \oplus\mathcal{F}(A_i)$ for every finite family (A_i) of étale k-algebras and $\mathcal{F}(k') \xrightarrow{\approx} \mathcal{F}(K)^{\mathrm{Gal}(K/k')}$ for every finite Galois extension K/k' of fields with k' of finite degree over k.

Choose a separable closure k^{sep} of k, and let $G = \mathrm{Gal}(k^{\mathrm{sep}}/k)$. For $\mathcal{F}$ a sheaf on $(\mathrm{Spec}\, k)_{et}$, define

$$M_{\mathcal{F}} = \varinjlim \mathcal{F}(k')$$

[16]Hint: Use that a square matrix with entries in a field whose columns are linearly independent is invertible.

where k' runs through the subfields k' of k^{sep} that are finite and Galois over k. Then $M_{\mathcal{F}}$ is a discrete G-module.

Conversely, if M is a discrete G-module, we define

$$\mathcal{F}_M(A) = \operatorname{Hom}_G(F(A), M)$$

where $F(A)$ is the G-set $\operatorname{Hom}_{k\text{-alg}}(A, k^{\text{sep}})$ (see §3). Then $\mathcal{F}_M$ is a sheaf on $\operatorname{Spec} k$.

The functors $\mathcal{F} \mapsto M_{\mathcal{F}}$ and $M \mapsto \mathcal{F}_M$ define an equivalence between the category of sheaves on $(\operatorname{Spec} k)_{et}$ and the category of discrete G-modules.

Stalks. Let X be a variety over an algebraically closed field, and let $\mathcal{F}$ be a presheaf or sheaf on X_{et}. The *stalk* of $\mathcal{F}$ at a point $x \in X$ is

$$\mathcal{F}_{\bar{x}} = \varinjlim{}_{(U,u)} \mathcal{F}(U),$$

where the limit is over the étale neighbourhoods of x.

Let X be a scheme and let $\mathcal{F}$ be a presheaf or sheaf on X_{et}. The *stalk* of $\mathcal{F}$ at a geometric point $\bar{x} \to X$ is

$$\mathcal{F}_{\bar{x}} = \varinjlim{}_{(U,u)} \mathcal{F}(U),$$

where the limit is over the étale neighbourhoods of $\bar{x}$.

Example 6.13. (a) The stalk of $\mathcal{O}_{X_{et}}$ at $\bar{x}$ is $\mathcal{O}_{X,\bar{x}}$, the strictly local ring at $\bar{x}$.

(b) The stalk of $\mathcal{F}_Z$ at $\bar{x}$, Z a variety or a scheme of finite type over X, is $Z(\mathcal{O}_{X,\bar{x}})$. For example, the stalks of μ_n, $\mathbb{G}_a$, $\mathbb{G}_m$, and GL_n at $\bar{x}$ are $\mu_n(\mathcal{O}_{X,\bar{x}})$, $\mathcal{O}_{X,\bar{x}}$ (regarded as an abelian group), $\mathcal{O}_{X,\bar{x}}^{\times}$, and $\mathrm{GL}_n(\mathcal{O}_{X,\bar{x}})$ respectively.

(c) The stalk of $\mathcal{M}^{et}$, $\mathcal{M}$ a coherent $\mathcal{O}_X$-module, at $\bar{x}$ is $\mathcal{M}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X,\bar{x}}$ where $\mathcal{M}_x$ is the stalk of $\mathcal{M}$ at x (as a sheaf for the Zariski topology).

(d) For a sheaf $\mathcal{F}$ on $\operatorname{Spec} k$, k a field, the stalk at $\bar{x} = \operatorname{Spec} k^{\text{sep}} \to \operatorname{Spec} k$ is $M_{\mathcal{F}}$ regarded as an abelian group.

Skyscraper sheaves. In general, a sheaf $\mathcal{F}$ is said to be a *skyscraper* sheaf if $\mathcal{F}_{\bar{x}} = 0$ except for a finite number of x. (Recall that $\bar{x}$ denotes a geometric point of X with image $x \in X$). We shall need some special skyscraper sheaves.

Let X be a Hausdorff topological space, and let $x \in X$. Let Λ be an abelian group. Define

$$\Lambda^x(U) = \begin{cases} \Lambda & \text{if } x \in U; \\ 0 & \text{otherwise.} \end{cases}$$

Then Λ^x is a sheaf on X. Obviously the stalk of Λ^x at $y \neq x$ is 0, and at x it is Λ. Let $\mathcal{F}$ be a sheaf on X. From the definition of direct limits, we see that to give a homomorphism $\mathcal{F}_x \overset{\text{df}}{=} \varinjlim_U \mathcal{F}(U) \to \Lambda$ is the same as to give a compatible family of maps $\mathcal{F}(U) \to \Lambda$, one for each open neighbourhood U of X, and to give such family is to give a map of sheaves $\mathcal{F} \to \Lambda^x$. Thus

$$\operatorname{Hom}(\mathcal{F}, \Lambda^x) \cong \operatorname{Hom}(\mathcal{F}_x, \Lambda).$$

Now let X be a variety over an algebraically closed field, and let $x \in X$. For an étale map $\varphi\colon U \to X$, define

$$\Lambda^x(U) = \oplus_{u \in \varphi^{-1}(x)} \Lambda.$$

Thus, $\Lambda^x(U) = 0$ unless $x \in \varphi(U)$, in which case it is a sum of copies of Λ indexed by the points of U mapping to x. Again, Λ^x is a sheaf, and its stalks are zero except at $\bar{x}$, where it has stalk Λ. Let $\mathcal{F}$ be a sheaf on X, and let $\mathcal{F}_{\bar{x}} \to \Lambda$ be a homomorphism of groups. A choice of a $u \in \varphi^{-1}(x)$ realizes (U, u) as an étale neighbourhood of x, and hence determines a map $\mathcal{F}(U) \to \mathcal{F}_{\bar{x}} \to \Lambda$. On combining these maps for $u \in \varphi^{-1}(x)$, we obtain a homomorphism

$$\mathcal{F}(U) \to \Lambda^x(U) \stackrel{\mathrm{df}}{=} \oplus_u \Lambda.$$

These are compatible with the restriction maps, and so define a homomorphism $\mathcal{F} \to \Lambda^x$. In this way, we again obtain an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \Lambda^x) \cong \mathrm{Hom}(\mathcal{F}_x, \Lambda).$$

Let X be scheme, and let $i\colon \bar{x} \to X$ be a geometric point of X. For any étale map $\varphi\colon U \to X$, we define

$$\Lambda^{\bar{x}}(U) = \oplus_{\mathrm{Hom}_X(\bar{x}, U)} \Lambda.$$

Again this gives a sheaf on X_{et}, and there is a natural isomorphism $\mathrm{Hom}(\mathcal{F}, \Lambda^{\bar{x}}) \to \mathrm{Hom}(\mathcal{F}_{\bar{x}}, \Lambda)$. However, if $x \stackrel{\mathrm{df}}{=} i(\bar{x})$ is not closed, then it need not be true that $(\Lambda^{\bar{x}})_{\bar{y}} = 0$ when the image of $\bar{y}$ is not equal to x. Thus, unless x is closed, $\Lambda^{\bar{x}}$ need not be a skyscraper sheaf.

Example 6.14. Let $X = \operatorname{Spec} k$, and let $\bar{x} \to X$ correspond to the inclusion of k into a separable closure k^{sep} of k. Then $\Lambda^{\bar{x}}$ is the sheaf on X_{et} corresponding to the discrete G-module consisting of the continuous maps $G \to \Lambda$ (an induced G-module — see Serre, Cohomologie Galoisienne, 2.5.)

Locally constant sheaves. Let X be a topological space. A sheaf $\mathcal{F}$ on X is said to be *locally constant* if $\mathcal{F}|U$ is constant for all U in some open covering of X.

When X is connected, locally arcwise connected, and locally simply connected, the locally constant sheaves are classified by the fundamental group of X.

Fix a point $x \in X$, and let $\mathcal{F}$ be a locally constant sheaf on X. Let $\gamma\colon [0,1] \to X$ be a continuous map with $\gamma(0) = x = \gamma(1)$, i.e., γ is a loop in X based at x. The inverse image $\gamma^*(\mathcal{F})$ of $\mathcal{F}$ on $[0,1]$ is constant. The choice of an isomorphism of $\gamma^*(\mathcal{F})$ with a constant sheaf Λ determines isomorphisms $\mathcal{F}_{\gamma(a)} \to \Lambda$ for each $0 \le a \le 1$. Since $\gamma(0) = \gamma(1) = x$, we get two isomorphisms $\mathcal{F}_x \to \Lambda$, which differ by an automorphism $\alpha(\gamma)$ of $\mathcal{F}_x$. The map $\gamma \mapsto \alpha(\gamma)$ defines a homomorphism $\pi_1(X, x) \to \mathrm{Aut}(\mathcal{F}_x)$. The following is well-known:

Proposition 6.15. *The map $\mathcal{F} \mapsto \mathcal{F}_x$ defines an equivalence between the category of locally constant sheaves of sets (resp. abelian groups) on X and the category of $\pi_1(X, x)$-sets (resp. modules).*

Let X be an algebraic variety (or a scheme). A sheaf $\mathcal{F}$ on X_{et} is *locally constant* if $\mathcal{F}|U$ is constant for all $U \to X$ in some étale covering of X.

Proposition 6.16. *Assume X is connected, and let $\bar{x}$ be a geometric point of X. The map $\mathcal{F} \mapsto \mathcal{F}_{\bar{x}}$ defines an equivalence between the category of locally constant sheaves of sets (resp. abelian groups) with finite stalks on X and the category of finite $\pi_1(X, \bar{x})$-sets (resp. modules).*

Proof. In fact, one shows that, if $Z \to X$ is a finite étale map, then the sheaf $\mathcal{F}_Z$ is locally constant with finite stalks, and every such sheaf is of this form for some Z; more precisely, $Z \mapsto \mathcal{F}_Z$ defines an equivalence from FEt/X to the category of locally constant sheaves on X_{et} with finite stalks. Now we can apply Theorem 3.1 to prove the proposition.

I sketch a proof of the above statement. Consider a surjective étale morphism $Z \to X$. As noted above 6.7, there exists a surjective finite étale map $Z' \to Z$ such that the composite $Z' \to X$ is a Galois covering. One shows that $Z \times_X Z'$ is a disjoint union of copies of Z'. The sheaf $\mathcal{F}_Z$ restricts to the sheaf $\mathcal{F}_{Z \times_X Z'}$ on Z', which is constant. This shows that $\mathcal{F}_Z$ is locally constant, and in fact becomes constant on a finite étale covering $Z' \to Z$.

Conversely, let $\mathcal{F}$ be a locally constant sheaf with finite fibres. By assumption, there is an étale covering $(U_i \to X)_{i \in I}$ such that $\mathcal{F}|U_i$ is constant, and hence of the form $\mathcal{F}_{Z_i}$ for some Z_i finite and étale over U_i (in fact, Z_i is a disjoint union of copies of U_i). The isomorphisms $(\mathcal{F}|U_i)|U_i \times_X U_j \xrightarrow{\approx} (\mathcal{F}|U_j)|U_i \times_X U_j$ induce isomorphisms $Z_i \times_{U_i} (U_i \times_X U_j) \to Z_j \times_{U_j} (U_i \times_X U_j)$, and descent theory (cf. EC I 2.23) shows that the system consisting of the Z_i and the isomorphisms arises from a $Z \to X$. $\square$

7. The Category of Sheaves on X_{et}.

In this section, we study the category of sheaves of abelian groups on X_{et}. In particular, we show that it is an abelian category.

Generalities on categories. Let $\mathbf{T}$ be an additive category — recall that this means that the sets $\mathrm{Hom}(A, B)$ are endowed with structures of abelian groups in such a way that the composition maps are bi-additive and that every finite collection of objects in $\mathbf{T}$ has a direct sum. A sequence

$$0 \to A \to B \xrightarrow{\alpha} C$$

in $\mathbf{T}$ is *exact* if

$$0 \to \mathrm{Hom}(T, A) \to \mathrm{Hom}(T, B) \to \mathrm{Hom}(T, C)$$

is exact for all objects T in $\mathbf{T}$, in which case A is called the *kernel* of α. A sequence

$$A \xrightarrow{\beta} B \to C \to 0$$

is *exact* if

$$0 \to \mathrm{Hom}(C, T) \to \mathrm{Hom}(B, T) \to \mathrm{Hom}(A, T)$$

is exact for all objects T in $\mathbf{T}$, in which case C is called the *cokernel* of β.

Let $\mathbf{T}$ be an additive category in which every morphism has both a kernel and cokernel. Let $\alpha \colon A \to B$ be a morphism in $\mathbf{T}$. The kernel of the cokernel of α is called the *image* of α, and the cokernel of the kernel of α is called the *co-image* of α. There is a canonical morphism from the co-image of α to the image of α, and if this is always an isomorphism, then $\mathbf{T}$ is called an *abelian category.*

Let $\mathbf{T}$ be an abelian category. On combining the two definitions, we obtain the notion of a short exact sequence

$$0 \to A \to B \to C \to 0,$$

and hence of a long exact sequence.

By a functor from one additive category to a second, we shall always mean an additive functor, i.e., we require the maps $\mathrm{Hom}(X, Y) \to \mathrm{Hom}(FX, FY)$ *to be homomorphisms of abelian groups.*

Now let $\mathbf{C}$ be a small[17] category, and let $\mathbf{T}$ be the category of contravariant functors $F \colon \mathbf{C} \to Ab$ from $\mathbf{C}$ to the category of abelian groups. A morphism $\alpha \colon F_1 \to F_2$ is a natural transformation. Thus, α attaches to each object U of $\mathbf{C}$ a homomorphism $\alpha(U) \colon F_1(U) \to F_2(U)$ in such a way that, for every morphism $\varphi \colon V \to U$, the diagram

$$\begin{array}{ccc} F_1(U) & \xrightarrow{\alpha(U)} & F_2(U) \\ \downarrow F_1(\varphi) & & \downarrow F_2(\varphi) \\ F_1(V) & \xrightarrow{\alpha(V)} & F_2(V) \end{array}$$

commutes.

[17] A category is small if its objects form a set (rather than a class).

Clearly, $\mathbf{T}$ becomes an additive category with the obvious group structure on $\mathrm{Hom}(F_1, F_2)$ and with direct sums defined by

$$(\oplus_i F_i)(U) = \oplus_i F_i(U).$$

We leave it as a (simple) exercise to the reader to verify the following statements:

7.1. (a) A sequence $\cdots \to F_i \to G_i \to H_i \to \cdots$ of contraviant functors on $\mathbf{C}$ is exact if and only if $\cdots \to F_i(U) \to G_i(U) \to H_i(U) \to \cdots$ is exact for all U in $\mathbf{C}$.

(b) The category of contravariant functors on $\mathbf{C}$ is abelian.

REMARK 7.2. In the definition of a category, $\mathrm{Hom}(X,Y)$ is required to be a set (not a class) for each pair of objects X and Y. If $\mathbf{C}$ is not small, then the natural transformations $\alpha\colon F \to G$ between two functors on $\mathbf{C}$ may not form a set.

Adjoint functors. Let $\mathbf{C}_1$ and $\mathbf{C}_2$ be abelian categories. Functors $L\colon \mathbf{C}_1 \to \mathbf{C}_2$ and $R\colon \mathbf{C}_2 \to \mathbf{C}_1$ are said to be *adjoint* if

$$\mathrm{Hom}_{\mathbf{C}_2}(LX_1, X_2) \cong \mathrm{Hom}_{\mathbf{C}_1}(X_1, RX_2),$$

functorially in X_1 and X_2. One also says that L is the *left adjoint* of R, and that R is the *right adjoint* of L. If L admits a right adjoint, then the right adjoint is unique (up to a unique isomorphism).

Recall that an object I of an additive category is said to be *injective* if $\mathrm{Hom}(\cdot, I)$ is an exact functor.

We leave it as a (simple) exercise to the reader to verify the following statements:

7.3. (a) A functor R that admits a left adjoint is left exact (and, in fact, preserves products and inverse limits).

(b) A functor L that admits a right adjoint is right exact (and, in fact, preserves direct sums and direct limits).

(c) A functor R that admits an exact left adjoint preserves injectives.

The category of presheaves. Let X be a variety (or scheme). By definition, the presheaves of abelian groups on X_{et} are the contravariant functors $Et/X \to Ab$. In order to apply the discussion in the above section to them, we should, strictly speaking, replace Et/X by a small category. For example, we could replace Et/X by the set of étale maps $U \to X$ of the following form: U is obtained by patching the varieties (or schemes) attached to quotients of rings of the form $A[T_1, T_2, \ldots]$ where $A = \Gamma(V, \mathcal{O}_X)$ for some open affine $V \subset X$ and $\{T_1, T_2, \ldots\}$ is a fixed set of symbols. We shall ignore this problem, and speak of the category $PreSh(X_{et})$ of presheaves of abelian groups on X_{et} as if it were the category of all contravariant functors on Et/X. It is an abelian category, in which $\mathcal{P}' \to \mathcal{P} \to \mathcal{P}''$ is exact if and only if $\mathcal{P}'(U) \to \mathcal{P}(U) \to \mathcal{P}''(U)$ is exact for all $U \to X$ étale. Kernels, cokernels, products, direct sum, inverse limits, direct limits, etc., are formed in $PreSh(X_{et})$ in the obvious way: construct the kernel, cokernel, ... for each $U \to X$ étale, and take the induced restriction maps.

The category of sheaves. Let X be a variety (or scheme). We define the category $Sh(X_{et})$ to be the full subcategory of $PreSh(X_{et})$ whose objects are the sheaves of abelian groups on X_{et}. Thus, to give a morphism $\mathcal{F}' \to \mathcal{F}$ of sheaves on X_{et} is to give a natural transformation $\mathcal{F}' \to \mathcal{F}$ of functors. Clearly, $Sh(X_{et})$ is an additive category. We examine what exactness means in $Sh(X_{et})$.

A morphism $\alpha\colon \mathcal{F} \to \mathcal{F}'$ of sheaves (or presheaves) is said to be *locally surjective* if, for every U and $s \in \mathcal{F}'(U)$, there exists a covering $(U_i \to U)$ such that $s|U_i$ is in the image of $\mathcal{F}(U_i) \to \mathcal{F}'(U_i)$ for each i.

LEMMA 7.4. *Let $\alpha\colon \mathcal{F} \to \mathcal{F}'$ be a homomorphism of sheaves on X_{et}. The following are equivalent:*

(a) *the sequence of sheaves $\mathcal{F} \xrightarrow{\alpha} \mathcal{F}' \to 0$ is exact;*
(b) *the map α is locally surjective;*
(c) *for each geometric point $\bar{x} \to X$, the map $\alpha_{\bar{x}}\colon \mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}}$ is surjective.*

PROOF. (b) $\Rightarrow$ (a). Let $\beta\colon \mathcal{F}' \to \mathcal{T}$ be a map of sheaves such that $\beta \circ \alpha = 0$; we have to prove that this implies that $\beta = 0$.

Let $s' \in \mathcal{F}'(U)$ for some étale $U \to X$. By assumption, there exists a covering $(U_i \to U)_{i\in I}$ and $s_i \in \mathcal{F}(U_i)$ such that $\alpha(s_i) = s'|U_i$. Now

$$\beta(s')|U_i = \beta(s'|U_i) = \beta \circ \alpha(s_i) = 0, \quad \text{all } i.$$

Since $\mathcal{T}$ is a sheaf, this implies that $\beta(s') = 0$.

(a) $\Rightarrow$ (c). Suppose $\alpha_{\bar{x}}$ is not surjective for some $\bar{x} \in X$, and let $\Lambda \neq 0$ be the cokernel of $\mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}}$. Let Λ^x be the sheaf defined in the last section; thus $\mathrm{Hom}(\mathcal{G}, \Lambda^x) = \mathrm{Hom}(\mathcal{G}_{\bar{x}}, \Lambda)$ for any sheaf $\mathcal{G}$ on X_{et}. The map $\mathcal{F}'_{\bar{x}} \to \Lambda$ defines a nonzero morphism $\mathcal{F}' \to \Lambda^x$, whose composite with $\mathcal{F} \to \mathcal{F}'$ is zero (because it corresponds to the composite $\mathcal{F}_{\bar{x}} \to \mathcal{F}'_{\bar{x}} \to \Lambda$). Therefore, $\mathcal{F} \xrightarrow{\alpha} \mathcal{F}' \to 0$ is not exact.

(c) $\Rightarrow$ (b). Let $U \to X$ be étale, and let $\bar{u} \to U$ be a geometric point of U. The composite $\bar{u} \to U \to X$ is a geometric point; let's denote it by $\bar{x}$. An étale neighbourhood of $\bar{u}$ gives, by composition with $U \to X$, and étale neighbourhood of $\bar{x}$, and the étale neighbourhoods of $\bar{x}$ arising in this fashion are cofinal; therefore $\mathcal{F}_{\bar{u}} \cong \mathcal{F}_{\bar{x}}$ for every sheaf $\mathcal{F}$ on X_{et}.

Thus, the hypothesis implies that $\mathcal{F}_{\bar{u}} \to \mathcal{F}'_{\bar{u}}$ is surjective for every geometric point $\bar{u} \to U$ of U. Let $s \in \mathcal{F}'(U)$. Let $u \in U$, and let $\bar{u} \to U$ be a geometric point of U with image u. Because $\mathcal{F}_{\bar{u}} \to \mathcal{F}'_{\bar{u}}$ is surjective, there exists an étale map $V \to U$ whose image contains u and which is such that $s|V$ is in the image $\mathcal{F}(V) \to \mathcal{F}'(V)$. On applying this statement for sufficiently many $u \in U$, we obtain the covering sought. $\square$

PROPOSITION 7.5. *Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

be a sequence of sheaves on X_{et}. The following are equivalent:

(a) *the sequence is exact in the category of sheaves;*
(b) *the sequence*

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$$

is exact for all étale $U \to X$;
(c) *the sequence*

$$0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}}$$

is exact for every geometric point $\bar{x} \to X$ of X.

Proof. In the next subsection, we prove that the functor $i\colon Sh(X_{et}) \to PreSh(X_{et})$ has a left adjoint a. Therefore (see 7.3), i is left exact, which proves the equivalence of (a) and (b).

Direct limits of exact sequences of abelian groups are exact, and so (b) implies (c). The converse is not difficult, using that $s \in \mathcal{F}(U)$ is zero if and only if $s_{\bar{u}} = 0$ for all geometric points $\bar{u}$ of U (cf. the proof of (c) implies (b) for the preceding proposition). □

Proposition 7.6. *Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

be a sequence of sheaves of abelian groups on X_{et}. The following are equivalent:

(a) *the sequence is exact in the category of sheaves;*
(b) *the map $\mathcal{F} \to \mathcal{F}''$ is locally surjective, and*

$$0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$$

is exact for all open $U \subset X$;
(c) *the sequence*

$$0 \to \mathcal{F}'_{\bar{x}} \to \mathcal{F}_{\bar{x}} \to \mathcal{F}''_{\bar{x}} \to 0$$

is exact for each geometric point $\bar{x} \to X$.

Proof. Combine the last two propositions. □

Remark 7.7. In (c) of the last three propositions, one need only check the condition for one geometric point $\bar{x} \to x \in X$ for each closed point x of a variety X, or for one geometric point $\bar{x} \to x \in X$ for each point x of a scheme X.

Proposition 7.8. *The category of sheaves of abelian groups on X_{et}, $Sh(X_{et})$, is abelian.*

Proof. The map from the co-image of a morphism to its image is an isomorphism because it is on stalks. □

Example 7.9. (a) (Kummer sequence). Let n be an integer that is not divisible by the characteristic of any residue field of X. For example, if X is a variety over a field k of characteristic $p \neq 0$, then we require that p not divide n. Consider the sequence

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m \to 0.$$

After (7.6) and (6.13b), in order to prove that this is exact, we have check that

$$0 \to \mu_n(A) \to A^\times \xrightarrow{t \mapsto t^n} A^\times \to 0$$

is exact for every strictly local ring $A = \mathcal{O}_{X,\bar{x}}$ of X. This is obvious except at the second $A^\times$, and here we have to show that every element of $A^\times$ is an nth power. But $\frac{d(T^n - a)}{dT} = nT^{n-1} \neq 0$ in the residue field of A, and so $T^n - a$ splits in $A[T]$.

(b) (Artin-Schreier sequence). Let X be a variety over a field k of characteristic $p \neq 0$, and consider the sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{t \mapsto t^p - t} \mathbb{G}_a \to 0.$$

Again, in order to prove that this sequence is exact, we have to check that

$$0 \to \mathbb{Z}/p\mathbb{Z} \to A^{\times} \xrightarrow{t \mapsto t^p - t} A^{\times} \to 0$$

is exact for every strictly local ring $A = \mathcal{O}_{X,\bar{x}}$ of X. This is obvious except at the second $A^{\times}$, but $\frac{d(T^p - T - a)}{dT} = -1 \neq 0$ in the residue field of A, and so $T^p - T - a$ splits in $A[T]$.

Remark 7.10. If p divides the characteristic of some residue field of X, then

$$0 \to \mu_p \to \mathbb{G}_m \xrightarrow{t \mapsto t^p} \mathbb{G}_m \to 0$$

will not be exact for the étale topology on X. However, it will be exact for the flat topology, because, for any $a \in \Gamma(U, \mathcal{O}_U)$, U affine, the equation $T^p - a$ defines a flat covering of U. Thus each element of $\mathbb{G}_m(U)$ is locally a pth power for the flat topology.

The sheaf associated with a presheaf. In this subsection, "sheaf" (or "presheaf") will mean sheaf (or presheaf) of sets.

Definition 7.11. Let $\mathcal{P} \to a\mathcal{P}$ be a homomorphism from a presheaf $\mathcal{P}$ to a sheaf (on some site); then $a\mathcal{P}$ is said to be the *sheaf associated with* $\mathcal{P}$ (or to be the sheafification of $\mathcal{P}$) if all other homomorphisms from $\mathcal{P}$ to a sheaf factor uniquely through $\mathcal{P} \to a\mathcal{P}$, i.e., if $\mathrm{Hom}(\mathcal{P}, \mathcal{F}) \cong \mathrm{Hom}(a\mathcal{P}, \mathcal{F})$ for all sheaves $\mathcal{F}$.

Clearly $a\mathcal{P}$, endowed with the map $\mathcal{P} \to a\mathcal{P}$, is unique up to a unique isomorphism if it exists. In the remainder of this subsection, we construct $a\mathcal{P}$ in various contexts. First we give a criterion for $\mathcal{F}$ to be the sheaf associated with $\mathcal{P}$.

Let $\mathcal{P}$ be a presheaf. Sections $s_1, s_2 \in \mathcal{P}(U)$ are said to be *locally equal* if $s_1|U_i = s_2|U_i$ for all U_i in some covering $(U_i \to U)_{i \in I}$ of U. The sheaf criterion implies that locally equal sections of a sheaf are equal.

Lemma 7.12. *Let $i \colon \mathcal{P} \to \mathcal{F}$ be a homomorphism from presheaf $\mathcal{P}$ to a sheaf $\mathcal{F}$. Assume:*

(a) *the only sections of $\mathcal{P}$ to have the same image in $\mathcal{F}(U)$ are those that are locally equal, and*
(b) *i is locally surjective.*

Then $(\mathcal{F}, i)$ is the sheaf associated with $\mathcal{P}$.

Proof. Let $i' \colon \mathcal{P} \to \mathcal{F}'$ be a second map from $\mathcal{P}$ into a sheaf, and let $s \in \mathcal{F}(U)$ for some U. Thanks to (b), we know that for some covering $(U_i \to U)$ of U, there exist $s_i \in \mathcal{P}(U_i)$ such that $i(s_i) = s|U_i$. Because of (a), $i'(s_i) \in \mathcal{F}'(U_i)$ is independent of the choice of s_i, and moreover that the restrictions of $i'(s_i)$ and $i'(s_j)$ to $\mathcal{F}'(U_i \times_U U_j)$ agree. We define $\alpha(s)$ to be the unique element of $\mathcal{F}'(U)$ that restricts to $i'(s_i)$ for all i. Then $s \mapsto \alpha(s) \colon \mathcal{F} \to \mathcal{F}'$ is the unique homomorphism such that $i \circ \alpha = i'$. □

Lemma 7.13. *Let $\mathcal{P}$ be a subpresheaf of a sheaf $\mathcal{F}$. For each U, let $\mathcal{P}'(U)$ be the set of $s \in \mathcal{F}(U)$ that are locally in $\mathcal{P}$ in the sense that there exists a covering $(U_i \to U)_{i\in I}$ such that $s|U_i \in \mathcal{P}(U_i)$ for each i. Then $\mathcal{P}'$ is a subsheaf of $\mathcal{F}$, and $\mathcal{P} \to \mathcal{P}'$ is locally surjective.*

Proof. The proof is trivial. □

We call $\mathcal{P}'$ the subsheaf of $\mathcal{F}$ *generated* by $\mathcal{P}$.

To construct $a\mathcal{P}$, it suffices to construct a sheaf $\mathcal{F}$ and a homomorphism $\mathcal{P} \to \mathcal{F}$ satisfying (7.12a), because then the subsheaf of $\mathcal{F}$ generated by the image of $\mathcal{P}$ will be the sheaf associated with $\mathcal{P}$.

We now work with sheaves on X_{et}.

Lemma 7.14. *If $i\colon \mathcal{P} \to \mathcal{F}$ satisfies the conditions (a) and (b) of Lemma 7.12, then*

$$i_{\bar{x}}\colon \mathcal{P}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$$

is an isomorphism for all geometric points $\bar{x}$.

Proof. This follows easily from the various definitions. □

For each $x \in X$, choose a geometric point $\bar{x} \to X$ with image x. For $\mathcal{P}$ a presheaf on X, define $\mathcal{P}^* = \prod(\mathcal{P}_{\bar{x}})^{\bar{x}}$ where $(\mathcal{P}_{\bar{x}})^{\bar{x}}$ is the sheaf associated with the abelian group $\mathcal{P}_{\bar{x}}$ as in §6. Then $\mathcal{P}^*$ is a sheaf, and the natural map $\mathcal{P} \to \mathcal{P}^*$ satisfies condition (a) of (7.12).

Theorem 7.15. *For any presheaf $\mathcal{P}$ on X_{et}, there exists an associated sheaf $i\colon \mathcal{P} \to a\mathcal{P}$. The map i induces isomorphisms $\mathcal{P}_{\bar{x}} \to (a\mathcal{P})_{\bar{x}}$ on the stalks. The functor $a\colon PreSh(X_{et}) \to Sh(X_{et})$ is exact.*

Proof. Take $a\mathcal{P}$ to be the subsheaf of $\mathcal{P}^*$ generated by $i(\mathcal{P})$. Then $i\colon \mathcal{P} \to a\mathcal{P}$ satisfies the conditions (a) and (b) of Lemma 7.12, from which the first two statements follow. If $\mathcal{P}' \to \mathcal{P} \to \mathcal{P}''$ is an exact sequence of abelian groups, then $\mathcal{P}'_{\bar{x}} \to \mathcal{P}_{\bar{x}} \to \mathcal{P}''_{\bar{x}}$ is exact for all $x \in X$ (because direct limits of exact sequences of abelian groups are exact). But the last sequence can be identified with $(a\mathcal{P}')_{\bar{x}} \to (a\mathcal{P})_{\bar{x}} \to (a\mathcal{P}'')_{\bar{x}}$, which shows that $a\mathcal{P}' \to a\mathcal{P} \to a\mathcal{P}''$ is exact. □

Example 7.16. A group Λ defines a *constant presheaf* $\mathcal{P}_\Lambda$ such that $\mathcal{P}_\Lambda(U) = \Lambda$ for all $U \neq \emptyset$. The sheaf associated with $\mathcal{P}_\Lambda$ is $\mathcal{F}_\Lambda$ (see §6).

If $\mathcal{P}$ is a sheaf of abelian groups, then so also is $a\mathcal{P}$.

Remark 7.17. Let $(\mathcal{F}_i)_{i\in I}$ be a family of sheaves of abelian groups on X_{et}.

Let $\mathcal{P}$ be the presheaf with $\mathcal{P}(U) = \prod_{i\in I} \mathcal{F}_i(U)$ for all $U \to X$ étale and the obvious restriction maps. Then $\mathcal{P}$ is a sheaf, and it is the product of the $\mathcal{F}_i$. A similar remark applies to inverse limits and kernels.

Let $\mathcal{P}$ be the presheaf with $\mathcal{P}(U) = \oplus_{i\in I}\mathcal{F}_i(U)$ for all $U \to X$ étale and the obvious restriction maps. In general, $\mathcal{P}$ will not be a sheaf, but $a\mathcal{P}$ is the direct sum of the $\mathcal{F}_i$ in $Sh(X_{et})$. A similar remark applies to direct limits and cokernels.

Lest the reader think that the category of sheaves is "just like the category of abelian groups", let me point out that in general it does not have enough projectives; nor is a product of exact sequences of sheaves necessarily exact.

ASIDE 7.18. Let $\mathcal{P}$ be a presheaf of sets on a topological space X. Define $E(\mathcal{P})$ to be $\coprod_{x\in X}\mathcal{P}_x$, and let $\pi\colon E(\mathcal{P})\to X$ be the map sending each element of $\mathcal{P}_x$ to x. An $s\in\mathcal{P}(U)$ defines a section $s\colon U\to E(\mathcal{P})$ to π over U, namely, $x\mapsto s_x$. Give $E(\mathcal{P})$ the finest topology for which all the maps $s\colon U\to E(\mathcal{P})$ are continuous: thus $V\subset E(\mathcal{P})$ is open if and only if, for all open $U\subset X$ and $s\in\mathcal{P}(U)$, $s^{-1}(V)$ is open in U.

[The reader is invited to draw a picture with X represented as a horizontal line and each stalk $\mathcal{P}_x$ represented as a short vertical line over x; the origin of the terminology "sheaf", "stalk", and "section" should now be clear.]

For each open subset U of X, let $\mathcal{F}(U)$ be the set of all continuous sections $s\colon U\to E(\mathcal{P})$ to π over U. The obvious map $\mathcal{P}\to\mathcal{F}$ realizes $\mathcal{F}$ as the sheaf associated with $\mathcal{P}$. [The space $E(\mathcal{P})$ is the "éspace étalé" associated with $\mathcal{P}$ — see Godement, R., *Théorie des Faisceaux*, Hermann, 1964, II 1.2. It is possible to avoid using these spaces — in fact, Grothendieck has banished them from mathematics — but they are quite useful, for example, for defining the inverse image of a sheaf. Interestingly, there is an analogue (requiring algebraic spaces) for the étale topology, which is useful, for example, for defining the actions of Frobenius maps on cohomology groups. This aspect is insufficiently emphasized in this version of the notes.

ASIDE 7.19. A presheaf of sets $\mathcal{P}$ on a site $\mathbf{T}$ is said to be *separated* if two sections are equal whenever they are locally equal. For a presheaf $\mathcal{P}$, define $\mathcal{P}^+$ to be the presheaf with

$$\mathcal{P}^+(U)=\varinjlim \operatorname{Ker}(\prod\mathcal{P}(U_i)\rightrightarrows\prod\mathcal{P}(U_i\times_X U_j)),$$

— limit over all coverings (assuming the limit exists). One proves without serious difficulty that:

(a) if $\mathcal{P}$ is separated, then $\mathcal{P}^+$ is a sheaf;
(b) for an arbitrary $\mathcal{P}$, $\mathcal{P}^+$ is separated.

Thus, for any presheaf $\mathcal{P}$, $\mathcal{P}^{++}$ is a sheaf. In fact, it is the sheaf associated with $\mathcal{P}$.

As Waterhouse has pointed out (Pac. J. Math 57 (1975), 597–610), there exist sites for which the limit does not exist: the problem is that the limit is over a class, not a set, and can not be replaced by a limit over a set. This doesn't seem to be a problem for any sites actually in use.

Algebraic spaces. The Zariski topology allows us to patch together affine varieties to obtain general varieties. One can ask whether the étale topology allows us to patch together affine varieties to obtain more general objects. The answer is yes!

For simplicity, I'll work over a fixed algebraically closed field k, and consider only k-varieties.

Endow Aff/k, the category of all affine k-varieties, with the étale topology. A sheaf on this site is a contravariant functor $\mathcal{A}\colon\mathrm{Aff}/k\to \mathit{Sets}$ that satisfies the sheaf condition for every surjective family of étale maps $(U_i\to U)_{i\in I}$ in Aff/k. Note that $\mathcal{A}$ can also be regarded as a covariant functor from affine k-algebras to *Sets*. For any k-variety V, the functor $A \mapsto V(A)$ sending an affine k-algebra to the set of points of V with coordinates in A is a sheaf on $(\mathrm{Aff}/k)_{et}$; moreover, because of the Yoneda Lemma, the category of k-varieties can be regarded as a full subcategory of the category of sheaves on $(\mathrm{Aff}/k)_{et}$.

Let U be an affine k-variety. An *étale equivalence relation* on U is a subvariety $R \subset U \times U$ such that:

(a) for each affine k-algebra A, $R(A)$ is an equivalence relation on $U(A)$ (in the usual sense—these are sets);
(b) the composites of the inclusion $R \hookrightarrow U \times U$ with the projection maps $U \times U \rightrightarrows U$ are surjective and étale.

A sheaf $\mathcal{A}$ on $(\mathrm{Aff}/k)_{et}$ is an *algebraic space* if there is an affine k-variety U, an étale equivalence relation R on U, and a map $U \to \mathcal{A}$ of sheaves realizing $\mathcal{A}$ as the sheaf-theoretic quotient of U by R. This last condition means that,

(a) for any affine k-algebra A and $s_1, s_2 \in U(A)$, s_1 and s_2 have the same image in $\mathcal{A}(A)$ if and only if $(s_1, s_2) \in R(A)$;
(b) the map $U \to \mathcal{A}$ of sheaves is locally surjective.

The category of algebraic varieties is a full subcategory of the category of algebraic spaces. Algebraic spaces have many advantages over algebraic varieties, of which I mention only two.

Quotients of algebraic spaces by group actions are more likely to exist. For example, Hironaka has constructed a nonsingular 3-dimensional variety V and a free action of a group G of order 2 on V such that the quotient V/G does not exist as an algebraic variety (the problem is that there exist orbits of G not contained in any affine). Quotients of algebraic spaces by free actions of finite groups always exist as algebraic spaces (Artin 1973, p109). For a very general theorem on the existence of quotients of algebraic spaces, see Keel, S., and Mori, S., Quotients by groupoids, Annals of Mathematics, **145** (1997), 193–213.

There is a natural fully faithful functor $V \to V(\mathbb{C})$ from the category of complete algebraic varieties over $\mathbb{C}$ to that of compact analytic spaces, but there is no convenient description of the essential image of the functor. For the category of complete algebraic spaces there is: it consists of compact analytic spaces M such that the field of meromorphic functions on M has transcendence degree equal to the dimension of M (these are the Moishezon spaces).

Addendum. Let X be a variety (or quasicompact scheme). Then every open subset U of X is quasicompact, i.e., any covering of U by open subsets $U = \cup U_i$ admits a finite subcovering. Moreover, because I defined an étale morphism to be of finite type (rather than locally of finite type), every U étale over X is quasicompact. Because étale morphisms are open, this implies that every étale covering $(U_i \to U)_{i \in I}$ has a finite subcovering. Sites such that every covering family contains a finite covering subfamily are said to be *Noetherian.* Note that the site defined by a topological space X will rarely be Noetherian: if X is Hausdorff, its site will be Noetherian if and only if every open subset of X is compact.

7.20. For a Noetherian site, in order to prove that a presheaf is a sheaf, it suffices to check the sheaf condition for finite coverings. It follows that direct sums and direct limits of sheaves formed in the naive (i.e., presheaf) way are again sheaves, and from this it follows that cohomology commutes with direct sums and direct limits (over directed sets). See EC III.3.

8. Direct and Inverse Images of Sheaves.

Direct images of sheaves. Let $\pi\colon Y \to X$ be a morphism of varieties (or schemes), and let $\mathcal{P}$ be a presheaf on Y_{et}. For $U \to X$ étale, define

$$\pi_*\mathcal{P}(U) = \mathcal{P}(U \times_X Y).$$

Since $U \times_X Y \to Y$ is étale (see 2.11), this definition makes sense. With the obvious restriction maps, $\pi_*\mathcal{P}$ becomes a presheaf on X_{et}.

Lemma 8.1. *If $\mathcal{F}$ is a sheaf, so also is $\pi_*\mathcal{F}$.*

Proof. For a variety (or scheme) V over X, let V_Y denote the variety (or scheme) $V \times_X Y$ over Y. Then $V \mapsto V_Y$ is a functor taking étale maps to étale maps, surjective families of maps to surjective families, and fibre products over X to fibre products over Y.

Let $(U_i \to U)$ be a surjective family of étale maps in Et/X. Then $(U_{iY} \to U_Y)$ is a surjective family of étale maps in Et/Y, and so

$$\mathcal{F}(U_Y) \to \prod \mathcal{F}(U_{iY}) \rightrightarrows \prod \mathcal{F}(U_{iY} \times_Y U_{jY})$$

is exact. But this is equal to the sequence

$$(\pi_*\mathcal{F})(U) \to \prod (\pi_*\mathcal{F})(U_i) \rightrightarrows \prod (\pi_*\mathcal{F})(U_i \times_X U_j),$$

which therefore is also exact. $\square$

Obviously, the functor

$$\pi_* : PreSh(Y_{et}) \to PreSh(X_{et})$$

is exact. Therefore, its restriction

$$\pi_* : Sh(Y_{et}) \to Sh(X_{et})$$

is left exact. It is *not* usually right exact: $\mathcal{F} \to \mathcal{F}'$ being locally surjective does not imply that $\pi_*\mathcal{F} \to \pi_*\mathcal{F}'$ is locally surjective. For example, if X is an algebraic variety over an algebraically closed field and $\pi\colon X \to P$ is the map from X to a point, then π_* is essentially the functor taking a sheaf $\mathcal{F}$ to its group of global sections $\Gamma(X, \mathcal{F})$, and $\mathcal{F} \to \mathcal{F}'$ being locally surjective does not imply that $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}')$ is surjective.

Example 8.2. Let $i\colon \bar{x} \to X$ be a geometric point of X. The functor $\mathcal{F} \mapsto \mathcal{F}(\bar{x})$ identifies the category of sheaves on $\bar{x}$ with the category of abelian groups. Let Λ be an abelian group regarded as a sheaf on $\bar{x}$. Then $i_*\Lambda = \Lambda^{\bar{x}}$, the skyscraper sheaf defined in the §6.

A geometric point $\bar{y} \to Y$ of Y defines a geometric point $\bar{y} \to Y \xrightarrow{\pi} X$ of X, which we denote $\bar{x}$ (or $\pi(\bar{y})$). Clearly

$$(\pi_*\mathcal{F})_{\bar{x}} = \varinjlim \mathcal{F}(V)$$

where the limit is over all étale neighbourhoods of $\bar{y}$ of the form U_Y for some étale neighbourhood of $\bar{x}$. Thus, there is a canonical map

$$(\pi_*\mathcal{F})_{\bar{x}} \to \mathcal{F}_{\bar{y}}.$$

In general, this will map will be neither injective nor surjective.

Proposition 8.3. (a) *Let $\pi \colon V \hookrightarrow X$ be an open immersion, i.e., the inclusion of an open subvariety (or subscheme) into X. Then*

$$(\pi_*\mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & x \in V; \\ ? & x \notin V. \end{cases}$$

(b) *Let $\pi \colon Z \hookrightarrow X$ be a closed immersion, i.e., the inclusion of a closed subvariety (or subscheme) into X. Then*

$$(\pi_*\mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & x \in Z; \\ 0 & x \notin Z. \end{cases}$$

(c) *Let $\pi \colon Y \to X$ be a finite map. Then*

$$(\pi_*\mathcal{F})_{\bar{x}} = \oplus_{y \mapsto x} \mathcal{F}_{\bar{y}}^{d(y)}$$

where $d(y)$ is the separable degree of $\kappa(y)$ over $\kappa(x)$. For example, if π is a finite étale map of degree d of varieties over an algebraically closed field, then

$$(\pi_*\mathcal{F})_{\bar{x}} = \mathcal{F}_{\bar{x}}^d.$$

Proof. When $\pi \colon V \hookrightarrow X$ is an immersion (either open or closed), the fibre product $U \times_X V = \varphi^{-1}(V)$ for any morphism $\varphi \colon U \to X$.

(a) If $x \in V$, then for any "sufficiently small" étale neighbourhood $\varphi \colon U \to X$ of $\bar{x}$, $\varphi(U) \subset V$, and so $U = \varphi^{-1}(V) = U \times_X V$. Thus the étale neighbourhoods of $\bar{x}$ of the form U_V form a cofinal set, which proves the first equality. Concerning the second, there is nothing to say except to point out that $(\pi_*\mathcal{F})_{\bar{x}}$ need not be zero when $x \notin V$ (see the examples below).

(b) If $x \notin Z$, then for any "sufficiently small" étale neighbourhood $\varphi \colon U \to X$ of $\bar{x}$, $\varphi(U) \cap Z = \emptyset$, and so $U \times_X Z = \varphi^{-1}(Z) = \emptyset$; thus $\mathcal{F}(U_Z) = 0$. When $x \in Z$, we have to see that every étale map $\bar{\varphi} \colon \bar{U} \to Z$ with $x \in \varphi(\bar{U})$ "extends" to an étale map $\varphi \colon U \to X$. In terms of rings, this amounts to showing that an étale homomorphism $\bar{A} \to \bar{B}$, $\bar{A} = A/\mathfrak{a}$, lifts to an étale homomorphism $A \to B$. But this is easy: we may assume that $\bar{B} = (\bar{A}[T]/(\bar{f}(T))_{\bar{b}}$ where $\bar{f}(T)$ is a monic polynomial such that $\bar{f}'(T)$ is invertible in $(\bar{A}[T]/(\bar{f}(T))_{\bar{b}}$—see §2 (standard étale morphisms); choose $f(T) \in A[T]$ lifting $\bar{f}(T)$, and set $B = (A[T]/(f(T))_b$ for an appropriate b.

(c) We omit the proof. □

Corollary 8.4. *The functor π_* is exact if π is finite or a closed immersion.*

Proof. This follows from the proposition and (7.6). □

Example 8.5. We list some examples, all analogues.

(a) Let X be an open disk of radius 1 centred at the origin in $\mathbb{R}^2$, let $U = X \smallsetminus \{(0,0)\}$, and let j be the inclusion $U \hookrightarrow X$. Let $\mathcal{F}$ be the locally constant sheaf on U corresponding to a $\pi_1(U,u)$-module F — recall $\pi_1(U,u) = \mathbb{Z}$ (see 6.15; here u is any point of U). Then

$$(j_*\mathcal{F})_{(0,0)} = F^{\pi_1(U,u)}(\text{elements fixed by } \pi_1).$$

(b) Let X be the affine line over an algebraically closed field, let $U = X \smallsetminus \{0\}$, and let j be the inclusion $U \hookrightarrow X$. Let $\mathcal{F}$ be the locally constant sheaf on U corresponding to a $\pi_1(U,\bar{u})$-module F — recall $\pi_1(U,\bar{u}) = \hat{\mathbb{Z}}$. Then

$$(j_*\mathcal{F})_{\bar{0}} = F^{\pi_1(U,\bar{u})}(\text{elements fixed by } \pi_1).$$

(c) Let $X = \operatorname{Spec} R$ where R is a Henselian discrete valuation ring, let $U = \operatorname{Spec} K$ where K is the field of fractions of R, and let j be the inclusion $U \hookrightarrow X$. Let $\mathcal{F}$ be the locally constant sheaf on U corresponding to a $\pi_1(U,\bar{u})$-module F — recall $\pi_1(U,\bar{u}) = \operatorname{Gal}(K^{\mathrm{sep}}/K)$. Here $\bar{u}$ is the geometric point corresponding to the inclusion $K \hookrightarrow K^{\mathrm{sep}}$. Then

$$(j_*\mathcal{F})_{\bar{x}} = F^I(\text{elements fixed by } I)$$

where x is the closed point of $\operatorname{Spec} R$.

Proposition 8.6. *For any morphisms* $Z \xrightarrow{\pi} Y \xrightarrow{\pi'} X$, $(\pi' \circ \pi)_* = \pi'_* \circ \pi_*$.

Proof. This is obvious from the definition. □

Inverse images of sheaves. Let $\pi : Y \to X$ be a morphism of varieties (or schemes). We shall define a left adjoint for the functor π_*. Let $\mathcal{P}$ be a presheaf on X_{et}. For $V \to Y$ étale, define

$$\mathcal{P}'(V) = \varinjlim \mathcal{P}(U)$$

where the direct limit is over the commutative diagrams

$$\begin{array}{ccc} V & \to & U \\ \downarrow & & \downarrow \\ Y & \to & X \end{array}$$

with $U \to X$ étale. One sees easily that, for any presheaf $\mathcal{Q}$ on Y, there are natural one-to-one correspondences between

- morphisms $\mathcal{P}' \to \mathcal{Q}$;
- families of maps $\mathcal{P}(U) \to \mathcal{Q}(V)$, indexed by commutative diagrams as above, compatible with restriction maps;
- morphisms $\mathcal{P} \to \pi_*\mathcal{Q}$.

Thus

$$\operatorname{Hom}_{Y_{et}}(\mathcal{P}', \mathcal{Q}) \cong \operatorname{Hom}_{X_{et}}(\mathcal{P}, \pi_*\mathcal{Q}),$$

functorially in $\mathcal{P}$ and $\mathcal{Q}$. Unfortunately, $\mathcal{P}'$ need not be a sheaf even when $\mathcal{P}$ is. Thus, for $\mathcal{F}$ a sheaf on X_{et}, we define $\pi^*\mathcal{F} = a(\mathcal{F}')$. Then, for any sheaf $\mathcal{G}$ on Y_{et},

$$\operatorname{Hom}_{Y_{et}}(\pi^*\mathcal{F}, \mathcal{G}) \cong \operatorname{Hom}_{Y_{et}}(\mathcal{F}', \mathcal{G}) \cong \operatorname{Hom}_{X_{et}}(\mathcal{F}, \mathcal{G}),$$

and so π^* is a left adjoint to $\pi_* : Sh(Y_{et}) \to Sh(X_{et})$.

Proposition 8.7. *For any morphisms* $Z \xrightarrow{\pi} Y \xrightarrow{\pi'} X$, $(\pi' \circ \pi)^* = \pi^* \circ \pi'^*$.

Proof. Both are left adjoints of $(\pi' \circ \pi)_* = \pi'_* \circ \pi_*$. □

EXAMPLE 8.8. Let $\pi : U \to X$ be an étale morphism. For any sheaves $\mathcal{F}$ on X_{et} and $\mathcal{G}$ on U_{et}, one sees easily that

$$\mathrm{Hom}(\mathcal{F}|U_{et}, \mathcal{G}) \cong \mathrm{Hom}(\mathcal{F}, \pi_* \mathcal{G})$$

—in both cases, to give a morphism is to give a family of maps $\mathcal{F}(V) \to \mathcal{G}(V)$ indexed by the étale maps $V \to U$ and compatible with restriction. Therefore, in this case $\pi^* : Sh(X_{et}) \to Sh(U_{et})$ is just the restriction map.

This can also be seen directly from the definition π^*, because for each $V \to U$, there is a final element in the family of diagrams over which the limit is taken, namely,

$$\begin{array}{ccc} V & = & V \\ \downarrow & & \downarrow \\ U & \to & X. \end{array}$$

REMARK 8.9. Let $i : \bar{x} \to X$ be a geometric point of X. For any sheaf $\mathcal{F}$ on X_{et}, $(i^*\mathcal{F})(\bar{x}) = \mathcal{F}_{\bar{x}}$ — this is clear from the definitions of i^* and $\mathcal{F}_{\bar{x}}$. Therefore, for any morphism $\pi\colon Y \to X$ and geometric point $i\colon \bar{y} \to Y$ of Y,

$$(\pi^*\mathcal{F})_{\bar{y}} = i^*(\pi^*\mathcal{F})(\bar{y}) = \mathcal{F}_{\bar{x}},$$

where $\bar{x}$ is the geometric point $\bar{y} \xrightarrow{i} Y \xrightarrow{\pi} X$ of X.

Since this is true for all geometric points of Y, we see that π^* is exact and therefore that π_* preserves injectives (by 7.3).

REMARK 8.10. Let $\mathcal{F}_Z$ be the sheaf on X_{et} defined by a variety Z. Then it is *not* always true that $\pi^*(\mathcal{F}_Z)$ is the sheaf defined by the variety $Z \times_Y X$. For example, if $\pi : X \to P$ is a map from a variety over an algebraically closed field k to a point, then $\pi^*(\mathbb{G}_a)$ is the constant sheaf defined by the group $\mathbb{G}_a(k) = k$.

However, it is true that $\pi^*(\mathcal{F}_Z) = \mathcal{F}_{Z \times_Y X}$ when π is étale or $Z \to X$ is étale.

REMARK 8.11. For a continuous map $\pi\colon Y \to X$ of topological spaces, there is a very simple description of the inverse image of a sheaf in terms of its associated éspace étalé, namely, $\pi^*(\mathcal{F})$ is the sheaf whose sections over an open subset $U \subset Y$ are the continous maps $s\colon U \to E(\mathcal{F})$ such that $s(u)$ lies in the stalk over $\pi(u)$ for all u.

Existence of enough injectives. Let X be a variety (or scheme).

PROPOSITION 8.12. *Every sheaf $\mathcal{F}$ on $Sh(X_{et})$ can be embedded into an injective sheaf.*

PROOF. For each $x \in X$, choose a geometric point $i_x : \bar{x} \to X$ with image x and an embedding $\mathcal{F}_{\bar{x}} \hookrightarrow I(x)$ of the abelian group $\mathcal{F}_{\bar{x}}$ into an injective abelian group. Then $\mathcal{I}^x \stackrel{\mathrm{df}}{=} i_{x*}(I(x))$ is injective (see 8.9). Since a product of injective objects is injective, $\mathcal{I} \stackrel{\mathrm{df}}{=} \prod \mathcal{I}^x$ will be an injective sheaf. The composite $\mathcal{P} \hookrightarrow \mathcal{P}^* \hookrightarrow \mathcal{I}$ is the embedding sought. □

Extension by zero. Let X be a variety (or scheme), and let $j : U \hookrightarrow X$ be an open immersion. As we noted above, for a sheaf $\mathcal{F}$ on U_{et}, the stalks of $j_*\mathcal{F}$ need not be zero at points outside U. We now define a functor $j_!$, "extension by zero", such that $j_!\mathcal{F}$ does have this property.

Let $\mathcal{P}$ be a presheaf on U_{et}. For any $\varphi : V \to X$ étale, define

$$\mathcal{P}_!(V) = \begin{cases} \mathcal{P}(V) & \text{if } \varphi(V) \subset U; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{P}_!$ is a presheaf on X_{et}, and for any other presheaf $\mathcal{Q}$ on X_{et}, a morphism $\mathcal{P} \to \mathcal{Q}|U$ extends uniquely to a morphism $\mathcal{P}_! \to \mathcal{Q}$ (obviously). Thus

$$\mathrm{Hom}_{X_{et}}(\mathcal{P}_!, \mathcal{Q}) \cong \mathrm{Hom}_{U_{et}}(\mathcal{P}, \mathcal{Q}|U)$$

functorially. Unfortunately, $\mathcal{P}_!$ need not be a sheaf even when $\mathcal{P}$ is. Thus, for $\mathcal{F}$ a sheaf on U_{et}, we define $j_!\mathcal{F}$ to be $a(\mathcal{F}_!)$. Then, for any sheaf $\mathcal{G}$ on X_{et},

$$\mathrm{Hom}_{X_{et}}(j_!\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{X_{et}}(\mathcal{F}_!, \mathcal{G}) \cong \mathrm{Hom}_{U_{et}}(\mathcal{F}, \mathcal{G}|U),$$

and so $j_!$ is a left adjoint to $\pi^* \colon Sh(Y_{et}) \to Sh(U_{et})$.

PROPOSITION 8.13. *Let $j\colon\ U \hookrightarrow X$ be an open immersion. For any sheaf $\mathcal{F}$ on U_{et} and geometric point $\bar{x} \to X$,*

$$(j_!\mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & x \in U; \\ 0 & x \notin U. \end{cases}$$

PROOF. Because of (7.14), it suffices to prove this with $j_!\mathcal{F}$ replaced by $\mathcal{F}_!$, in which case it follows immediately from the definitions. □

COROLLARY 8.14. *The functor $j_!\colon\ Sh(U_{et}) \to Sh(X_{et})$ is exact, and j^* preserves injectives.*

PROOF. The first part of the statement follows from the proposition and (7.6), and the second part follows from the first part and (7.3). □

Let Z be the complement of U in X, and denote the inclusion $Z \hookrightarrow X$ by i. Let $\mathcal{F}$ be a sheaf on X_{et}. There is a canonical morphism $j_!j^*\mathcal{F} \to \mathcal{F}$, corresponding by adjointness to the identity map on $j^*\mathcal{F}$, and a canonical morphism $\mathcal{F} \to i_*i^*\mathcal{F}$, corresponding by adjointness to the identity map on $i^*\mathcal{F}$.

PROPOSITION 8.15. *For any sheaf $\mathcal{F}$ on X, the sequence*

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

is exact.

PROOF. This can be checked on stalks. For $x \in U$, the sequence of stalks is

$$0 \to \mathcal{F}_{\bar{x}} \xrightarrow{\mathrm{id}} \mathcal{F}_{\bar{x}} \to 0 \to 0,$$

and for $x \notin U$, the sequence of stalks is

$$0 \to 0 \to \mathcal{F}_{\bar{x}} \xrightarrow{\mathrm{id}} \mathcal{F}_{\bar{x}} \to 0.$$

Both are visibly exact. □

REMARK 8.16. It is possible to define $j_!$ for any étale map $j\colon\ U \to X$. Let $\mathcal{F}$ be a sheaf on U_{et}. For any $\varphi\colon\ V \to X$ étale, define

$$\mathcal{F}_!(V) = \oplus_\alpha \mathcal{F}(V)$$

where the sum is over the morphisms $\alpha\colon\ V \to U$ such that $j \circ \alpha = \varphi$. Then $\mathcal{F}_!$ is a presheaf on X_{et}, and we define $j_!\mathcal{F}$ to be its associated sheaf. Again $j_!$ is the left adjoint of j^* and is exact. Hence j^* preserves injectives.

Sheaves on $X = U \cup Z$. Let X be a variety (or scheme), and let

$$U \xrightarrow{j} X \xleftarrow{i} Z$$

be morphisms with j an open immersion, i a closed immersion, and $X = j(U) \cup i(Z)$. We identify U and Z with subvarieties (or subschemes) of X.

From a sheaf $\mathcal{F}$ on X_{et}, we obtain sheaves $\mathcal{F}_1 \overset{\text{df}}{=} i^*\mathcal{F}$ on Z and $\mathcal{F}_2 = j^*\mathcal{F}$ on U. Moreover, there is a canonical homomorphism $\mathcal{F} \to j_*j^*\mathcal{F}$ corresponding by adjointness to the identity map of $j^*\mathcal{F}$. On applying i^* to it, we obtain a morphism $\phi_{\mathcal{F}}\colon\ \mathcal{F}_1 \to i^*j_*\mathcal{F}_2$.

Let $Tr(X, U, Z)$ be the category[18] whose objects are the triples $(\mathcal{F}_1, \mathcal{F}_2, \phi)$ with $\mathcal{F}_1$ a sheaf on Z_{et}, $\mathcal{F}_2$ a sheaf on U_{et}, and ϕ a map $\mathcal{F}_1 \to i^*j_*\mathcal{F}_2$. A morphism $(\mathcal{F}_1, \mathcal{F}_2, \phi) \to (\mathcal{F}_1', \mathcal{F}_2', \phi')$ is a pair of morphisms $\psi_1\colon\ \mathcal{F}_1 \to \mathcal{F}_1'$ and $\psi_2\colon\ \mathcal{F}_2 \to \mathcal{F}_2'$ such that the following diagram commutes

$$\begin{array}{ccc}
\mathcal{F}_1 & \xrightarrow{\phi} & i^*j_*\mathcal{F}_2 \\
\downarrow & & \downarrow \\
\mathcal{F}_1' & \xrightarrow{\phi'} & i^*j_*\mathcal{F}_2'.
\end{array}$$

Proposition 8.17. *The functor $\mathcal{F} \mapsto (\mathcal{F}_1, \mathcal{F}_2, \phi_{\mathcal{F}})$ is an equivalence $Sh(X_{et}) \to Tr(X, U, Z)$.*

Proof. An essential inverse is provided by the functor

$$(\mathcal{F}_1, \mathcal{F}_2, \phi) \mapsto i_*\mathcal{F}_1 \times_{i_*i^*j_*\mathcal{F}_2} j_*\mathcal{F}_2.$$

See EC p74 for the details. □

Under this category equivalence,

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to 0$$

corresponds to the exact sequence

$$0 \to (0, j^*\mathcal{F}, 0) \to (i^*\mathcal{F}, j^*\mathcal{F}, \phi_{\mathcal{F}}) \to (i^*\mathcal{F}, 0, 0) \to 0.$$

The sequence splits if and only if $\phi_{\mathcal{F}} = 0$.

Let Y be a subset of X. We say that a sheaf $\mathcal{F}$ has *support* in Y if $\mathcal{F}_{\bar{x}} = 0$ whenever $x \notin Y$.

Corollary 8.18. *For any closed immersion $i\colon\ Z \hookrightarrow X$, i_* defines an equivalence between the category of sheaves on Z_{et} and the category of sheaves on X_{et} with support in Z.*

Proof. This follows from the proposition and the obvious fact that $\mathcal{F}$ has support in Z if and only if it corresponds to a triple of the form $(\mathcal{F}_1, 0, 0)$. □

[18] Working mathematicians will recognize this as a comma category.

9. Cohomology: Definition and the Basic Properties.

In the last section, we showed that the category of sheaves of abelian groups $Sh(X_{et})$ is an abelian category with enough injectives. The functor

$$\mathcal{F} \mapsto \Gamma(X, \mathcal{F})\colon Sh(X_{et}) \to Ab$$

is left exact, and we define $H^r(X_{et}, -)$ to be its rth right derived functor. Explicitly, for a sheaf $\mathcal{F}$, choose an injective resolution

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \mathcal{I}^2 \to \cdots,$$

and apply the functor $\Gamma(X, -)$ to obtain a complex

$$\Gamma(X, \mathcal{I}^0) \to \Gamma(X, \mathcal{I}^1) \to \Gamma(X, \mathcal{I}^2) \to \cdots.$$

This is no longer exact (in general), and $H^r(X_{et}, \mathcal{F})$ is defined to be its r^{th} cohomology group. The theory of derived functors shows:

(a) for any sheaf $\mathcal{F}$, $H^0(X_{et}, \mathcal{F}) = \Gamma(X, \mathcal{F})$;
(b) if $\mathcal{I}$ is injective, then $H^r(X_{et}, \mathcal{I}) = 0$ for $r > 0$;
(c) a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

gives rise to a long exact sequence

$$0 \to H^0(X_{et}, \mathcal{F}') \to H^0(X_{et}, \mathcal{F}) \to H^0(X_{et}, \mathcal{F}'') \to H^1(X_{et}, \mathcal{F}') \to \cdots,$$

and the association of the long exact sequence with the short exact sequence is functorial.

Moreover, the functors $H^r(X_{et}, -)$ are uniquely determined (up to a unique isomorphism) by the properties (a), (b), (c).

Remark 9.1. We shall make frequent use of the following statement:

Let $L = L_2 \circ L_1$ where L_1 and L_2 are both left exact functors from abelian categories with enough injectives. If L_1 preserves injectives and $(R^r L_1)(X) = 0$ for some object X, then $(R^r L)(X) = (R^r L_2)(L_1 X)$.

The proof is obvious: choose an injective resolution $X \to I^\bullet$ of X, and note that the hypotheses on L_1 imply that $L_1 X \to L_1 I^\bullet$ is an injective resolution of $L_1 X$, which can be used to compute $(R^r L_2)(L_1 X)$. Now both $(R^r L)(X)$ and $(R^r L_2)(L_1 X)$ are the rth cohomology groups of $L_2(L_1 I^\bullet) = L I^\bullet$.

Remark 9.2. Let $\varphi\colon\ U \to X$ be an étale morphism. As we noted in (8.16), $\varphi^*\colon\ Sh(X_{et}) \to Sh(U_{et})$ is exact and preserves injectives. Since the composite

$$Sh(X_{et}) \xrightarrow{\varphi^*} Sh(U_{et}) \xrightarrow{\Gamma(U,-)} Ab$$

is $\Gamma(U, -)$ (recall that in this situation, φ^* is just restriction), we see that the right derived functors of $\mathcal{F} \mapsto \mathcal{F}(U)\colon\ Sh(X_{et}) \to Ab$ are $\mathcal{F} \mapsto H^r(U_{et}, \mathcal{F}|U)$. We often denote $H^r(U_{et}, \mathcal{F}|U)$ by $H^r(U_{et}, \mathcal{F})$.

In the remainder of this section, we verify that analogues of the Eilenberg-Steenrod axioms hold.

The dimension axiom. Let $x = \operatorname{Spec} k$ for some field k, and let $\bar{x} = \operatorname{Spec} k^{\mathrm{sep}}$ for some separable closure k^{sep} of k. As we observed in §6, the functor

$$\mathcal{F} \mapsto M_{\mathcal{F}} \overset{\mathrm{df}}{=} \mathcal{F}_{\bar{x}}$$

defines an equivalence from the category of sheaves on x_{et} to the category of discrete G-modules where $G = \operatorname{Gal}(k^{\mathrm{sep}}/k)$. Since $(M_{\mathcal{F}})^G = \Gamma(x, \mathcal{F})$, the derived functors of $M \mapsto M^G$ and $\mathcal{F} \mapsto \Gamma(x, \mathcal{F})$ correspond. Thus

$$H^r(x, \mathcal{F}) \cong H^r(G, M_{\mathcal{F}}).$$

In order to have

$$H^r(x, \mathcal{F}) = 0, \text{for } r > 0, \text{for all } \mathcal{F},$$

as the dimension axiom demands, we must take x to be geometric point, i.e., the spectrum of a separably closed field. Thus, as we have already seen in other contexts, it is a *geometric* point that plays the role for the étale site that a point plays for a topological space.

The exactness axiom. Let Z be a closed subvariety (or subscheme) of X, and let $U = X \smallsetminus Z$. For any sheaf $\mathcal{F}$ on X_{et}, define

$$\Gamma_Z(X, \mathcal{F}) = \operatorname{Ker}(\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F})).$$

Thus $\Gamma_Z(X, \mathcal{F})$ is the group of sections of $\mathcal{F}$ with support on Z. We sometimes omit the X from the notation. The functor $\mathcal{F} \mapsto \Gamma_Z(X, \mathcal{F})$ is obviously left exact, and we denote its r^{th} right derived functor by $H^r_Z(X, -)$ *(cohomology of $\mathcal{F}$ with support on Z).*

THEOREM 9.3. *For any sheaf $\mathcal{F}$ on X_{et} and closed $Z \subset X$, there is a long exact sequence*

$$\cdots \to H^r_Z(X, \mathcal{F}) \to H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F}) \to H^{r+1}_Z(X, \mathcal{F}) \to \cdots .$$

The sequence is functorial in the pairs $(X, X \setminus Z)$ and $\mathcal{F}$.

We shall prove this in the next subsection.

Ext**-groups.** For a fixed sheaf $\mathcal{F}_0$,

$$\mathcal{F} \mapsto \operatorname{Hom}_X(\mathcal{F}_0, \mathcal{F}) \colon Sh(X_{et}) \to Ab.$$

is left exact, and we denote its rth right derived functor by $\operatorname{Ext}^r(\mathcal{F}_0, -)$. Thus,

(a) for any sheaf $\mathcal{F}$, $\operatorname{Ext}^0(\mathcal{F}_0, \mathcal{F}) = \operatorname{Hom}(\mathcal{F}_0, \mathcal{F})$;
(b) if $\mathcal{I}$ is injective, then $\operatorname{Ext}^r(\mathcal{F}_0, \mathcal{I}) = 0$ for $r > 0$;
(c) a short exact sequence of sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

gives rise to a long exact sequence

$$\cdots \to \operatorname{Ext}^r_X(\mathcal{F}_0, \mathcal{F}') \to \operatorname{Ext}^r_X(\mathcal{F}_0, \mathcal{F}) \to \operatorname{Ext}^r_X(\mathcal{F}_0, \mathcal{F}'') \to \cdots$$

and the association of the long exact sequence with the short exact sequence is functorial.

EXAMPLE 9.4. Let $\mathbb{Z}$ denote the constant sheaf on X. For any sheaf $\mathcal{F}$ on X, the map $\alpha \mapsto \alpha(1)$ is an isomorphism $\mathrm{Hom}_X(\mathbb{Z}, \mathcal{F}) \xrightarrow{\approx} \mathcal{F}(X)$. Thus $\mathrm{Hom}_X(\mathbb{Z}, -) \cong \Gamma(X, -)$, and so $\mathrm{Ext}^r_X(\mathbb{Z}, -) \cong H^r(X_{et}, -)$.

Because $\mathrm{Hom}_X(\mathcal{F}_0, -)$ is functorial in $\mathcal{F}_0$, so also is $\mathrm{Ext}^r_X(\mathcal{F}_0, -)$.

PROPOSITION 9.5. *A short exact sequence*

$$0 \to \mathcal{F}_0' \to \mathcal{F}_0 \to \mathcal{F}_0'' \to 0$$

of sheaves on X_{et} gives rise to a long exact sequence

$$\cdots \to \mathrm{Ext}^r_X(\mathcal{F}_0'', \mathcal{F}) \to \mathrm{Ext}^r_X(\mathcal{F}_0, \mathcal{F}) \to \mathrm{Ext}^r_X(\mathcal{F}_0', \mathcal{F}) \to \cdots$$

for any sheaf $\mathcal{F}$.

PROOF. If $\mathcal{I}$ is injective, then

$$0 \to \mathrm{Hom}_X(\mathcal{F}_0'', \mathcal{I}) \to \mathrm{Hom}_X(\mathcal{F}_0, \mathcal{I}) \to \mathrm{Hom}_X(\mathcal{F}_0', \mathcal{I}) \to 0$$

is exact. For any injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$ of $\mathcal{F}$,

$$0 \to \mathrm{Hom}_X(\mathcal{F}_0'', \mathcal{I}^\bullet) \to \mathrm{Hom}_X(\mathcal{F}_0, \mathcal{I}^\bullet) \to \mathrm{Hom}_X(\mathcal{F}_0', \mathcal{I}^\bullet) \to 0$$

is an exact sequence of complexes, which, according to a standard result in homological algebra, gives rise to a long exact sequence of cohomology groups,

$$\cdots \to \mathrm{Ext}^r_X(\mathcal{F}_0'', \mathcal{F}) \to \mathrm{Ext}^r_X(\mathcal{F}_0, \mathcal{F}) \to \mathrm{Ext}^r_X(\mathcal{F}_0', \mathcal{F}) \to \cdots .$$

□

We now prove Theorem 9.3. Let

$$U \xrightarrow{j} X \xleftarrow{i} Z$$

be as in the statement of the Theorem. Let $\mathbb{Z}$ denote the constant sheaf on X defined by $\mathbb{Z}$, and consider the exact sequence (8.15)

$$0 \to j_! j^* \mathbb{Z} \to \mathbb{Z} \to i_* i^* \mathbb{Z} \to 0. \qquad (*)$$

For any sheaf $\mathcal{F}$ on X_{et},

$$\mathrm{Hom}_X(j_! j^* \mathbb{Z}, \mathcal{F}) = \mathrm{Hom}_U(j^* \mathbb{Z}, j^* \mathcal{F}) = \mathcal{F}(U),$$

and so $\mathrm{Ext}^r_X(j_! j^* \mathbb{Z}, \mathcal{F}) = H^r(U_{et}, \mathcal{F})$. From the exact sequence

$$0 \to \mathrm{Hom}(i_* i^* \mathbb{Z}, \mathcal{F}) \to \mathrm{Hom}(\mathbb{Z}, \mathcal{F}) \to \mathrm{Hom}(j_! j^* \mathbb{Z}, \mathcal{F})$$

we find that $\mathrm{Hom}_X(i_* i^* \mathbb{Z}, \mathcal{F}) = \Gamma_Z(X, \mathcal{F})$, and so $\mathrm{Ext}^r_X(i_* i^* \mathbb{Z}, \mathcal{F}) = H^r_Z(X, \mathcal{F})$. Therefore, the long exact sequence sequence of Ext's corresponding to $(*)$, as in (9.5), is the sequence required for Theorem 9.3.

Excision. Excision for topological spaces says that cohomology with support on Z should depend only on a neighbourhood of Z in X, e.g., replacing X with an open neighbourhood of Z shouldn't change $H^r_Z(X, \mathcal{F})$. The following is the analogous statementfor the étale topology.

THEOREM 9.6 (Excision). *Let $\pi\colon X' \to X$ be an étale map and let $Z' \subset X'$ be a closed subvariety (or scheme) of X' such that*

(a) *$Z \overset{df}{=} \pi(Z')$ is closed in X, and the restriction of π to Z' is an isomorphism of Z' onto Z, and*
(b) *$\pi(X' \smallsetminus Z') \subset X \smallsetminus Z$.*

Then, for any sheaf $\mathcal{F}$ on X_{et}, the canonical map $H^r_Z(X_{et}, \mathcal{F}) \to H^r_{Z'}(X'_{et}, \mathcal{F}|X')$ is an isomorphism for all r.

PROOF. Let $U' = X' \setminus Z'$ and $U = X \setminus Z$. We have a commutative diagram

$$\begin{array}{ccccc} U' & \xrightarrow{j'} & X' & \xleftarrow{i'} & Z' \\ \downarrow & & \downarrow \pi & & \downarrow\approx \\ U & \xrightarrow{j} & X & \xleftarrow{i} & Z. \end{array}$$

This gives rise to a diagram

$$\begin{array}{ccccccc} 0 & \to & \Gamma_{Z'}(X', \pi^*\mathcal{F}) & \to & \Gamma(X', \pi^*\mathcal{F}) & \to & \Gamma(U', \pi^*\mathcal{F}) \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \to & \Gamma_Z(X, \mathcal{F}) & \to & \Gamma(X, \mathcal{F}) & \to & \Gamma(U, \mathcal{F}). \end{array}$$

The first vertical arrow is induced by the remaining two. Because π^* is exact and preserves injectives, it suffices to prove the theorem for $r = 0$, i.e., to prove that the map $\Gamma_Z(X, \mathcal{F}) \to \Gamma_{Z'}(Z', \mathcal{F}|X')$ in the above diagram is an isomorphism.

Suppose $s \in \Gamma_Z(X, \mathcal{F})$ maps to zero in $\Gamma_{Z'}(X', \mathcal{F}|X')$. Then s, regarded as an element of $\Gamma(X, \mathcal{F})$, restricts to zero in $\Gamma(X', \mathcal{F})$ and $\Gamma(U, \mathcal{F})$. As the pair of maps $(X' \to X, U \to X)$ is a covering of X and $\mathcal{F}$ is a sheaf, this implies that $s = 0$.

Let $s' \in \Gamma_{Z'}(X', \mathcal{F}|X')$, and regard it as an element of $\Gamma(X', \mathcal{F})$. We have to find an $s \in \Gamma(X, \mathcal{F})$ restricting to s' in $\Gamma(X', \mathcal{F})$ and 0 in $\Gamma(U, \mathcal{F})$. The sheaf criterion will provide us with such a section once we have checked that the two section s' and 0 agree on "overlaps". Note first that $U \times_X X' = \pi^{-1}(U) = U'$, and both sections restrict to zero on U'. It remains to check that the restrictions of s' under the two maps $X' \leftleftarrows X' \times_X X'$ are equal, and this we can do on stalks. For a point in $U' \times_X U'$, the two restrictions are zero. The two maps $Z' \leftleftarrows Z' \times_X Z'$ are equal, and so the two restrictions agree at a point of $Z' \times_X Z'$. Since $X' \times_X X' = U' \times_X U' \cup Z' \times_X Z'$, this completes the proof. □

The situation in the theorem arises when $X' \to X$ is étale and X' contains a closed subscheme for which there is a morphism $s\colon Z \to X$ such that $\pi \circ s = \mathrm{id}_Z$ and $\pi^{-1}(Z) = s(Z)$.

COROLLARY 9.7. *Let x be a closed point of X. For any sheaf $\mathcal{F}$ on X, there is an isomorphism $H^r_x(X, \mathcal{F}) \to H^r_x(\operatorname{Spec} \mathcal{O}^h_{X,x}, \mathcal{F})$ where $\mathcal{O}^h_{X,x}$ is the Henselization of $\mathcal{O}_{X,x}$.*

PROOF. According to the theorem, $H^r_x(X, \mathcal{F}) = H^r_u(U, \mathcal{F})$ for any étale neighbourhood (U, u) of x such that u is the only point of U mapping to x. Such étale neighbourhoods are cofinal, and on passing to the limit over them, we obtain the required isomorphism (see below 10.8 for the behaviour of cohomology when one passes to an inverse limit over schemes). □

The homotopy axiom. Recall that the homotopy axiom says that homotopic maps induce the same maps on cohomology. There is an analogue of this statement in which homotopy equivalence is replaced by rational equivalence.

For simplicity, let X be an algebraic variety over an algebraically closed field k. Let Z be a closed subvariety of $X \times \mathbb{P}^1$ whose image under the projection map $\pi\colon\ X \times \mathbb{P}^1 \to \mathbb{P}^1$ is dense in $\mathbb{P}^1$. For any closed point $t \in \mathbb{P}^1$, $Z(t) \stackrel{\text{df}}{=} Z \cap (X \times \{t\})$ is a closed subvariety of $X \times \{t\} = X$ (the intersection should be formed in the sense of intersection theory, i.e., allowing multiplicities—see later). We can regard the $Z(t)$ for $t \in \pi(Z)$ as a continuous family of algebraic cycles on X, and we write $Z_1 \sim Z_2$ if Z_1 and Z_2 occur in such a family. We say that two algebraic cycles Z and Z' are *rationally equivalent* if there exist algebraic cycles $Z_1, Z_2, \ldots, Z_r$ such that

$$Z \sim Z_1 \sim \cdots \sim Z_r \sim Z'.$$

THEOREM 9.8. *Two morphisms φ, φ' define the same map on étale cohomology if their graphs are rationally equivalent.*

PROOF. See later §23. □

10. Čech Cohomology.

It is not practical to use the definition of the cohomology groups in terms of derived functors to compute them directly. Under mild hypotheses on X, the derived functor groups agree with the Čech groups, which are sometimes more manageable.

Definition of the Čech groups. Let $\mathcal{U} = (U_i \to X)_{i \in I}$ be an étale covering of X, and let $\mathcal{P}$ be a presheaf of abelian groups on X_{et}. Define

$$C^r(\mathcal{U}, \mathcal{P}) = \prod_{(i_0, \dots, i_r) \in I^{r+1}} \mathcal{P}(U_{i_0 \dots i_r}), \quad \text{where } U_{i_0 \dots i_r} = U_{i_0} \times_X \cdots \times_X U_{i_r}.$$

For $s = (s_{i_0 \dots i_r}) \in C^r(\mathcal{U}, \mathcal{P})$, define $d^r s \in C^{r+1}(\mathcal{U}, \mathcal{P})$ by the rule

$$(d^r s)_{i_0 \dots i_{r+1}} = \sum_{j=0}^{r+1} (-1)^j \mathrm{res}_j (s_{i_0 \dots i_{j-1} i_{j+1} \dots i_{r+1}})$$

where res_j is the restriction map corresponding to the projection map

$$U_{i_0 \dots i_{r+1}} \to U_{i_0 \dots i_{j-1} i_{j+1} \dots i_{r+1}}.$$

As in the classical case, one verifies by a straightforward calculation that

$$C^\bullet(\mathcal{U}, \mathcal{P}) \stackrel{\mathrm{df}}{=} C^0(\mathcal{U}, \mathcal{P}) \to \cdots \to C^r(\mathcal{U}, \mathcal{P}) \xrightarrow{d^r} C^{r+1}(\mathcal{U}, \mathcal{P}) \to \cdots$$

is a complex. Define

$$\check{H}^r(\mathcal{U}, \mathcal{P}) = H^r(C^\bullet(\mathcal{U}, \mathcal{P})).$$

It is called the r^{th} *Čech cohomology group* of $\mathcal{P}$ relative to the covering $\mathcal{U}$.

Note that

$$\check{H}^0(\mathcal{U}, \mathcal{P}) = \mathrm{Ker}(\prod \mathcal{P}(U_i) \rightrightarrows \prod \mathcal{P}(U_{ij})).$$

Therefore, for a sheaf $\mathcal{F}$,

$$\check{H}^0(\mathcal{U}, \mathcal{F}) = \Gamma(X, \mathcal{F}).$$

Example 10.1. Let $\mathcal{U}$ be the covering of X consisting of a single Galois covering $Y \to X$ with Galois group G. If $\mathcal{P}$ is a presheaf on X carrying disjoint unions to products, then

$$\check{H}^r(X, \mathcal{P}) = H^r(G, \mathcal{P}(Y)) (\text{group cohomology});$$

see Exercise 6.5.

A second covering $\mathcal{V} = (V_j \to X)_{j \in J}$ of X is called a *refinement* of $\mathcal{U}$ if there is a map $\tau \colon J \to I$ such that $V_j \to X$ factors through $U_{\tau j} \to X$ for all $j \in J$. The choice of a τ and X-morphisms $\varphi_j \colon V_j \to U_{\tau j}$ for each j determines a map of complexes

$$\tau^\bullet \colon C^\bullet(\mathcal{U}, \mathcal{P}) \to C^\bullet(\mathcal{V}, \mathcal{P}), \quad (\tau^r s)_{j_0 \dots j_r} = s_{\tau j_0 \dots \tau j_r} | V_{j_0 \dots j_r}.$$

As in the classical case, one verifies that the map on cohomology groups

$$\rho(\mathcal{V}, \mathcal{U}) \colon \check{H}^r(\mathcal{U}, \mathcal{P}) \to \check{H}^r(\mathcal{V}, \mathcal{P})$$

is independent of all choices. We may pass to the limit over all coverings, and so obtain the Čech cohomology groups

$$\check{H}^r(X, \mathcal{P}) \stackrel{\mathrm{df}}{=} \varinjlim_{\mathcal{U}} \check{H}^r(\mathcal{U}, \mathcal{P}).$$

They have the following properties:

(a) $\check{H}^0(X,\mathcal{F})=\Gamma(X,\mathcal{F})$ for any sheaf $\mathcal{F}$ on X;
(b) $\check{H}^r(X,\mathcal{I})=0$, $r>0$, for all injective sheaves $\mathcal{I}$.

Statement (a) is obvious. Statement (b) is proved by showing that, for each covering $(U_i\to X)$, there is an exact sequence of presheaves of abelian groups $\mathcal{Z}_\bullet$ such that

$$C^\bullet(\mathcal{U},\mathcal{F})=\mathrm{Hom}(\mathcal{Z}_\bullet,\mathcal{F})$$

for all sheaves $\mathcal{F}$. If $\mathcal{I}$ is injective as a sheaf, then it is injective as a presheaf (because a is an exact left adjoint to the inclusion functor; cf. 7.3c), and so $\mathrm{Hom}(\mathcal{Z}_\bullet,\mathcal{I})$ is exact. For the details, see the proof of EC III 2.4.

It follows that the isomorphism $\check{H}^0(X,\mathcal{F})\cong H^0(X,\mathcal{F})$ extends to an isomorphism for all r and $\mathcal{F}$ if and only if every short exact sequence

$$0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$$

of sheaves gives a long exact sequence

$$\cdots\to\check{H}^r(X,\mathcal{F}')\to\check{H}^r(X,\mathcal{F})\to\check{H}^r(X,\mathcal{F}'')\to\cdots$$

of Čech cohomology groups.

Theorem 10.2. *Assume that every finite subset of X is contained in an open affine and that X is quasi-compact (for example, X could be a quasi-projective variety). Then, for any short exact sequence of sheaves*

$$0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0,$$

the direct limit of the complexes

$$0\to C^\bullet(\mathcal{U},\mathcal{F}')\to C^\bullet(\mathcal{U},\mathcal{F})\to C^\bullet(\mathcal{U},\mathcal{F}'')\to 0$$

over the étale coverings of X is exact, and so gives rise to a long exact sequence of Čech cohomology groups. Thus

$$\check{H}^r(X,\mathcal{F})\cong H^r(X,\mathcal{F})$$

for all r and all sheaves $\mathcal{F}$.

The difficulty in proving the exactness of the direct limit of the Čech complexes is the following: because $\mathcal{F}\to\mathcal{F}''\to 0$ is exact, we know that $\mathcal{F}(U_{i_0\ldots i_n})\to\mathcal{F}''(U_{i_0\cdots i_n})$ is locally surjective, i.e., that for each $s\in\mathcal{F}''(U_{i_0\ldots i_n})$, there exists a covering $(V_j\to U_{i_0\ldots i_n})_{j\in J}$ such that $s|V_j$ lifts to $\mathcal{F}(V_j)$ for each j; the problem is that we don't know in general that the V_j can be chosen to be of the form $V_{i_0\ldots i_n}=V_{i_0}\times_X\cdots\times_X V_{i_n}$ with V_{i_m} mapping to U_{i_m}.

The key to the proof of the theorem is the following result of M. Artin (Advances in Math. 7 (1971), 282–296.) (See also Hochster and Huneke, Ann. of Math., 135 (1992), Theorem 9.2.):

> Let A be a ring, let $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ be prime ideals in A, and let $A_1,\ldots,A_r$ be the strict Henselizations of the local rings $A_{\mathfrak{p}_1},\ldots,A_{\mathfrak{p}_r}$. Then $A'\overset{\mathrm{df}}{=}A_1\otimes_A\cdots\otimes_A A_r$ has the property that any faithfully flat étale map $A'\to B$ has a section $B\to A'$.

Almost by definition, a strictly local ring has this property. The interest of the statement is that it holds for tensor products of strictly local rings obtained from a single ring.

For the rest of the proof of Theorem 10.2, see the paper of Artin or EC III 2.17.

REMARK 10.3. (a) For the Zariski topology, only a weaker result is true: for a variety or separated scheme, the Čech cohomology of a coherent $\mathcal{O}_X$-module agrees with the derived functor cohomology. This follows from the fact that, for any open affine U and exact sequence

$$0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$$

of coherent $\mathcal{O}_X$-modules,

$$0 \to \Gamma(U, \mathcal{M}') \to \Gamma(U, \mathcal{M}) \to \Gamma(U, \mathcal{M}'') \to 0$$

is exact. Thus the Čech complex corresponding to a covering $X = \cup U_i$ of X by open affines will be exact (because a finite intersection of open affines in a separated space is affine — AG 3.26)

Of course, the Čech cohomology groups of constant sheaves agree with the derived functor cohomology groups: both are zero when X is irreducible.

(b) Most of the formalism concerning the Čech cohomology of topological spaces applies in the setting of the étale site, but not all. For example, Čech cohomology can *not* be computed using alternating cochains, as Example 10.1 demonstrates. The usual proof breaks down because there may be several maps from one étale X-variety to a second.

Comparison of the first cohomology groups. On any site, the first Čech cohomology group equals the first derived functor group. We sketch the proof.

PROPOSITION 10.4. *For a sheaf $\mathcal{F}$ on X_{et}, let $\mathcal{H}^r(\mathcal{F})$ be the presheaf $U \mapsto H^r(U, \mathcal{F}|U)$. For all $r > 0$, the sheaf associated with $\mathcal{H}^r(\mathcal{F})$ is* 0.

PROOF. Consider the functors

$$Sh(X_{et}) \xrightarrow{i} Presh(X_{et}) \xrightarrow{a} Sh(X_{et}).$$

Recall from §7 that i is left exact and that a is exact. Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Then

$$\mathcal{H}^r(\mathcal{F}) = H^r(i\mathcal{I}^\bullet).$$

Because a is exact and $a \circ i = \mathrm{id}$,

$$a(\mathcal{H}^r(\mathcal{F})) \stackrel{\mathrm{df}}{=} a(H^r(i\mathcal{I}^\bullet)) = H^r(ai\mathcal{I}^\bullet) = H^r(\mathcal{I}^\bullet) = 0 \text{ for } r > 0$$

□

COROLLARY 10.5. *Let $s \in H^r(X, \mathcal{F})$ for some $r > 0$. Then there exists a covering $(U_i \to X)$ such that the image of s in each group $H^r(U_i, \mathcal{F})$ is zero.*

PROOF. Recall that, for any presheaf $\mathcal{P}$, the only sections of $\mathcal{P}$ mapping to zero in $a\mathcal{P}$ are those that are locally zero, i.e., become zero when restricted to the U_i in some covering of X. □

We now define a map $H^1(X,\mathcal{F}) \to \check{H}^1(X,\mathcal{F})$. Choose an injective embedding $\mathcal{F} \hookrightarrow \mathcal{I}$ of $\mathcal{F}$, and let $\mathcal{G}$ be the cokernel. Thus, we have an exact sequence

$$0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{G} \to 0,$$

which gives rise to an exact cohomology sequence

$$0 \to \mathcal{F}(X) \to \mathcal{I}(X) \to \mathcal{G}(X) \to H^1(X,\mathcal{F}) \to 0.$$

Let $s \in H^1(X,\mathcal{F})$, and let $t \in \mathcal{G}(X)$ map to s. According to the above corollary, there is a covering $(U_i \to X)$ such that s restricts to zero on each U_i, and so $t|U_i$ lifts to an element $\tilde{t}_i \in \mathcal{I}(U_i)$. Let $s_{ij} = \tilde{t}_j|U_{ij} - \tilde{t}_i|U_{ij}$ regarded as an element of $\mathcal{F}(U_{ij})$. One checks easily that s_{ij} is a one-cocycle.

Proposition 10.6. *The map $s \mapsto (s_{ij})$ defines an isomorphism $H^1(X,\mathcal{F}) \to \check{H}^1(X,\mathcal{F})$.*

Proof. We leave it as an exercise to the reader to find a direct proof. Alternatively, we give a proof below involving spectral sequences. □

The spectral sequence relating Čech and derived-functor cohomology. We sketch the derivation of this sequence.

One can show that the category of presheaves on X_{et} has enough injectives. The same argument as in the sheaf case shows that, for $\mathcal{I}$ an injective presheaf, $\check{H}^r(X_{et},\mathcal{I}) = 0$ for $r > 0$. Since it is obvious that a short exact sequence of presheaves gives a long exact sequence of Čech cohomology groups, we see that $\check{H}^r(X_{et},-)$ is the rth right derived functor of

$$\mathcal{P} \mapsto \check{H}^0(X_{et},\mathcal{P})\colon Presh(X_{et}) \to Ab.$$

Consider the sequence of functors

$$Sh(X_{et}) \xrightarrow{i} Presh(X_{et}) \xrightarrow{\check{H}^0(X_{et},-)} Ab.$$

The rth right derived functor of i is $\mathcal{H}^r(-)$. We have already noted that i preserves injectives, and so a theorem of Grothendieck (Weibel 1994, 5.8.3) provides us with a spectral sequence:

$$\check{H}^r(X_{et},\mathcal{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X_{et},\mathcal{F}).$$

According to (10.5), $\check{H}^0(X_{et},\mathcal{H}^s(\mathcal{F})) = 0$ for $s > 0$. Thus, for a sheaf $\mathcal{F}$,

$$\check{H}^r(X,\mathcal{F}) \cong H^r(X,\mathcal{F}) \qquad \text{for } r = 0, 1,$$

and there is an exact sequence

$$0 \to \check{H}^2(X,\mathcal{F}) \to H^2(X,\mathcal{F}) \to \check{H}^1(X,\mathcal{H}^1(\mathcal{F})) \to \check{H}^3(X,\mathcal{F}) \to H^3(X,\mathcal{F}).$$

Similarly, for any étale covering $\mathcal{U} = (U_i \to X)$ of X, there exists a spectral sequence

$$\check{H}^r(\mathcal{U},\mathcal{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X_{et},\mathcal{F}).$$

The Mayer-Vietoris sequence. When $\mathcal{U} = (U_i \to X)$ is a open covering of X (in the Zariski sense), then the Čech cohomology groups can be computed using alternating cochains. For example, if $X = U_0 \cup U_1$, then the Čech cohomology groups of a presheaf $\mathcal{P}$ are the cohomology groups of the complex

$$\Gamma(U_0, \mathcal{P}) \times \Gamma(U_1, \mathcal{P}) \to \Gamma(U_0 \cap U_1, \mathcal{P});$$

in particular, $\check{H}^r(\mathcal{U}, \mathcal{P}) = 0$ for $r \geq 2$.

THEOREM 10.7. *Let $X = U_0 \cup U_1$ (union of two open subsets). For any sheaf $\mathcal{F}$ on X_{et}, there is an infinite exact sequence*

$$\cdots \to H^s(X, \mathcal{F}) \to H^s(U_0, \mathcal{F}) \oplus H^s(U_1, \mathcal{F}) \to H^s(U_0 \cap U_1, \mathcal{F}) \to H^{s+1}(X, \mathcal{F}) \to \cdots.$$

PROOF. On taking $\mathcal{P} = \mathcal{H}^s(\mathcal{F})$, $s > 0$, in the above discussion, we obtain an exact sequence

$$0 \to \check{H}^0(\mathcal{U}, H^s(\mathcal{F})) \to H^s(U_0, \mathcal{F}) \oplus H^s(U_1, \mathcal{F}) \to H^s(U_0 \cap U_1, \mathcal{F}) \to \check{H}^1(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) \to 0.$$

In the spectral sequence

$$\check{H}^r(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) \Rightarrow H^{r+s}(X, \mathcal{F}),$$

$\check{H}^r(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) = 0$ unless $r = 0, 1$. The spectral sequence therefore gives exact sequences

$$0 \to \check{H}^1(\mathcal{U}, \mathcal{H}^s(\mathcal{F})) \to H^{s+1}(X, \mathcal{F}) \to \check{H}^0(\mathcal{U}, \mathcal{H}^{s+1}(\mathcal{F})) \to 0, \quad \text{all } s \geq 0.$$

We can splice the sequences together to get the sequence in the statement of the theorem. □

Behaviour of cohomology with respect to inverse limits of schemes. Let I be a directed set, and let $(X_i)_{i \in I}$ be a projective system of schemes indexed by I. If the transition maps

$$X_i \xleftarrow{\varphi_i^j} X_j, \quad i \leq j,$$

are affine, i.e., $(\varphi_i^j)^{-1}(U)$ is affine for any open affine $U \subset X_i$, then the inverse limit scheme $X_\infty = \varprojlim X_i$ exists. For example, if all the X_i are affine, say, $X_i = \operatorname{Spec}(A_i)$, then we have a direct system of rings $(A_i)_{i \in I}$ and $X_\infty = \operatorname{Spec} A_\infty$ where $A = \varinjlim A_i$.

THEOREM 10.8. *Let I be a directed set, and let $(X_i)_{i \in I}$ be an inverse system of X-schemes. Assume that all the X_i are quasicompact and that the maps $X_i \leftarrow X_j$ are all affine. Let $X_\infty = \varprojlim X_i$, and, for any sheaf $\mathcal{F}$ on X, let $\mathcal{F}_i$ be its inverse image on X_i, $i \in I \cup \{\infty\}$. Then*

$$\varinjlim{}_i H^r(X_{i,et}, \mathcal{F}_i) \cong H^r(X_\infty, \mathcal{F}_\infty).$$

PROOF. The proof in the general case is rather difficult. It depends on the fact that the category of étale schemes of finite type over X_∞ is the direct limit of the categories of such schemes over the X_i. See SGA4, VII.5.8, or Artin 1962, III.3.

However, it is much easier to prove the theorem for the Čech cohomology (see EC III.3.17). When 10.2 applies, one can recover the theorem for the derived functor groups. □

11. Principal Homogeneous Spaces and H^1.

In the last section, we showed that the $H^1(X_{et}, \mathcal{F})$ coincides with $\check{H}^1(X_{et}, \mathcal{F})$. In this section, we explain how to interpret $\check{H}^1(X_{et}, \mathcal{F})$ as the group of principal homogeneous spaces for $\mathcal{F}$. Since it is no more difficult, except that we have to be a little more careful with the definitions, we work with sheaves of noncommutative groups.

Definition of the first Čech group. Let $\mathcal{U} = (U_i \to X)_{i\in I}$ be an étale covering of X, and let $\mathcal{G}$ be a sheaf of groups on X_{et} (not necessarily commutative). As in the last section, we write $U_{ij\ldots}$ for $U_i \times_X U_j \times_X \cdots$. A 1-*cocycle for* $\mathcal{U}$ *with values* in $\mathcal{G}$ is a family $(g_{ij})_{(i,j)\in I\times I}$ with $g_{ij} \in \mathcal{G}(U_{ij})$ such that

$$(g_{ij}|U_{ijk}) \cdot (g_{jk}|U_{ijk}) = g_{ik}|U_{ijk}, \text{ all } i, j, k.$$

Two cocycles g and g' are *cohomologous*, denoted $g \sim g'$, if there is a family $(h_i)_{i\in I}$ with $h_i \in \mathcal{G}(U_i)$ such that

$$g'_{ij} = (h_i|U_{ij}) \cdot g_{ij} \cdot (h_j|U_{ij})^{-1}, \text{ all } i, j.$$

The set of 1-cocycles modulo $\sim$ is denoted $\check{H}^1(X_{et}, \mathcal{G})$. It is not in general a group, but it does have a distinguished element represented by the 1-cocycle (g_{ij}) with $g_{ij} = 1$ for all i, j.

A sequence

$$1 \to \mathcal{G}' \to \mathcal{G} \to \mathcal{G}'' \to 0$$

of sheaves of groups is said to be *exact* if

$$1 \to \mathcal{G}'(U) \to \mathcal{G}(U) \to \mathcal{G}''(U)$$

is exact for all $U \to X$ étale and $\mathcal{G} \to \mathcal{G}''$ is locally surjective. Such a sequence gives rise to a sequence of sets

$$1 \to \mathcal{G}'(X) \to \mathcal{G}(X) \to \mathcal{G}''(X) \to \check{H}^1(X, \mathcal{G}') \to \check{H}^1(X, \mathcal{G}) \to \check{H}^1(X, \mathcal{G}'')$$

that is exact in the following sense: the image of each arrow is exactly the set mapped to the distinguished element by the following arrow.

Principal homogeneous spaces. Let G be a group and S a set on which G acts on the right. Then S is said to be a principal homogenous space (or torsor) for G if, for one (hence every) $s \in S$, the map $g \mapsto gs\colon\ G \to S$ is a bijection.

Let $\mathcal{G}$ be a sheaf of groups on X_{et}, and let $\mathcal{S}$ be a sheaf of sets on which $\mathcal{G}$ acts on the right. Then $\mathcal{S}$ is called a *principal homogeneous space* for $\mathcal{G}$ if

(a) there exists a étale covering $(U_i \to X)_{i\in I}$ of X such that, for all i, $\mathcal{S}(U_i) \neq \emptyset$; and
(b) for every $U \to X$ étale and $s \in \Gamma(U, \mathcal{S})$, the map $g \mapsto sg\colon\ \mathcal{G}|U \to \mathcal{S}|U$ is an isomorphism of sheaves.

A principal homogeneous space $\mathcal{S}$ is *trivial* if it is isomorphic (as a sheaf with a right action of $\mathcal{G}$) to $\mathcal{G}$ acting on itself by right multiplication, or, equivalently, if $\mathcal{S}(X) \neq \emptyset$. The axioms require $\mathcal{S}$ to be locally isomorphic to the trivial principal homogenous space.

We say that the covering $(U_i \to X)_{i\in I}$ *splits* $\mathcal{S}$ if $\mathcal{S}(U_i) \neq \emptyset$.

Let $\mathcal{S}$ be a principal homogenous space for $\mathcal{G}$. Let $(U_i \to X)_{i\in I}$ be an étale covering of X that splits $\mathcal{S}$, and choose an $s_i \in \mathcal{S}(U_i)$ for each i. Because of condition (b), there exists a unique $g_{ij} \in \mathcal{G}(U_{ij})$, such that

$$(s_i|U_{ij}) \cdot g_{ij} = s_j|U_{ij}.$$

Then $(g_{ij})_{I\times I}$ is a cocycle, because (omitting the restrictions signs)

$$s_i \cdot g_{ij} \cdot g_{jk} = s_k = s_i \cdot g_{ik}.$$

Moreover, replacing s_i with $s_i' = s_i \cdot h_i$, $h_i \in \mathcal{G}(U_i)$ leads to a cohomologous cocycle. Thus, $\mathcal{S}$ defines a class $c(\mathcal{S})$ in $\check{H}^1(\mathcal{U}, \mathcal{G})$ where $\mathcal{U} = (U_i \to X)_{i\in I}$.

Proposition 11.1. *The map $\mathcal{S} \mapsto c(\mathcal{S})$ defines a bijection from the set of isomorphism classes of principal homogenous spaces for $\mathcal{G}$ split by $\mathcal{U}$ to $\check{H}^1(\mathcal{U}, \mathcal{G})$.*

Proof. Let $\alpha\colon\ \mathcal{S} \to \mathcal{S}'$ be an isomorphism of $\mathcal{G}$-sheaves, and choose $s_i \in \mathcal{S}(U_i)$. Then $\alpha(s_i) \in \mathcal{S}'$, and (omitting the restriction signs)

$$s_i \cdot g_{ij} = s_j \Rightarrow \alpha(s_i) \cdot g_{ij} = \alpha(s_j).$$

Therefore the 1-cocycle defined by the family $(\alpha(s_i))$ equals that defined by (s_i). This shows that $c(\mathcal{S})$ depends only on the isomorphism class of $\mathcal{S}$, and so $\mathcal{S} \mapsto c(\mathcal{S})$ does define a map from the set of isomorphism classes.

Suppose that $c(\mathcal{S}) = c(\mathcal{S}')$. Then we may choose sections $s_i \in \mathcal{S}(U_i)$ and $s_i' \in \mathcal{S}'(U_i)$ that define the same 1-cocycle (g_{ij}). Suppose there exists a $t \in \mathcal{S}(X)$. Then

$$t|U_i = s_i \cdot g_i$$

for a unique $g_i \in \mathcal{G}(U_i)$; from the equality $(t|U_i)|U_{ij} = (t|U_j)|U_{ji}$, we find that

$$(g_i|U_{ij}) = g_{ij} \cdot (g_j|U_{ij}) \qquad (*)$$

Because $\mathcal{S}$ is a sheaf, $t \mapsto (g_i)_{i\in I}$ is a bijection from $\mathcal{S}(X)$ onto the set of families $(g_i)_{i\in I}$, $g_i \in \mathcal{G}(U_i)$, satisfying $(*)$. A similar statement holds for $\mathcal{S}'$, and so there is a canonical bijection $\mathcal{S}(X) \to \mathcal{S}'(X)$. For any $V \to X$, we can apply the same argument to the covering $(U_i \times_X V \to V)$ of V and the elements $s_i|U_i \times_X V$ and $s_i'|U_i \times_X V$ to obtain a canonical bijection $\mathcal{S}(V) \to \mathcal{S}'(V)$. The family of these bijections is an isomorphism of $\mathcal{G}$-sheaves $\mathcal{S} \to \mathcal{S}'$.

Thus, the map is an injection into $\check{H}^1(\mathcal{U}, \mathcal{G})$, and it remains to prove that it is surjective. Let $(g_{ij})_{I\times I}$ be a 1-cocycle for the covering $\mathcal{U} = (U_i \to X)$. For any $V \to X$ étale, let $(V_i \to V)$ be the covering of V with $V_i = U_i \times_X V$. Define $\mathcal{S}(V)$ to be the set of families $(g_i)_{i\in I}$, $g_i \in \mathcal{G}(V_i)$, such that

$$(g_i|V_{ij}) = g_{ij} \cdot (g_j|V_{ij}).$$

Showing that this defines a sheaf of $\mathcal{G}$-sets, and that $c(\mathcal{S})$ is represented by (g_{ij}) involves only routine checking. □

Remark 11.2. When $\mathcal{G}$ is the sheaf defined by a group scheme G over X, one would like to know that every principal homogeneous space $\mathcal{S}$ for $\mathcal{G}$ is the sheaf defined by a scheme. In general, this can be a difficult question in descent theory: we know that $\mathcal{S}|U_i$ is represented by a scheme over U_i for each $i \in I$, namely, by G_{U_i}, and would like to know that this implies that $\mathcal{S}$ itself is represented by a scheme S over X.

It is known that $\mathcal{S}$ is defined by a scheme when, for example, $\mathcal{G}$ is defined by an scheme affine over X. See EC III 4.3 for a summary of what was known in 1978—not much more is known today.

EXAMPLE 11.3. Let $\mathcal{G}$ be the constant sheaf on X_{et} defined by a finite group G. A Galois covering of X with group G is a principal homogeneous space for $\mathcal{G}$, and every principal homogeneous space arises from a Galois covering. When X is connected, the Galois coverings of X with group G are classified by the continuous homomorphisms $\pi_1(X, \bar{x}) \to G$, where $\bar{x} \to X$ is any geometric point of X. Thus, for a connected X, there is a canonical isomorphism

$$H^1(X_{et}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{conts}}(\pi_1(X, \bar{x}), G).$$

Interpretation of $\check{H}^1(X, \mathrm{GL}_n)$. Let $L_n(X)$ be the set of isomorphism classes of locally free sheaves of $\mathcal{O}_X$-modules of rank n on X for the Zariski topology.

THEOREM 11.4. *There are natural bijections*

$$L_n(X_{zar}) \leftrightarrow \check{H}^1(X_{zar}, \mathrm{GL}_n) \leftrightarrow \check{H}^1(X_{et}, \mathrm{GL}_n) \leftrightarrow \check{H}^1(X_{fl}, \mathrm{GL}_n).$$

PROOF. (Sketch). Let $\mathfrak{M}$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank n on X_{zar}. By definition, this means that there is an open covering $X = \cup U_i$ such that $\mathcal{M}|U_i \approx \mathcal{O}_{U_i}^n$ for all n.

Observe, that we may take the U_i to be affine, say, $U_i = \operatorname{Spec} A_i$, and then $\mathcal{M}|U_i$ is the sheaf associated with the A_i-module A_i^n—in particular, $\mathcal{M}$ is coherent.

Now choose isomorphisms $\theta_i \colon \mathcal{M}|U_i \to \mathcal{O}_{U_i}^n$ for each i. Then the family

$$\theta_{ij} \stackrel{\mathrm{df}}{=} (\theta_i|U_{ij}) \circ (\theta_j|U_{ij})^{-1} \in GL_n(U_{ij}), \quad (i,j) \in I \times I$$

is a one-cocycle, because (omitting the restriction signs)

$$\theta_{ij} \circ \theta_{jk} = \theta_i \circ \theta_j^{-1} \circ \theta_j \circ \theta_k^{-1} = \theta_i \circ \theta_k^{-1} = \theta_{ik} \quad (\text{on } U_{ijk}).$$

Moreover, if θ_i is replaced by $h_i \circ \theta_i$, then θ_{ij} is replaced by

$$\theta'_{ij} = h_i \circ \theta_{ij} \circ h_j^{-1},$$

and so the class of (θ_{ij}) in $\check{H}^1(\mathcal{U}, GL_n)$, $\mathcal{U} = (U_i)$, depends only on $\mathcal{M}$. In this way, we get a well-defined map from $L_n(X_{zar}) \to \check{H}^1(X_{zar}, GL_n)$, and it is not difficult to show that this is a bijection (cf. the proof of 11.1).

We can define $L_n(X_{et})$ and $L_n(X_{fl})$ similarly to $L_n(X_{zar})$, and a similar argument shows that there are bijections $L_n(X_*) \to \check{H}^1(X_*, \mathrm{GL}_n)$, $* = et$ or fl. Thus, it remains to show that the maps $\mathcal{M} \mapsto \mathcal{M}^{et}$ and $\mathcal{M} \mapsto \mathcal{M}^{fl}$ give[19] bijections $L_n(X_{zar}) \to L_n(X_{et})$ and $L_n(X_{zar}) \to L_n(X_{fl})$. We treat the flat case. We have to show:

(a) every locally free sheaf of $\mathcal{O}_{X_{fl}}$-modules is of the form $\mathcal{M}^{fl}$ for some coherent sheaf $\mathcal{M}$ of $\mathcal{O}_{X_{zar}}$-modules;
(b) let $\mathcal{M}$ be a coherent sheaf of $\mathcal{O}_{X_{zar}}$-modules; if $\mathcal{M}^{fl}$ is locally free, then so also is $\mathcal{M}$.
(c) for $\mathcal{M}$ and $\mathcal{N}$ in $L_n(X_{zar})$, $\mathcal{M} \approx \mathcal{N}$ if and only if $\mathcal{M}^{fl} \approx \mathcal{N}^{fl}$.

[19]The sheaf $\mathcal{M}^{et}$ is defined in §6; the definition of $\mathcal{M}^{fl}$ is similar.

Clearly, it suffices to verify these statements for X affine.

In order to prove (b), we shall need the following result from commutative algebra.

11.5. Let A be a ring (commutative and Noetherian, as always), and let M be a finitely generated A-module. The following are equivalent:

(a) for all maximal ideals (and hence for all prime ideals) $\mathfrak{m}$ of A, the $A_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$ is free;
(b) there is a finite set of elements $f_1, \ldots, f_n$ in A such that $A = (f_1, \ldots, f_n)$ and M_{f_i} is a free A_{f_i}-module for all i;
(c) M is projective.

See, for example, Eisenbud[20] 1995, A3.2. Of course, (b) says that the coherent sheaf $\mathcal{M}$ on $\operatorname{Spec} A$ defined by M is locally free (for the Zariski topology).

Now let M be a finitely generated A-module, and let $A \to B$ be a faithfully flat homomorphism. Recall that for M to be projective means that $\operatorname{Hom}_A(M, -)$ is an exact functor. But, for any A-module N,

$$B \otimes_A \operatorname{Hom}_A(M, N) = \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N),$$

and so if $B \otimes_A M$ is projective, then so also is M.

Now suppose that the coherent sheaf $\mathcal{M}$ of $\mathcal{O}_X$-modules defined by M becomes free on some flat covering $(U_i \xrightarrow{\varphi_i} X)_{i \in I}$ of X. At the cost of possibly enlarging I, we may suppose that each U_i is affine. But $\varphi_i(U_i)$ is open in X, and X is quasicompact, and so we may assume I to be finite. Then $Y = \coprod U_i$ is affine. Thus, there we have a faithfully flat map $A \to B$ such that $B \otimes_A M$ is free. In particular, $B \otimes_A M$ is projective, and so M is projective. This proves (b).

In order to prove (a), we shall again need a result from commutative algebra. Let $A \to B$ be a faithfully flat homomorphism. For a B-module N, let N_0 and N_1 be the $B \otimes_A B$-modules obtained from N by tensoring with

$$b \mapsto 1 \otimes b \colon B \to B \otimes_A B, \quad b \mapsto b \otimes 1 \colon B \to B \otimes_A B$$

respectively. To give an A-module M such that $B \otimes_A M = N$ is to give an isomorphism $\theta\colon\ N_0 \to N_1$ satisfying a certain natural cocycle condition (see Waterhouse[21] 1979, 17.2). Since the composites of the two maps with $A \to B$ are equal, $(B \otimes_A M)_0 = (B \otimes_A M)_1$ and so θ for $N = B \otimes_A M$ can be taken to be the identity map. Conversely, given a θ satisfying the cocycle condition, we define M to be the submodule of N on which the maps $N \xrightarrow{\text{canonical}} N_1$ and $N \xrightarrow{\text{canonical}} N_0 \xrightarrow{\theta} N_1$ agree.

Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{X_{fl}}$-modules, and suppose that $\mathcal{M}$ becomes equal to $\mathcal{N}^{fl}$ on some flat covering, which (as before) we can take to consist of a single map $Y \to X$ with Y affine. Thus $Y \to X$ corresponds to a faithfully flat map $A \to B$, and $\mathcal{M}|Y$ is the sheaf defined by a B-module N. The composites $Y \times_X Y \rightrightarrows Y \to X$ are equal, and so the two restrictions of $\mathcal{M}$ to $Y \times_X Y$ are equal. The identity map $\mathcal{M}|Y \times_X Y \to \mathcal{M}|Y \times_X Y$ defines an isomorphism $\theta\colon\ N_0 \to N_1$ which satisfies the cocycle condition in the last paragraph. Therefore, the pair (N, θ) arises from an A-module M. Finally, because of the construction of M, $\mathcal{M}$ is the sheaf associated with M.

[20] *Commutative Algebra*, Springer.
[21] *Introduction to Affine Group Schemes*, Springer.

This completes the sketch of the proof of (a), and we omit the proof of (c). $\square$

COROLLARY 11.6. *There is a canonical isomorphism* $H^1(X_{et}, \mathbb{G}_m) \cong Pic(X)$.

PROOF. By definition, $Pic(X)$ is the group of isomorphism classes of locally free sheaves of $\mathcal{O}_X$-modules of rank 1 for the Zariski topology. Thus the case $n = 1$ of the theorem shows that $Pic(X) \cong \check{H}^1(X_{et}, \mathbb{G}_m)$. But, because $\mathbb{G}_m$ is commutative, this equals $H^1(X_{et}, \mathbb{G}_m)$. $\square$

In the case $X = \operatorname{Spec} K$, K a field, the corollary says that $H^1(X_{et}, \mathbb{G}_m) = 0$. This statement is often called Hilbert's Theorem 90, although it is a considerable generalization of the original theorem (see FT 5.19 for the original theorem). The corollary is often referred to as Hilbert's Theorem 90 also.

REMARK 11.7. It is in fact slightly easier to prove the corollary than the theorem. Corresponding to the continuous morphism $\pi\colon X_{fl} \to X_{zar}$, there is a Leray spectral sequence $H^r(X_{zar}, R^s\pi_*\mathbb{G}_m) \to H^{r+s}(X_{fl}, \mathbb{G}_m)$ where $R^s\pi_*\mathbb{G}_m$ is the sheaf associated with the presheaf $U \mapsto H^r(U_{zar}, \mathbb{G}_m)$ (see the next section). This spectral sequence gives a map $H^1(X_{zar}, \mathbb{G}_m) \to H^1(X_{fl}, \mathbb{G}_m)$, which will be an isomorphism if $R^1\pi_*\mathbb{G}_m = 0$. To check this, it suffices to show that the stalks of $R^1\pi_*\mathbb{G}_m$ are zero, and this amounts to proving that $H^1(Y_{fl}, \mathbb{G}_m) = 0$ when $Y = \operatorname{Spec} \mathcal{O}_{X,x}$ for some $x \in X$. Thus, we have to give the argument in the above proof only in the case that A is a local ring and $n = 1$. This is a little easier.

Nonabelian H^2. It is possible to define $H^2(X_{et}, \mathcal{G})$ for any sheaf of groups $\mathcal{G}$, not necessarily abelian (in fact, for a more general object called a band (lien in French)). The set classifies equivalence classes of gerbs (gerbes in French) bound by $\mathcal{G}$. For a brief, but not entirely reliable[22], summary of this theory, see Deligne et al, *Hodge Cycles, Motives, and Shimura Varieties*, Lecture Notes in Math. 900, Springer, 1982, pp 220–228.

[22] Some of the statements in the first paragraph on p223 apply only when F is commutative.

12. Higher Direct Images; the Leray Spectral Sequence.

Higher direct images. Let $\pi\colon\ Y \to X$ be a morphism of varieties (or schemes). Recall that, for a sheaf $\mathcal{F}$ on Y_{et}, we defined $\pi_*\mathcal{F}$ to be the sheaf on X_{et} with

$$\Gamma(U, \pi_*\mathcal{F}) = \Gamma(U_Y, \mathcal{F}), \quad U_Y \overset{\text{df}}{=} U \times_X Y.$$

The functor $\pi_*\colon\ Sh(Y_{et}) \to Sh(X_{et})$ is left exact, and hence we can consider its right derived functors $R^r\pi_*$. We call the sheaves $R^r\pi_*\mathcal{F}$ the *higher direct images* of $\mathcal{F}$.

Proposition 12.1. *For any $\pi\colon\ Y \to X$ and sheaf $\mathcal{F}$ on Y_{et}, $R^r\pi_*\mathcal{F}$ is the sheaf on X_{et} associated with the presheaf $U \mapsto H^r(U_Y, \mathcal{F})$.*

Proof. Let π_p be the functor $Presh(Y_{et}) \to Presh(X_{et})$ sending a presheaf $\mathcal{P}$ on Y_{et} to the presheaf $U \mapsto \Gamma(U_Y, \mathcal{P})$ on X_{et} — it is obviously exact. From the definition of π_*,

$$\begin{array}{ccc} Presh(Y_{et}) & \xrightarrow{\pi_p} & Presh(X_{et}) \\ \uparrow i & & \downarrow a \\ Sh(X_{et}) & \xrightarrow{\pi_*} & Sh(X_{et}) \end{array}$$

commutes (i is the functor "regard a sheaf as a presheaf"). Let $\mathcal{F} \to \mathcal{I}^\bullet$ be an injective resolution of $\mathcal{F}$. Then, because a and π_p are exact,

$$R^r\pi_*\mathcal{F} \overset{\text{df}}{=} H^r(\pi_*\mathcal{I}) = H^r(a \circ \pi_p \circ i\mathcal{I}^\bullet) = a \circ \pi_p(H^r(i\mathcal{I}^\bullet)).$$

As we have already noted (proof of 10.4), $H^r(i\mathcal{I}^\bullet)$ is the presheaf $U \mapsto H^r(U, \mathcal{F})$, and so $\pi_p(H^r(i\mathcal{I}^\cdot))$ is the presheaf $U \mapsto H^r(U_Y, \mathcal{F})$. □

Corollary 12.2. *The stalk of $R^r\pi_*\mathcal{F}$ at $\bar{x} \to X$ is $\varinjlim H^r(U_Y, \mathcal{F})$ where the limit is over all étale neighbourhoods (U, u) of $\bar{x}$.*

Proof. By definition, $\varinjlim H^r(U_Y, \mathcal{F})$ is the stalk of $\pi_p(R^r i(\mathcal{F}))$, which equals the stalk of $a\pi_p(R^r i(\mathcal{F}))$ by (7.15). □

Example 12.3. If $\pi\colon\ Y \hookrightarrow X$ is a closed immersion, then π_* is exact, and so $R^r\pi_*\mathcal{F} = 0$ for $r > 0$. More generally, if $\pi\colon\ Y \to X$ is a finite map, then $R^r\pi_*\mathcal{F} = 0$ for $r > 0$ (again π_* is exact 8.4).

Example 12.4. Assume X is connected and normal, and let $g\colon\ \eta \to X$ be the inclusion of the generic point of X. Then

$$(R^r g_*\mathcal{F})_{\bar{x}} = H^r(\operatorname{Spec} K_{\bar{x}}, \mathcal{F})$$

where $K_{\bar{x}}$ is the field of fractions of $\mathcal{O}_{X,\bar{x}}$. Moreover, in this case g_* takes a constant sheaf on η to a constant sheaf, and a locally constant sheaf on η to a locally constant sheaf on X. We make this explicit.

Let K^{sep} be a separable closure of K, and let $G = \operatorname{Gal}(K^{\text{sep}}/K)$. Let $M = M_{\mathcal{F}}$, the G-module corresponding to $\mathcal{F}$ (as in §6). Then $\mathcal{F}$ is constant if G acts trivially on M and locally constant if the action of G on M factors through a finite quotient.

The map $\operatorname{Spec} K^{\text{sep}} \to \operatorname{Spec} K \to X$ is a geometric point of X, which we denote $\bar{\eta}$. The strictly local ring $\mathcal{O}_{X,\bar{\eta}}$ is K^{sep} because the normalization of X in any finite extension L of K contained in K^{sep} will be étale over X on some nonempty open subset. Thus $(R^r g_*\mathcal{F})_{\bar{\eta}} = M$ if $r = 0$, and is 0 otherwise.

In general, $K_{\bar{x}}$ will be the union of all finite extensions L of K contained in K^{sep} such that the normalization of X in L is unramified at *some* point lying over x. Thus $(R^r g_*\mathcal{F})_{\bar{x}} = H^r(H, M)$ where $H = \text{Gal}(K^{\text{sep}}/K_{\bar{x}})$.

For example, let $X = \operatorname{Spec} A$ with A a Dedekind domain. Let $\tilde{A}$ be the integral closure of A in K^{sep}. A closed point x of X is a nonzero prime ideal $\mathfrak{p}$ of A, and the choice of a prime ideal $\tilde{\mathfrak{p}}$ of $\tilde{A}$ lying over $\mathfrak{p}$ determines a geometric point $\bar{x} \to x \to X$ of X. In this case, $K_{\bar{x}} = (K^{\text{sep}})^{I(\tilde{\mathfrak{p}})}$ where $I(\tilde{\mathfrak{p}}) \subset G$ is the inertia group of $\tilde{\mathfrak{p}}$. Thus $(R^r g_*\mathcal{F})_{\bar{x}} = H^r(I(\tilde{\mathfrak{p}}), M)$.

Example 12.5. Assume X is integral but not necessarily normal. Then $g\colon \eta \to X$ will factor as $g\colon \eta \to \tilde{X} \to X$ where $\tilde{X}$ is the normalization of X in η. For example:

$$\begin{aligned} X &= \operatorname{Spec} A, && A \text{an integral domain;} \\ \eta &= \operatorname{Spec} K, && K \text{the field of fractions of } A; \\ \tilde{X} &= \operatorname{Spec} \tilde{A}, && \tilde{A} \text{the integral closure of } A \text{in } K. \end{aligned}$$

Since $\tilde{X} \to X$ is finite, this shows that g factors into a composite of maps of the type considered in the two examples above.

For the map $\tilde{X} \to X$, the direct image of a constant sheaf need not be constant. Consider for example the map $\pi\colon \mathbb{A}^1 \to \{Y^2 = X^3\}$, $t \mapsto (t^2, t^3)$. For a constant sheaf Λ on $\mathbb{A}^1$, $(\pi_*\Lambda)_{\bar{x}} = \Lambda$ unless $x = 0$ in which case it is $\Lambda \oplus \Lambda$.

Example 12.6. Let X be an integral scheme, and consider the inclusion $i\colon z \to X$ of a point of z of X, not necessarily generic or closed. Let Z be the closure of z in X, so that Z is irreducible and z is its generic point. Then i factors into

$$z \to \tilde{Z} \to Z \hookrightarrow X,$$

which is a composite of maps of the types considered in the last three examples.

The Leray spectral sequence. Since we shall need to use it several times, I state Grothendieck's theorem on the existence of spectral sequences.

Theorem 12.7. *Let* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{C}$ *be abelian categories, and assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *have enough injectives. Let* $F\colon \mathbf{A} \to \mathbf{B}$ *and* $G\colon \mathbf{B} \to \mathbf{C}$ *be left exact functors, and assume that* $(R^r G)(FI) = 0$ *for* $r > 0$ *if* I *is injective (this is true for example if* F *takes injectives to injectives). Then there is a spectral sequence*

$$E_2^{rs} = (R^r G)(R^s F)(A) \Rightarrow R^{r+s}(FG)(A).$$

There is a three page explanation of spectral sequences in EC pp307–309, a 14 page explanation in Shatz, Profinite Groups, Arithmetic, and Geometry, Princeton, 1972, II.4, and a 45 page explanation in Weibel, Chapter V.

Theorem 12.8 (Leray spectral sequence). *Let* $\pi\colon Y \to X$ *be a morphism of varieties (or schemes). For any sheaf* $\mathcal{F}$ *on* Y_{et}, *there is a spectral sequence*

$$H^r(X_{et}, R^s\pi_*\mathcal{F}) \Rightarrow H^{r+s}(Y_{et}, \mathcal{F}).$$

Proof. The functors $\pi_*\colon Sh(Y_{et}) \to Sh(X_{et})$ and $\Gamma(X, -)\colon Sh(X_{et})$ are both left exact, and their composite is $\Gamma(Y, -)$. Since π_* preserves injectives, this theorem is a special case of the preceding theorem. $\square$

The original Leray spectral sequence has exactly the same form as the above sequence: for any continuous map $\pi\colon\ Y \to X$ of topological spaces and sheaf $\mathcal{F}$ on Y, there is a spectral sequence

$$H^r(X, R^s\pi_*\mathcal{F}) \Rightarrow H^{r+s}(Y, \mathcal{F}).$$

EXAMPLE 12.9. Let X be a variety over a field k, and let $\pi\colon\ X \to P$ be the map from X to a point $P = \mathrm{Specm}(k)$. Let $\bar{k}$ be the separable closure of k, and let $\bar{X}$ be the variety over $\bar{k}$ obtained from X by base change. Let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. When we identify $Sh(P_{et})$ with the category $\mathbf{Mod}_\Gamma$ of discrete Γ-modules, π_* becomes identified with the functor $Sh(X_{et}) \to \mathbf{Mod}_\Gamma$, $\mathcal{F} \mapsto \mathcal{F}(\bar{X}) \stackrel{\mathrm{df}}{=} \varinjlim_{k'} \mathcal{F}(X_{k'})$ (limit over the subfields k' of $\bar{k}$ finite over k). Thus, in this case, the Leray spectral sequence becomes

$$H^r(\Gamma, H^s(\bar{X}_{et}, \mathcal{F})) \to H^{r+s}(X, \mathcal{F})$$

where

$$H^s(\bar{X}, \mathcal{F}) \stackrel{\mathrm{df}}{=} \varinjlim_{k'} H^s(X_{k'}, \mathcal{F}|X_{k'}) = H^s(\bar{X}, \varphi^*\mathcal{F}), \quad \varphi\colon \bar{X} \to X.$$

13. The Weil-Divisor Exact Sequence and the Cohomology of $\mathbb{G}_m$.

We saw in the last section that $H^1(X_{et}, \mathbb{G}_m) = H^1(X_{zar}, \mathcal{O}_X^\times) = \mathrm{Pic}(X)$, and so this group is known (more accurately, it has a name, and so we can pretend we know it). We wish now to try to compute $H^r(X_{et}, \mathbb{G}_m)$ for all r. This is a key case, because once we know $H^r(X_{et}, \mathbb{G}_m)$, we will be able to use the Kummer sequence (7.9) to compute $H^r(X_{et}, \mu_n)$ for all n prime to the residue characteristics.

The Weil-divisor exact sequence.

The exact sequence for rings. We shall need to use some results from commutative algebra.

13.1. Let A be an integrally closed integral domain. Then

$$A = \cap_{\mathrm{ht}(\mathfrak{p})=1} A_{\mathfrak{p}}$$

(intersection in the field of fractions K of A).

Recall that the height $\mathrm{ht}(\mathfrak{p})$ of a prime ideal in a Noetherian ring is the maximal length of a chain $\mathfrak{p} = \mathfrak{p}_h \supset \mathfrak{p}_{h-1} \supset \cdots$ of prime ideals. Therefore, the prime ideals of height one in an integral domain are the minimal nonzero prime ideals. For such a prime ideal $\mathfrak{p}$, $A_{\mathfrak{p}}$ has exactly one nonzero prime ideal. Since $A_{\mathfrak{p}}$ is again integrally closed, it is a discrete valuation ring. Let $\mathrm{ord}_{\mathfrak{p}}$ be the valuation on K defined by $A_{\mathfrak{p}}$, so that $A_{\mathfrak{p}} = \{a \in K \mid \mathrm{ord}_{\mathfrak{p}}(a) \geq 0\}$. Then (13.1) says that $A = \{a \in K \mid \mathrm{ord}_{\mathfrak{p}}(a) \geq 0$ all $\mathfrak{p}\}$, which implies that $A^\times = \{a \in K \mid \mathrm{ord}_{\mathfrak{p}}(a) = 0\text{all } \mathfrak{p}\}$. In other words, the sequence

$$\begin{array}{ccccccc} 0 & \to & A^\times & \to & K^\times & \to & \oplus_{\mathrm{ht}(\mathfrak{p})=1}\mathbb{Z} \\ & & & & a & \mapsto & (\mathrm{ord}_{\mathfrak{p}}(a)) \end{array}$$

is exact. The second map will not in general be surjective. For example, when A is a Dedekind domain, its cokernel is the ideal class group of A in the sense of algebraic number theory.

We shall need two further results from commutative algebra.

13.2. (a) A (Noetherian) integral domain A is a unique factorization domain if and only if every prime ideal $\mathfrak{p}$ of height 1 in A is principal.
(b) A regular local ring is a unique factorization domain.

Thus, when A is an integral domain,

$$0 \to A^\times \to K^\times \xrightarrow{(\mathrm{ord}_{\mathfrak{p}})} \oplus_{\mathrm{ht}(\mathfrak{p})=1} \mathbb{Z} \to 0$$

is exact if and only if A is a unique factorization domain.

Recall that a unique factorization domain is integrally closed.

The exact sequence for the Zariski topology. [23]Recall that a variety (or scheme) is said to be normal if $\Gamma(U, \mathcal{O}_X)$ is an integrally closed integral domain for every connected open affine $U \subset X$, or, equivalently, if $\mathcal{O}_{X,x}$ is an integrally closed integral domain for all x in X.

[23]See also AG §10.

In the remainder of this subsection, we assume X to be connected and normal. Then there is a field K of rational functions on X that is the field of fractions of $\Gamma(U, \mathcal{O}_X)$ for any open affine $U \subset X$—when X is a variety, K is denoted $k(X)$, and when X is a scheme, it is denoted $R(X)$.

A *prime (Weil-) divisor* on X is a closed irreducible subvariety (or closed integral subscheme) Z of codimension 1, and a *(Weil-) divisor* on X is an element $D = \sum n_Z Z$ of the free abelian group generated by the prime divisors. For any nonempty open subset U of X, the map $Z \mapsto Z \cap U$ is a bijection from the set of prime divisors of X meeting U to the set of prime divisors of U—the inverse map sends a prime divisor of U to its closure in X.

If U is an open affine subset in X, with $\Gamma(U, \mathcal{O}_X) = A$ say, then the map $\mathfrak{p} \mapsto V(\mathfrak{p})$ (zero set of $\mathfrak{p}$) is a bijection from the set of prime ideals of A of height one to the set of prime divisors of U—the inverse map sends a prime divisor Z of U to the ideal $I(Z)$ of functions zero on Z.

In particular, every prime divisor Z on X defines a discrete valuation ord_Z on K, namely, that corresponding the ideal $I(Z) \subset \Gamma(U, \mathcal{O}_X)$ where U is an open affine meeting Z. Intuitively, for $f \in K$, $\mathrm{ord}_Z(f)$ is the order of the zero (or pole) of f along Z.

PROPOSITION 13.3. *There is a sequence of sheaves on* X_{zar}

$$0 \to \mathcal{O}_X^\times \to K^\times \to \mathit{Div} \to 0$$

where $\Gamma(U, K^\times) = K^\times$ *for all nonempty open* U *and* $\mathit{Div}(U)$ *is the group of divisors on* U. *The sequence is always left exact, and it is exact when* X *is regular (e.g., a nonsingular variety).*

PROOF. For any open affine U in X, with $\Gamma(U, \mathcal{O}_X) = A$ say, the sequence of sections over U is the sequence

$$0 \to A^\times \to K^\times \to \oplus_{\mathrm{ht}(\mathfrak{p})=1} \mathbb{Z} \to 0$$

discussed earlier. For any $x \in X$, the sequence of stalks at x has the same form with A replaced by $\mathcal{O}_{X,x}$. Since $\mathcal{O}_{X,x}$ is an integrally closed integral domain, this sequence is always left exact, and it is exact if $\mathcal{O}_{X,x}$ is regular. $\square$

Recall that, when considered as a scheme, an irreducible variety X has a generic point η which has the property that it belongs to all nonempty open subsets of X. Thus, if $K^\times$ denotes the constant sheaf on the point η (Zariski topology), then $\Gamma(U, g_* K^\times) = K^\times$ for all nonempty open $U \subset X$ where $g\colon\ \eta \to X$ is the inclusion map. Similarly, let z be the generic point of a prime divisor $Z \subset X$, and let $i_z\colon\ z \to X$ be the inclusion of z into X. For an open subset $U \subset X$, $z \in U$ if and only if $U \cap Z$ is nonempty. By definition, the Zariski closure of z is Z, which has codimension 1, and so one says that z has codimension 1. Therefore,

$$\Gamma(U, \oplus_{\mathrm{codim}(z)=1} i_{z*}\mathbb{Z}) = \mathrm{Div}(U).$$

On combining these remarks, we see that the sequence in (13.3) can be rewritten

$$0 \to \mathcal{O}_X^\times \to g_* K^\times \to \oplus_{\mathrm{codim}(z)=1} i_{z*}\mathbb{Z} \to 0.$$

The exact sequence for étale topology.

PROPOSITION 13.4. *For any connected normal variety (or scheme) X, there is a sequence of sheaves on X_{et}*

$$0 \to \mathbb{G}_m \to g_* \mathbb{G}_{m,K} \to \oplus_{codim(z)=1} i_{z*} \mathbb{Z} \to 0.$$

It is always left exact, and it is exact if X is regular (e.g., a nonsingular variety).

PROOF. For any étale $U \to X$ with U connected, the restriction of the sequence to U_{zar} is the sequence in (13.3). Since U is regular if X is (see 2.12), the statement follows. □

Application: the cohomology of $\mathbb{G}_m$ on a curve. Let X be a complete nonsingular algebraic curve over an algebraically closed field k. We shall use the Weil-divisor exact sequence to compute the cohomology of the sheaf $\mathbb{G}_m$ on X.

In order to do this, we shall need to make use of some results from number theory.

A field k is said to be[24] *quasi-algebraically closed* if every nonconstant homogeneous polynomial $f(T_1, \ldots, T_n) \in k[T_1, \ldots, T_n]$ of degree $d < n$ has a nontrivial zero in k^n. An algebraically closed field clearly satisfies this condition, because if T_1 really occurs in $f(T_1, \ldots, T_n)$, the polynomial $f(T_1, c_2, \ldots, c_n)$ will have degree > 1 for a suitable choice of values c_i for the T_i, and so will have a zero in k.

13.5. The following fields are quasi-algebraically closed:

(a) a finite field;
(b) a function field of dimension 1 over an algebraically closed field;
(c) the field of fractions K of a Henselian ring R with algebraically closed residue field provided that the completion of K is separable over K.

A field K is said to be a *function field in n variables* over a subfield k if it is finitely generated (as a field) over k and of transcendence degree n. Thus, such fields arise as the field of rational functions on a connected variety of dimension n over k. The separability condition in (c) holds, for example, when $R = \mathcal{O}^h_{X,x}$ for some point x on a scheme of finite type over a field and (of course) when K has characteristic zero.

Statement (a) was conjectured by E. Artin, and proved by Chevalley (Shatz 1972, p109). Statement (b) is a reformulation of the theorem of Tsen, and is usually referred to as Tsen's theorem (ib. pp 108–109). Statement (c) is a theorem of Lang (ib. p116).

The relevance of these results to Galois cohomology is shown by the following proposition.

PROPOSITION 13.6. *Let k be a quasi-algebraically closed field, and let $G = \mathrm{Gal}(k^{sep}/k)$. Then:*

(a) *the Brauer group of k is zero, i.e., $H^2(G, (k^{sep})^\times) = 0$;*
(b) *$H^r(G, M) = 0$ for $r > 1$ and any torsion discrete G-module M;*
(c) *$H^r(G, M) = 0$ for $r > 2$ and any discrete G-module M.*

PROOF. (a). In order to prove (a), we must show that every central division algebra D over k has degree 1 (see CFT Chapter IV). Let $[D\colon\ k] = n^2$, and choose a basis

[24] Such a field is also said to be C_1; a field is C_r if it satisfies the condition but with $d^r < n$.

for $e_1, \ldots, e_{n^2}$ for D as a k-vector space. Then there is a homogeneous polynomial $f(X_1, \ldots, X_{n^2})$ of degree n such that $f(a_1, \ldots, a_{n^2})$ is the reduced norm of the element $\alpha = \sum a_i e_i$ of D. The reduced norm of α in D/k is $\mathrm{Nm}_{\mathbb{Q}[\alpha]/\mathbb{Q}}(\alpha)^r$, $r = \frac{n}{[\mathbb{Q}[\alpha]\colon \mathbb{Q}]}$, which is nonzero if $\alpha \neq 0$ (because $\mathbb{Q}[\alpha]$ is a field). Thus $f(X_1, \ldots, X_{n^2})$ has no nontrivial zero, which, because k is quasi-algebraically closed, implies that $n \geq n^2$. This is possible only if $n = 1$.

(b) One first shows that if k is quasi-algebraically closed, then so also is any finite extension of k (Shatz 1972, p107). Together with (a), this remark shows that, for any finite Galois extension L/K of finite extensions of k, $H^r(\mathrm{Gal}(L/K), L^\times) = 0$ for $r = 1, 2$. Now Tate's Theorem (CFT II 2.17) implies that $H^r(\mathrm{Gal}(L/k), L^\times) = 0$ for all $r > 0$ and any L/k finite and Galois. On passing to the inverse limit, one finds that $H^r(G, (k^{\mathrm{sep}})^\times) = 0$ for $r > 0$. From the cohomology sequence of the Kummer sequence

$$0 \to \mu_n \to (k^{\mathrm{sep}})^\times \xrightarrow{n} (k^{\mathrm{sep}})^\times \to 0$$

we find that $H^r(G, \mu_n) = 0$ for all $r > 1$ and n relatively prime to the characteristic of k. Let p be a prime $\neq \mathrm{char}(k)$. There exists a finite Galois extension K of k of degree prime to p such that K contains a primitive pth root of 1. The composite

$$H^r(G, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\mathrm{Res}} H^r(H, \mathbb{Z}/p\mathbb{Z}) \xrightarrow{\text{corollary}} H^r(G, \mathbb{Z}/p\mathbb{Z}), \quad H = \mathrm{Gal}(k^{\mathrm{sep}}/K),$$

is multiplication by $[K\colon\ k]$ (see CFT II 1.30), and so is an isomorphism. Because $H^r(H, \mathbb{Z}/p\mathbb{Z}) = H^r(H, \mu_p) = 0$, $H^r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r > 1$ (and the first remark then shows that this is also true for any open subgroup of G).

Directly from the Artin-Scheier exact sequence

$$0 \to \mathbb{Z}/p\mathbb{Z} \to k \xrightarrow{t \mapsto t^p - t} k \to 0$$

and (CFT II 1.23) one finds that $H^r(G, \mathbb{Z}/p\mathbb{Z}) = 0$ for $r > 1$ for any field k of characteristic p.

Now let M be a finite G-module. We want to show that $H^r(G, M) = 0$ for $r > 1$. We may suppose that M has order a power of a prime p. The Sylow theorems and the restriction-corestriction argument used above allows us to assume that G acts on M through a finite p-group $\bar{G}$. Now a standard result shows that the only simple $\bar{G}$-module of p-power order is $\mathbb{Z}/p\mathbb{Z}$ with the trivial action, and so M has a composition series whose quotients are all $\mathbb{Z}/p\mathbb{Z}$. An induction argument now shows that $H^r(G, M) = 0$ for $r > 1$.

As any torsion G-module is a union of its finite submodules, and cohomology commutes with direct limits, this completes the proof.

(c) We omit the proof, since we don't need the result. It is perhaps worth noting however that, for k a finite field,

$$H^2(G, \mathbb{Z}) \cong H^1(G, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{conts}}(G, \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \neq 0.$$

□

THEOREM 13.7. *For a connected nonsingular curve X over an algebraically closed field,*

$$H^r(X_{et}, \mathbb{G}_m) = \begin{cases} \Gamma(X, \mathcal{O}_X^\times), & r = 0 \\ Pic(X), & r = 1 \\ 0, & r > 0. \end{cases}$$

This will follow from the Weil-divisor exact sequence once we have proved the next lemma (because, for any regular scheme X, $Pic(X) = \{\text{divisors}\}/\{\text{principal divisors}\}$—see Hartshorne, II.6.16).

LEMMA 13.8. *The cohomology groups $H^r(X_{et}, g_*\mathbb{G}_{m,\eta})$ and $H^r(X_{et}, Div_X)$ are zero for all $r > 0$.*

PROOF. For x a closed point of X, i_{x*} is exact, and so $H^r(X_{et}, i_{x*}\mathcal{F}) = H^r(x_{et}, \mathcal{F}) = 0$ for any sheaf $\mathcal{F}$ on x_{et}. Hence $H^r(X_{et}, \mathrm{Div}_X) = 0$ for $r > 0$.

Now consider $R^r g_*\mathbb{G}_{m,\eta}$. According to (12.4),

$$(R^r g_*\mathbb{G}_{m,\eta})_{\bar{y}} = \begin{cases} 0 & \text{if } y = \eta \\ H^r(\operatorname{Spec} K_{\bar{x}}, \mathbb{G}_m) & \text{if } y = x \neq \eta. \end{cases}$$

Here $K_{\bar{x}}$ is the field of fractions of the Henselian discrete valuation ring $\mathcal{O}_{X,\bar{x}}$, and so Lang's Theorem 13.5c shows that $H^r(\operatorname{Spec} K_{\bar{x}}, \mathbb{G}_m) = 0$ for $r > 0$. Therefore $R^r g_*\mathbb{G}_{m,\eta} = 0$ for $r > 0$, and so the Leray spectral sequence for g shows that $H^r(X_{et}, g_*\mathbb{G}_{m,\eta}) = H^r(G, (K^{\text{sep}})^\times)$ for all r, where $G = \mathrm{Gal}(K^{\text{sep}}/K)$. Now $H^r(G, (K^{\text{sep}})^\times) = 0$ for $r = 1$ by Hilbert's Theorem 90, and $H^r(G, (K^{\text{sep}})^\times) = 0$ for $r > 1$ by Tsen's Theorem 13.5b. $\square$

Let X be a connected nonsingular variety (or a regular integral quasi-compact scheme) and let $K = k(X)$ (or $K = R(X)$). Similar arguments to the above show that there is are exact sequences

$$0 \to H^0(X_{et}, \mathbb{G}_m) \to K^\times \to \oplus_{\operatorname{codim}(x)=1}\mathbb{Z} \to H^1(X_{et}, \mathbb{G}_m) \to 0$$

and

$$0 \to H^2(X_{et}, \mathbb{G}_m) \to H^2(K, \mathbb{G}_{m,\eta}).$$

Moreover, $H^r(X_{et}, \mathbb{G}_m)$ is torsion for $r > 1$. Here (and elsewhere) $H^r(K, -) = H^r(\operatorname{Spec}(K)_{et}, -)$. (See EC III 2.22.)

Now assume that X has dimension 1, that its residue fields are perfect, and that either K has characteristic zero or X is an algebraic curve over a field (so that the separability condition in Lang's Theorem holds). Then the last exact sequence extends to a long sequence

$$0 \to H^2(X_{et}, \mathbb{G}_m) \to H^2(K, \mathbb{G}_{m,\eta}) \to \oplus_v H^1(\kappa(v), \mathbb{Q}/\mathbb{Z}) \to \cdots$$
$$\cdots \to H^r(X_{et}, \mathbb{G}_m) \to H^r(K, \mathbb{G}_m) \to \oplus_v H^{r-1}(\kappa(v), \mathbb{Q}/\mathbb{Z}) \to \cdots$$

Here the sums are over the closed points v of X, and $\kappa(v)$ is the residue field at v. When $\kappa(v)$ is finite, $H^r(\kappa(v), \mathbb{Q}/\mathbb{Z}) = 0$ for $r > 1$.

Let $X = \operatorname{Spec} R$, where R is the ring of integers in a totally imaginary number field K. On comparing the above sequence with the fundamental sequence in global class field theory (CFT VIII 2.3), namely, with

$$0 \to H^2(K, \mathbb{G}_m) \to \oplus_v H^2(K_v, \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \to 0,$$

one finds that

$$H^2(X_{et}, \mathbb{G}_m) = 0 \text{ and } H^3(X_{et}, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}.$$

14. The Cohomology of Curves.

In this section, we study the étale cohomology of curves. Not only is this a paradigm for the study of the étale cohomology of a general variety, but many proofs in the general case are by induction on the dimension of a variety starting from the case of a curve.

The Picard group of a curve. We saw in the last section that, for a connected nonsingular curve U over an algebraically closed field k,

$$H^r(U_{et}, \mathbb{G}_m) = \begin{cases} k^\times, & r = 0, \\ Pic(U), & r = 1, \\ 0, & r > 1. \end{cases}$$

We wish to use the Kummer sequence to compute the cohomology of μ_n, but first we need to know the structure of $Pic(U)$.

The Picard group of U can be defined by the exact sequence

$$K^\times \to \oplus_{x \in U} \mathbb{Z} \to Pic(U) \to 0.$$

Here $K = k(U)$, the field of rational functions on U, and the sum is over all closed points of U. For a closed point x of U, let $[x]$ be the divisor corresponding to it. Thus any element of $\mathrm{Div}(U)$ can be written as a finite sum $D = \sum_{x \in U} n_x[x]$, $n_x \in \mathbb{Z}$. The *degree of* $D = \sum n_x[x]$ is $\sum n_x$.

Now let X be a complete connected nonsingular curve over an algebraically closed field. The divisor $div(f)$ of a nonzero rational function f on X has degree zero (a rational function has as many poles as zeros counting multiplicities). Let $\mathrm{Div}^0(X)$ denote the group of divisors of degree 0, and $Pic^0(X)$ the quotient of $\mathrm{Div}^0(X)$ by the subgroup of principal divisors.

Proposition 14.1. *The sequence*

$$0 \to Pic^0(X) \to Pic(X) \to \mathbb{Z} \to 0$$

is exact. For any integer n relatively prime to the characteristic of k,

$$z \mapsto nz \colon Pic^0(X) \to Pic^0(X)$$

is surjective with kernel equal to a free $\mathbb{Z}/n\mathbb{Z}$-module of rank $2g$, where g is the genus of X.

Proof. The sequence is part of the kernel-cokernel sequence[25] of the pair of maps

$$K^\times \overset{\mathrm{div}}{\to} \mathrm{Div}^0(X) \hookrightarrow \mathrm{Div}(X).$$

The proof of the second statement is more difficult. Assume first that $k = \mathbb{C}$. Choose a basis $\omega_1, \ldots, \omega_g$ for the holomorphic differentials on the Riemann surface $X(\mathbb{C})$ and

[25]The kernel-cokernel exact sequence of the pair $A \xrightarrow{f} B \xrightarrow{g} C$ is

$$0 \to \mathrm{Ker}(f) \to \mathrm{Ker}(g \circ f) \to \mathrm{Ker}(g) \to \mathrm{Coker}(f) \to \mathrm{Coker}(g \circ f) \to \mathrm{Coker}(g) \to 0$$

(CFT II 4.2).

a basis $\gamma_1, \dots, \gamma_{2g}$ for $H_1(X(\mathbb{C}), \mathbb{Z})$. Let Λ be the subgroup of $\mathbb{C}^g$ generated by the vectors

$$\left(\int_{\gamma_i} \omega_1, \dots, \int_{\gamma_i} \omega_g\right), \quad i = 1, \dots, 2g.$$

For each pair of points $z_0, z_1 \in X(\mathbb{C})$, choose a path $\gamma(z_0, z_1)$ from z_0 to z_1, and let

$$I(z_0, z_1) = \left(\int_{\gamma(z_0,z_1)} \omega_1, \dots, \int_{\gamma(z_0,z_1)} \omega_g\right) \in \mathbb{C}^g.$$

Its image in $\mathbb{C}^g/\Lambda$ is independent of the choice of the path $\gamma(z_0, z_1)$, and the map $[z_1] - [z_0] \mapsto I(z_0, z_1)$ extends by linearity to a homomorphism

$$i\colon \operatorname{Div}^0(X) \to \mathbb{C}^g/\Lambda.$$

The famous theorem of Abel (Fulton, W., Algebraic Topology, Springer 1995, 21.18) says that $i(D) = 0$ if and only if D is principal, and the equally famous Jacobi Inversion Theorem says that i is onto (ib. 21.32). Therefore, i induces an isomorphism

$$Pic^0(X) \xrightarrow{\approx} \mathbb{C}^g/\Lambda.$$

Clearly, for any integer n, $x \mapsto nx\colon\ \mathbb{C}^g/\Lambda \to \mathbb{C}^g/\Lambda$ is surjective with kernel

$$\frac{1}{n}\Lambda/\Lambda \approx (\frac{1}{n}\mathbb{Z}/\mathbb{Z})^{2g} \cong (\mathbb{Z}/n\mathbb{Z})^{2g}.$$

This completes the proof of the proposition in the case $k = \mathbb{C}$.

For an arbitrary algebraically closed field k, the proof is similar, but requires the algebraic theory of the Jacobian variety. Briefly, there is an abelian variety J of dimension g, called the Jacobian variety of X, such that $Pic^0(X) \cong J(k)$. As for any abelian variety J of dimension g over an algebraically closed field k and integer n prime to the characteristic of k, $n\colon\ J(k) \to J(k)$ is surjective with kernel a free $\mathbb{Z}/n\mathbb{Z}$-module of rank n^{2g} (see, for example, my articles in *Arithmetic Geometry*, Springer, 1986). □

The cohomology of μ_n.

PROPOSITION 14.2. *Let X be a complete connected nonsingular curve over an algebraically closed field k. For any n prime to the characteristic of k,*

$$H^0(X, \mu_n) = \mu_n(k), \quad H^1(X, \mu_n) \approx (\mathbb{Z}/n\mathbb{Z})^{2g}, \quad H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z},$$

and $H^r(X, \mu_n) = 0, \quad r > 2.$

PROOF. From the Kummer sequence (7.9), we obtain an exact sequence

$$\cdots \to H^r(X_{et}, \mu_n) \to H^r(X_{et}, \mathbb{G}_m) \xrightarrow{n} H^r(X_{et}, \mathbb{G}_m) \to \cdots.$$

Thus, the statement can be read off from (14.1) and (13.7). □

PROPOSITION 14.3. *Let U be a nonsingular curve over an algebraically closed field k. For any n prime to the characteristic of k and closed point $x \in U$,*

$$H^2_x(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}, \quad H^r_x(U, \mu_n) = 0 for\ r \neq 2.$$

PROOF. Let R be the Henselization of $\mathcal{O}_{U,u}$. By excision (9.7),

$$H^r_x(U,\mu_n) \cong H^r_x(V,\mu_n), \quad V = \operatorname{Spec} R.$$

Now $H^r(V,\mathbb{G}_m) = 0$ for $r > 0$ (Theorem 13.7also applies to V), and so the exact sequence of the pair $(V, V \setminus x)$

$$\cdots \to H^r_x(V,\mathbb{G}_m) \to H^r(V,\mathbb{G}_m) \to H^r(V \setminus x,\mathbb{G}_m) \to \cdots$$

provides us with isomorphisms

$$H^{r-1}(V \setminus x,\mathbb{G}_m) \to H^r_x(V,\mathbb{G}_m)$$

for all $r \neq 1$. But $V \setminus x = \operatorname{Spec} K$, where K is the field of fractions of R, and Lang's Theorem (13.5c) and (13.6) show that $H^r(K,\mathbb{G}_m) = 0$ for $r \geq 1$. Hence

$$H^1_x(V,\mathbb{G}_m) \cong H^0(K,\mathbb{G}_m)/H^0(V,\mathbb{G}_m) \cong \mathbb{Z}, \quad H^r_x(V,\mathbb{G}_m) = 0 \text{for } r \neq 1.$$

The proposition now follows from the exact sequence of the Kummer sequence

$$\cdots \to H^r_x(V,\mu_n) \to H^r_x(V,\mathbb{G}_m) \xrightarrow{n} H^r_x(V,\mathbb{G}_m) \to \cdots .$$

□

REMARK 14.4. Let M be a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 1, and let M denote also the constant sheaf on a variety (or scheme) Y defined by M. Then

$$H^r(Y_{et}, M) \cong H^r(Y_{et},\mathbb{Z}/n\mathbb{Z}) \otimes M \approx H^r(Y_{et},\mathbb{Z}/n\mathbb{Z}).$$

This remark applies to $M = \mu_n$ when Y is a variety over a field k containing n distinct roots of 1.

Cohomology with compact support for curves. We first recall some definitions. A function field in one variable over a field k is a field $K \supset k$ of transcendence degree 1 over k and finitely generated over k, i.e., K is a finite extension of $k(T)$ for some T transcendental over k. A curve U over k is regular if $\mathcal{O}_{U,u}$ is a discrete valuation ring for all u. A nonsingular curve is regular, but a regular curve need not be nonsingular unless k is perfect (by definition a variety U over a field k is nonsingular if $U_{k^{\mathrm{al}}}$ is nonsingular over k^{al} — in the language of schemes, U is smooth over $\operatorname{Spec} k$).

Let U be a connected regular curve over a field k. Then $K \overset{\mathrm{df}}{=} k(U)$ is a function field in one variable over k, and the map $u \mapsto \mathcal{O}_{U,u}$ is an injection from U into the set of discrete valuation rings $R \subset K$ satisfying the condition:

($*$) R contains k and has field of fractions K.

The proof of injectivity uses that varieties are separated (see Notations and Conventions) — the map is not injective for the "affine line with the origin doubled" (AG 3.10).

Conversely, every function field K in one variable has a connected complete regular curve X canonically associated with it. In fact, we can define X to be the set of all discrete valuation rings in K satisfying ($*$), and endow it with the topology for which the proper closed subsets are the finite sets. For each open U in X, define

$$\Gamma(U,\mathcal{O}_X) = \cap_{R \in U} R.$$

Then $(X,\mathcal{O}_X)$ is a complete curve with K as its field of rational functions, and the local ring at the point R of X is R itself. (See Hartshorne 1977, I.6.)

Let U be a connected regular curve over k, and let X be the connected complete regular curve canonically associated with $k(U)$. Then the map

$$j\colon U \to X, \quad j(u) = \mathcal{O}_{U,u},$$

realizes U as an open subvariety of X. For a sheaf $\mathcal{F}$ on U, we define

$$H^r_c(U, \mathcal{F}) = H^r(X, j_!\mathcal{F})$$

and refer to the $H^r_c(U, \mathcal{F})$ as the *cohomology groups of $\mathcal{F}$ with compact support.* For an explanation of this terminology, see §18 below.

Because $j_!$ is exact, a short exact sequence of sheaves on U gives a long exact sequence of cohomology groups. However, because $j_!$ doesn't preserve injectives, $H^r_c(U, \mathcal{F})$ is *not* the r^{th} right derived functor of $H^0_c(U, -)$.

PROPOSITION 14.5. *For any connected regular curve U over an algebraically closed field k and integer n not divisible by the characteristic of k, there is a canonical isomorphism*

$$H^2_c(U, \mu_n) \to \mathbb{Z}/n\mathbb{Z}.$$

PROOF. Let $j\colon\ U \hookrightarrow X$ be the canonical inclusion of U into a complete regular curve, and let $i\colon\ Z \hookrightarrow X$ be the complement of U in X. Regard μ_n as a sheaf on X. From the sequence (see 8.15)

$$0 \to j_!j^*\mu_n \to \mu_n \to i_*i^*\mu_n \to 0$$

we obtain an exact sequence

$$\cdots \to H^r_c(U, \mu_n) \to H^r(X, \mu_n) \to H^r(X, i_*i^*\mu_n) \to \cdots .$$

But

$$H^r(X, i_*i^*\mu_n) \cong H^r(Z, i^*\mu_n) = 0 \quad \text{for } r > 0,$$

and so

$$H^2_c(U, \mu_n) \cong H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

□

REMARK 14.6. For any $x \in U$, sheaf $\mathcal{F}$ on U, and $r \geq 0$, there is a canonical map $H^r_x(U, \mathcal{F}) \to H^r_c(U, \mathcal{F})$. For $\mathcal{F} = \mu_n$ and $r = 2$, the map is compatible with the isomorphisms in (14.3) and (14.5).

The Poincaré duality theorem. Throughout this subsection, U is a connected regular curve over an algebraically closed field k, and n is an integer not divisible by the characteristic of k.

By a finite locally constant sheaf $\mathcal{F}$, I shall mean a locally constant sheaf, killed by n, that has finite stalks. Thus, for some finite étale covering $U' \to U$, $\mathcal{F}|U'$ is the constant sheaf defined by a finite $\mathbb{Z}/n\mathbb{Z}$-module M, and to give a finite locally constant sheaf $\mathcal{F}$ on U is to give a finite $\mathbb{Z}/n\mathbb{Z}$-module endowed with a continuous action of $\pi_1(U, \bar{u})$. (See §6.)

THEOREM 14.7 (Poincaré Duality). *For any finite locally constant sheaf $\mathcal{F}$ on U and integer $r \geq 0$, there is a canonical perfect pairing of finite groups*

$$H^r_c(U, \mathcal{F}) \times H^{2-r}(U, \check{\mathcal{F}}(1)) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

A pairing $M \times N \to C$ is said to be *perfect* if the induced maps

$$M \to \mathrm{Hom}(N, C), \quad N \to \mathrm{Hom}(M, C)$$

are isomorphisms. The sheaf $\check{\mathcal{F}}(1)$ is

$$V \mapsto \mathrm{Hom}_V(\mathcal{F}|V, \mu_n|V).$$

Note that, if $\mathcal{G} = \check{\mathcal{F}}(1)$, then $\mathcal{F} = \check{\mathcal{G}}(1)$.

For a discussion of the relation of the above theorem to the usual Poincaré duality theorem for topological spaces, see the §24.

I defer a description of the pairing until later. In the sketch of the proof of the theorem, I omit proofs that the diagrams commute. It is important to understand though, that the theorem is about a specific pairing, i.e., it doesn't just say that the groups are dual, but that they are dual with a *specific pairing.*

Example 14.8. Let X be a complete connected nonsingular curve over an algebraically closed field k. In this case, the pairing is

$$H^r(X, \mathcal{F}) \times H^{2-r}(X, \check{\mathcal{F}}(1)) \to H^2(X, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

On taking $\mathcal{F} = \mathbb{Z}/n\mathbb{Z}$ and tensoring two of the groups with μ_n (cf. 14.4), we obtain a pairing

$$H^r(X, \mu_n) \times H^{2-r}(X, \mu_n) \to H^2(X, \mu_n \otimes \mu_n) \cong \mu_n.$$

For $r = 1$, this can be identified with the Weil pairing

$$Jac(X)_n \times Jac(X)_n \to \mu_n$$

on the points of order n on the Jacobian of X (AV §12).

Sketch of the proof of Poincaré duality. In the proof, we make frequent use of the the five-lemma in the following form. Consider a commutative diagram with exact rows,

$$\begin{array}{ccccccc} \bullet & \to & \bullet & \to & \bullet & \to & \bullet \\ \downarrow a & & \downarrow b & & \downarrow c & & \downarrow d \\ \bullet & \to & \bullet & \to & \bullet & \to & \bullet \end{array} ;$$

then

$$b, d \text{ injective, } a \text{ surjective } \Rightarrow c \text{ injective;}$$

$$a, c \text{ surjective, } d \text{ injective } \Rightarrow b \text{ surjective.}$$

Step 0. For any torsion sheaf $\mathcal{F}$, the groups $H^r(U, \mathcal{F})$ and $H^r_c(U, \mathcal{F})$ are zero for $r > 2$.

Proof. For the vanishing of $H^r(U, \mathcal{F}) = 0$, see Theorem 15.1 below. It implies that $H^r_c(U, \mathcal{F}) \overset{\mathrm{df}}{=} H^r(X, j_!\mathcal{F}) = 0$ for $r > 2$. □

Thus the theorem is true for $r \neq 0, 1, 2$.

For convenience, write $T^r(U, \mathcal{F}) = H^{2-r}_c(U, \check{\mathcal{F}}(1))^\vee$ (the last $^\vee$ means dual in the sense $\mathrm{Hom}(-, \mathbb{Z}/n\mathbb{Z})$). We have to show that the map

$$\phi^r(U, \mathcal{F}) \colon H^r(U, \mathcal{F}) \to T^r(U, \mathcal{F})$$

defined by the pairing in the theorem (with $\mathcal{F}$ replaced by $\check{\mathcal{F}}(1)$) is an isomorphism of finite groups for all finite locally constant sheaves $\mathcal{F}$ on U.

Because $\mathrm{Hom}_{\mathbb{Z}/n\mathbb{Z}}(-, \mathbb{Z}/n\mathbb{Z})$ preserves the exactness of sequences of $\mathbb{Z}/n\mathbb{Z}$-modules, so also does $\mathcal{F} \mapsto \check{\mathcal{F}}(1)$ (look on stalks). It follows that an exact sequence of finite locally constant sheaves

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

gives rise to an exact sequence

$$\cdots \to T^s(U, \mathcal{F}') \to T^s(U, \mathcal{F}) \to T^s(U, \mathcal{F}'') \to T^{s+1}(U, \mathcal{F}') \to \cdots .$$

Step 1: Let $\pi\colon\ U' \to U$ be a finite map. The theorem is true for $\mathcal{F}$ on U' if and only if it is true for $\pi_*\mathcal{F}$ on U.

PROOF. Recall that π_* is exact and preserves injectives, and so $H^r(U, \pi_*\mathcal{F}) \cong H^r(U', \mathcal{F})$ for all r. Since a similar statement is true for T^s, we see that $\phi^r(U, \pi_*\mathcal{F})$ can be identified with $\phi^r(U', \mathcal{F})$. □

Step 2: Let $V = U \setminus x$ for some point $x \in U$. Then there is an exact commutative diagram with isomorphisms where indicated:

$$\begin{tikzcd}
H^r_x(U, \mu_n) \arrow[r] \arrow[d, "\approx"] & H^r(U, \mu_n) \arrow[r] \arrow[d, "\phi^r(U,\mu_n)"] & H^r(V, \mu_n) \arrow[r] \arrow[d, "\phi^r(V,\mu_n)"] & H^{r+1}_x(U, \mu_n) \arrow[d, "\approx"] \\
H^{2-r}(x, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[r] & H^{2-r}_c(U, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[r] & H^{2-r}_c(V, \mathbb{Z}/n\mathbb{Z})^\vee \arrow[r] & H^{3-r}(x, \mathbb{Z}/n\mathbb{Z})^\vee .
\end{tikzcd}$$

PROOF. The upper sequence is the exact sequence of the pair (U, V). According to (14.3), $H^r_x(V, \mu_n) = H^0(x, \mathbb{Z}/n\mathbb{Z})$ when $r = 2$ and is zero otherwise.

The lower sequence is the compact cohomology sequence of

$$0 \to j_!\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to i_*i^*\mathbb{Z}/n\mathbb{Z} \to 0,$$

where j and i are the inclusions of V and x into U respectively (see 8.16).

For $r = 2$, the unlabelled vertical map is the isomorphism given by the pairing

$$H^0(x, \mathbb{Z}/n\mathbb{Z}) \times H^0(x, \mathbb{Z}/n\mathbb{Z}) \to H^0(x, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

(Poincaré duality for the point x), and for $r \neq 2$, it is a map between zero groups. □

Step 3: The map $\phi^0(U, \mathbb{Z}/n\mathbb{Z})$ is an isomorphism of finite groups.

PROOF. In this case, the pairing is

$$H^0(U, \mathbb{Z}/n\mathbb{Z}) \times H^2_c(U, \mu_n) \to H^2_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

Here $H^0(U, \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$, and its action on $H^2_c(U, \mu_n)$ is defined by the natural $\mathbb{Z}/n\mathbb{Z}$-module structure on $H^2_c(U, \mu_n)$. □

Step 4: The theorem is true for $r = 0$ and $\mathcal{F}$ locally constant.

PROOF. Let $U' \to U$ be a finite étale covering such that $\mathcal{F}|U'$ is constant. We can embed $\mathcal{F}|U'$ into a sheaf $\mathcal{F}' = (\mathbb{Z}/n\mathbb{Z})^s$ on U' for some s. On applying π_* to $\mathcal{F}|U' \hookrightarrow \mathcal{F}'$ and composing the result with the natural inclusion $\mathcal{F} \hookrightarrow \pi_*\pi^*\mathcal{F}$, we obtain the first map in the sequence

$$0 \to \mathcal{F} \to \pi_*\mathcal{F}' \to \mathcal{F}'' \to 0.$$

The cokernel $\mathcal{F}''$ is again locally constant (when $U' \to U$ is chosen to be Galois, $\mathcal{F}''$ becomes constant on U'). Consider

$$\begin{array}{ccccccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & H^0(U,\mathcal{F}) & \longrightarrow & H^0(U,\pi_*\mathcal{F}') & \longrightarrow & H^0(U,\mathcal{F}'') \\
\downarrow\approx & & \downarrow\approx & & \downarrow & & \downarrow\approx & & \downarrow \\
\bullet & \longrightarrow & \bullet & \longrightarrow & T^0(U,\mathcal{F}) & \longrightarrow & T^0(U,\pi_*\mathcal{F}') & \longrightarrow & T^0(U,\mathcal{F}'').
\end{array}$$

The groups "$\bullet$" are in fact zero, but all we need is that the arrows are isomorphisms. The map $\phi^0(U,\pi_*\mathcal{F}')$ is an isomorphism by Steps 2 and 3. The five-lemma shows that $\phi^0(U,\mathcal{F})$ is a injective. Since this is true for all locally constant sheaves $\mathcal{F}$, $\phi^0(U,\mathcal{F}'')$ is injective, and the five-lemma now shows that $\phi^0(U,\mathcal{F})$ is surjective. Finally, $H^0(U,\mathcal{F})$ is obviously finite. □

Step 5: The map $\phi^1(U,\mu_n)$ is surjective.

PROOF. Recall (11.3) that $H^1(U,\mathbb{Z}/n\mathbb{Z}) = \mathrm{Hom}_{\mathrm{conts}}(\pi_1(U,\bar{u}),\mathbb{Z}/n\mathbb{Z})$. Let $s \in H^1(U,\mathbb{Z}/n\mathbb{Z})$, and let $\pi\colon U' \to U$ be the Galois covering corresponding to the kernel of s. Then s maps to zero in $H^1(U',\mathbb{Z}/n\mathbb{Z})$, which is isomorphic to $H^1(U,\pi_*\mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{F}''$ be the cokernel of $\mathbb{Z}/n\mathbb{Z} \to \pi_*(\mathbb{Z}/n\mathbb{Z})$, and consider

$$\begin{array}{ccccccc}
H^0(U,\pi_*\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^0(U,\mathcal{F}'') & \longrightarrow & H^1(U,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & H^1(U,\pi_*\mathbb{Z}/n\mathbb{Z}) \\
\downarrow\approx & & \downarrow\approx & & \downarrow & & \downarrow \\
T^0(U,\pi_*\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & T^0(U,\mathcal{F}'') & \longrightarrow & T^1(U,\mathbb{Z}/n\mathbb{Z}) & \longrightarrow & T^1(U,\pi_*\mathbb{Z}/n\mathbb{Z}).
\end{array}$$

From our choice of π, s maps to zero in $H^1(U,\pi_*\mathbb{Z}/n\mathbb{Z})$, and a diagram chase shows that, if s also maps to zero in $T^1(U,\mathbb{Z}/n\mathbb{Z})$, then it is zero. □

Step 6. The maps $\phi^r(U,\mathbb{Z}/n\mathbb{Z})$ are isomorphisms of finite groups for $r = 1,2$.

PROOF. First take $U = X$. For $r = 1$, ϕ is a map $H^1(X,\mathbb{Z}/n\mathbb{Z}) \to H^1(X,\mu_n)^\vee$. Because $\phi^1(X,\mathbb{Z}/n\mathbb{Z})$ is injective (Step 5), and $H^1(X,\mathbb{Z}/n\mathbb{Z})$ is finite of the same order as $H^1(X,\mu_n)$ (14.2, 14.4), it is an isomorphism. For $r = 2$, we have to show that the pairing

$$H^2(X,\mathbb{Z}/n\mathbb{Z}) \times \mu_n(k) \to H^2(X,\mu_n)$$

is perfect, but this follows from (14.4).

We have shown that $\phi^r(X,\mathbb{Z}/n\mathbb{Z})$ is an isomorphism of finite groups for $r = 1,2$. To deduce the same statement for $\phi^r(U,\mathbb{Z}/n\mathbb{Z})$, remove the points of $X \setminus U$ one at a time, and apply the five-lemma to the diagram in Step 2. □

Step 7. The maps $\phi^r(U,\mathcal{F})$ are isomorphisms of finite groups for $r = 1,2$ and $\mathcal{F}$ locally constant.

PROOF. Apply the argument in Step 4, twice. □

This completes the sketch of the proof of Theorem 14.7.

The Hochschild-Serre spectral sequence. In this subsection, X is an arbitrary variety (or scheme).

THEOREM 14.9. *Let $\pi\colon\ Y \to X$ be a Galois covering with Galois group G. For any sheaf $\mathcal{F}$ on X_{et}, there is a spectral sequence*

$$H^r(G, H^s(Y_{et}, \mathcal{F}|Y)) \Rightarrow H^{r+s}(X_{et}, \mathcal{F}).$$

PROOF. For any sheaf $\mathcal{F}$ on X, $\mathcal{F}(Y)$ is a left G-module, and $\mathcal{F}(X) = \mathcal{F}(Y)^G$ (see 6.4). Therefore the composite of the functors

$$Sh(X_{et}) \xrightarrow{\mathcal{F}\mapsto\mathcal{F}(Y)} G\text{-Mod} \xrightarrow{M\mapsto M^G} Ab$$

is $\Gamma(X, -)$. The theorem will follow from (12.7) once we show that $\mathcal{I}$ injective implies that $H^r(Y, \mathcal{I}(Y)) = 0$ for $r > 0$. But if $\mathcal{I}$ is injective as a sheaf, then it is injective as a presheaf, and so $\check{H}^r(Y/X, \mathcal{I}) = 0$ for $r > 0$ (see §10). As we observed in (10.1), $\check{H}^r(Y/X, \mathcal{I}) = \check{H}^r(G, \mathcal{I}(Y))$. □

REMARK 14.10. Let

$$\cdots \to Y_i \to Y_{i-1} \to \cdots \to Y_1 \to Y_0 = X$$

be a tower in which each map $Y_i \to Y_j$ is Galois. Let G_i be the Galois group of Y_i over X, and let $G = \varprojlim G_i$. For each i, there is a spectral sequence

$$H^r(G_i, H^s(Y_i, \mathcal{F}_i)) \Rightarrow H^{r+s}(X, \mathcal{F}), \quad \mathcal{F}_i = \mathcal{F}|Y_i.$$

On passing to the inverse limit, we obtain a spectral sequence

$$H^r(G, H^s(Y_\infty, \mathcal{F}_\infty)) \Rightarrow H^{r+s}(X, \mathcal{F}).$$

Here $H^r(G, -)$ is the cohomology group of the profinite group G computed using continuous cochains, $Y_\infty = \varprojlim Y_i$, and $\mathcal{F}_\infty$ is the inverse image of $\mathcal{F}$ on Y_∞ (cf. 10.8and CFT II.3.3).

EXAMPLE 14.11. Let A be a Dedekind domain with field of fractions K, and let K^{sep} be a separable closure of K. Let K^{un} be the composite of all subfields L of K^{sep} such that $[L\colon\ K] < \infty$ and the integral closure of A in L is unramified over A at all primes. Let $G = \text{Gal}(K^{\text{un}}/K)$ — it is $\pi_1(U, \bar{\eta})$ where $U = \operatorname{Spec} A$ and $\bar{\eta}$ is the geometric point $\operatorname{Spec} K^{\text{sep}} \to U$. Let $\mathcal{F}_M$ be the locally constant sheaf on U corresponding to the discrete π_1-module M. Then there is a spectral sequence

$$H^r(\pi_1(U, \bar{\eta}), H^s(\tilde{U}, \mathcal{F}_M)) \Rightarrow H^{r+s}(U, \mathcal{F}_M).$$

If $H^s(\tilde{U}, M) = 0$ for $r > 0$, then this gives isomorphisms

$$H^r(\pi_1(U, \bar{\eta}), M) \cong H^r(U, \mathcal{F}_M)$$

for all r.

Cohomology of locally constant sheaves on affine curves. Let X be a complete connected nonsingular curve over a field k. For any nonempty finite set S of closed points of X, $U = X \setminus S$ is affine. (The Riemann-Roch theorem provides us with a nonconstant function f on U, i.e., a nonconstant function $f\colon\ U \to \mathbb{A}^1$.)

Proposition 14.12. *Let U be a nonsingular affine curve over an algebraically closed field. For any locally constant torsion sheaf $\mathcal{F}$ on U, $H^r(U_{et}, \mathcal{F}) = 0$ for $r \geq 2$.*

Proof. After Step 0 of the proof of Theorem 14.7, it suffices to prove this for $r = 2$.

We first show that $H^2(U, \mu_n) = 0$. From the Weil-divisor sequence, we found that $H^1(U, \mathbb{G}_m) = Pic(U)$, and that $H^r(U, \mathbb{G}_m) = 0$ for $r > 0$, and so the Kummer sequence shows that $H^2(U, \mu_n) = Pic(U)/nPic(U)$ and that $H^r(U, \mu_n) = 0$ for $r > 2$. Let X be the complete nonsingular curve containing U. Since U omits at least one point of X, the projection map

$$\begin{array}{ccc} \mathrm{Div}(X) & \to & \mathrm{Div}(U) \\ \| & & \| \\ \oplus_{x \in X}\mathbb{Z} & & \oplus_{x\in U}\mathbb{Z} \end{array}$$

maps $\mathrm{Div}^0(X) \overset{\mathrm{df}}{=} \{\sum n_x[x] \mid \sum n_x = 0\}$ onto $\mathrm{Div}(U)$. Since the projection map sends principal divisors to principal divisors, we see that $\mathrm{Pic}(U)$ is a quotient of $\mathrm{Pic}^0(X)$, which is divisible. Therefore $\mathrm{Pic}(U)$ is also divisible, and $H^2(U, \mu_n) = 0$.

Next, let $\mathcal{F}$ be a locally constant sheaf on U with finite fibres. There exists a finite Galois covering $U' \to U$ such that $\mathcal{F}|U'$ is constant, and a surjective "trace map" $\pi_*\pi^*\mathcal{F} \to \mathcal{F}$. Because $H^2(U, \pi_*\pi^*\mathcal{F}) = 0$, it follows that $H^2(U, \mathcal{F}) = 0$. (Let G be the Galois group of U' over U. For any étale map $V \to U$, $\Gamma(V, \pi_*\pi^*\mathcal{F}) = \Gamma(U' \times_U V, \mathcal{F})$, which is a G-module with fixed module $\Gamma(V, \mathcal{F})$. The trace map is $s \mapsto \sum_{g\in G} gs$. To see that it is surjective, look on the stalks.)

Finally, a locally constant torsion sheaf is a direct limit of locally constant sheaves with finite fibres. □

Proposition 14.13. *Let U be a connected variety (or scheme) with $\pi_1(U, \bar{u}) = 1$ for one (hence, every) geometric point $\bar{u} \to U$. Then $H^1(U, \mathcal{F}) = 0$ for any locally constant sheaf on U.*

Proof. Because of (6.16), any locally constant sheaf is constant, and because of (11.3), $H^1(U, \mathcal{F}_M) = \mathrm{Hom}(\pi_1(U, \bar{u}), M) = 0$. □

Theorem 14.14. *Let U be a smooth affine curve over a field k, and let $\mathcal{F}$ be the locally constant sheaf on U corresponding to a discrete $\pi_1(U, \bar{u})$-module M. Then $H^r(U, \mathcal{F}) \cong H^r(\pi_1(U, \bar{u}), M)$ for all r.*

Proof. Let $\tilde{U}$ be the universal covering variety of U. Then $H^r(\tilde{U}, \mathcal{F}) = 0$ for $r \geq 2$ by (14.12), and $H^1(\tilde{U}, \mathcal{F}) = 0$ because $\tilde{U}$ is simply connected. We can now apply (14.11). □

Curves over finite fields. Let k be a finite field, and let $\Gamma = \mathrm{Gal}(k^{\mathrm{al}}/k)$. Then $\Gamma \cong \hat{\mathbb{Z}}$, with $1 \in \hat{\mathbb{Z}}$ corresponding to the Frobenius element F in Γ. There are canonical isomorphisms

$$H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \to \mathrm{Hom}_{\mathrm{conts}}(\Gamma, \mathbb{Z}/n\mathbb{Z}) \xrightarrow{\varphi \mapsto \varphi(F)} \mathbb{Z}/n\mathbb{Z}.$$

PROPOSITION 14.15. *Let M be a finite discrete Γ-module, and let $\check{M} = \mathrm{Hom}(M, \mathbb{Z}/n\mathbb{Z})$ (dual abelian group). Then the cup-product pairing*

$$H^r(\Gamma, M) \times H^{1-r}(\Gamma, \check{M}) \to H^1(\Gamma, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

is a perfect pairing of finite groups.

PROOF. By definition, $H^0(\Gamma, M) = M^\Gamma$, the largest subgroup of M on which Γ acts trivially. The map sending a crossed homomorphism $f\colon\ \Gamma \to M$ to its value on the Frobenius element induces an isomorphism of $H^1(\Gamma, M)$ with M_Γ, the largest quotient of M on which Γ acts trivially. In the duality $M \leftrightarrow \check{M}$, the largest subgroup on which Γ acts trivially corresponds to the largest quotient group on which it acts trivially, i.e., the perfect pairing

$$M \times \check{M} \to \mathbb{Z}/n\mathbb{Z}$$

induces a perfect pairing

$$M^\Gamma \times (\check{M})_\Gamma \to \mathbb{Z}/n\mathbb{Z}.$$

□

The proposition can be stated more succinctly as: for any finite Γ-module M killed by n,

$$(M^\Gamma)^\vee = (M^\vee)_\Gamma, \quad (M_\Gamma)^\vee = (M^\vee)^\Gamma.$$

Let U be a connected nonsingular curve over a finite field k, and let $\bar{U}$ be the curve over k^{al} obtained by base change. For any torsion Γ-module M,

$$H^r(\Gamma, M) = M^\Gamma, M_\Gamma, 0$$

respectively for $r = 0, 1, \geq 2$. Therefore, for any torsion sheaf $\mathcal{F}$ on U, the Hochschild-Serre spectral sequence gives short exact sequences

$$0 \to H^{r-1}(\bar{U}, \mathcal{F})_\Gamma \to H^r(U, \mathcal{F}) \to H^r(\bar{U}, \mathcal{F})^\Gamma \to 0,$$

and similarly for the cohomology groups with compact support,

$$0 \to H_c^{r-1}(\bar{U}, \mathcal{F})_\Gamma \to H_c^r(U, \mathcal{F}) \to H_c^r(\bar{U}, \mathcal{F})^\Gamma \to 0.$$

If $\mathcal{F}$ is a finite locally constant sheaf on U, then the groups in this sequence are finite, and according to Proposition 14.15, its dual is

$$0 \to (H_c^r(\bar{U}, \mathcal{F})^\vee)_\Gamma \to H_c^r(U, \mathcal{F})^\vee \to (H_c^{r-1}(\bar{U}, \mathcal{F})^\vee)^\Gamma \to 0.$$

On using Theorem 14.7to replace two of the groups in this sequence, we obtain the sequence

$$0 \to H^{2-r}(\bar{U}, \check{\mathcal{F}}(1))_\Gamma \to H_c^r(U, \mathcal{F})^\vee \to H^{3-r}(\bar{U}, \check{\mathcal{F}}(1))^\Gamma \to 0.$$

This suggests:

Theorem 14.16 (Poincaré Duality). *For any finite locally constant sheaf $\mathcal{F}$ on U and integer $r \geq 0$, there is a canonical perfect pairing of finite groups*

$$H^r_c(U,\mathcal{F}) \times H^{3-r}(U,\check{\mathcal{F}}(1)) \to H^3_c(U,\mu_n) \cong \mathbb{Z}/n\mathbb{Z}.$$

Proof. The only difficulty is in defining the pairing, and verifying that it is compatible with the pairing over $\bar{U}$, i.e., in showing that

$$\begin{tikzcd}
0 \arrow[r] & H^{r-1}_c(\bar{U},\mathcal{F})_\Gamma \arrow[r] \arrow[d, "\approx"] & H^r_c(U,\mathcal{F}) \arrow[r] \arrow[d] & H^r_c(\bar{U},\mathcal{F})^\Gamma \arrow[r] \arrow[d, "\approx"] & {} \\
0 \arrow[r] & (H^{3-r}(\bar{U},\check{\mathcal{F}}(1))^\Gamma)^\vee \arrow[r] & H^{3-r}(U,\check{\mathcal{F}}(1))^\vee \arrow[r] & (H^{2-r}(\bar{U},\check{\mathcal{F}}(1))_\Gamma)^\vee \arrow[r] & {}
\end{tikzcd}$$

commutes. □

An exact sequence in the Galois cohomology of a function field over a finite field. Let X be a complete nonsingular connected curve over a finite field k, and let U be an open subset of X. Let $\mathcal{F}$ be the sheaf on U corresponding to a finite $\pi_1(U,\bar{u})$-module M of order prime to the characteristic of k. We wish to interpret the exact sequence

$$\cdots \to H^r_Z(U,j_!\mathcal{F}) \to H^r(X,j_!\mathcal{F}) \to H^r(U,\mathcal{F}) \to \cdots$$

purely in terms of Galois cohomology groups.

Let $G = \pi_1(U,\bar{u})$ and for all $v \in X \setminus U$, let G_v be the decomposition group at v. Then

$$\begin{aligned} H^r(U,\mathcal{F}) &\cong H^r(G,M) \\ H^r(X,j_!\mathcal{F}) \overset{\text{df}}{=} H^r_c(U,\mathcal{F}) &\cong H^{3-r}(U,\check{\mathcal{F}}(1))^\vee \cong H^{3-r}(U,\check{M}(1))^\vee. \end{aligned}$$

It remains to compute $H^r_Z(U,j_!\mathcal{F})$. By excision, $H^r_Z(U,j_!\mathcal{F}) = \oplus H^r_z(U_z,j_!\mathcal{F})$ where $U_z = \operatorname{Spec}\mathcal{O}^h_{X,z}$.

Lemma 14.17. *Let $V = \operatorname{Spec} A$ with A a discrete valuation ring, and let u and z respectively be the open and closed points of V. For any sheaf $\mathcal{F}$ on u, $H^r(V,j_!\mathcal{F}) = 0$ for all r, and so the boundary map in the exact sequence of the pair (V,u),*

$$H^{r-1}(u,\mathcal{F}) \to H^r_z(V,\mathcal{F})$$

is an isomorphism for all r.

Proof. See my book *Arithmetic Duality Theorems*, II 1.1. □

Thus

$$H^r_Z(U,j_!\mathcal{F}) = \oplus_{v\in Z} H^{r-1}(G_v,M).$$

Theorem 14.18. *For any finite discrete G-module M of order relatively prime to the characteristic of k, there is a canonical exact sequence*

$$\begin{array}{ccccccc} 0 & \to & H^0(G,M) & \to & \oplus_{v\in Z}H^0(G_v,M) & \to & H^2(G,\check{M}(1))^\vee \\ & & & & & & \downarrow \\ & & H^1(G,\check{M}(1))^\vee & \leftarrow & \oplus_{v\in Z}H^1(G_v,M) & \leftarrow & H^1(G,M) \\ & & \downarrow & & & & \\ & & H^2(G,M) & \to & \oplus_{v\in Z}H^2(G_v,M) & \to & H^0(G,\check{M}(1))^\vee \to 0. \end{array}$$

Proof. Make the substitutions described above in the exact sequence of $j_!\mathcal{F}$ for the pair (X,U), where $\mathcal{F}$ is the locally constant sheaf on U corresponding to M. □

An exact sequence in the Galois cohomology of a number field. Let K be a number field, and let S be a finite set of primes of K including the infinite primes. Let R_S be the ring of S-integers in K, i.e., the ring of elements a of K such that $\mathrm{ord}_v(a)\geq 0$ for $v\notin S$.

Theorem 14.19. *Let G_S be the Galois group of the maximal extension of K unramified outside S, and let M be a finite discrete G_S-module whose order is a unit in R_S. There is a canonical exact sequence*

$$\begin{array}{ccccccc} 0 & \to & H^0(G_S,M) & \to & \oplus_{v\in Z}H^0(G_v,M) & \to & H^2(G_S,\check{M}(1))^\vee \\ & & & & & & \downarrow \\ & & H^1(G_S,\check{M}(1))^\vee & \leftarrow & \oplus_{v\in Z}H^1(G_v,M) & \leftarrow & H^1(G_S,M) \\ & & \downarrow & & & & \\ & & H^2(G_S,M) & \to & \oplus_{v\in Z}H^2(G_v,M) & \to & H^0(G_S,\check{M}(1))^\vee \to 0. \end{array}$$

Theorem 14.19is a very important result of Tate — see his talk at ICM 1962 (Poitou obtained similar results about the same time). For a direct proof of it (based on an unpublished proof of Tate), see I.4 of my book, *Arithmetic Duality Theorems.* Alternatively, it can be recovered from a theorem in the étale cohomology of R_S (see (14.22) below).

Pairings and Ext-groups. In this subsection, we explain how to define the pairings in the above duality theorems, and also how to extend the theorems to more general sheaves.

We first state a theorem generalizing (14.7), and then explain it.

Theorem 14.20. *Let U be a nonsingular curve over an algebraically closed field k. For all constructible sheaves $\mathcal{F}$ of $\mathbb{Z}/n\mathbb{Z}$-modules on U and all $r\geq 0$, there is a canonical perfect pairing of finite groups*

$$H^r_c(U,\mathcal{F})\times \mathrm{Ext}^{2-r}_{U,n}(\mathcal{F},\mu_n)\to H^r_c(U,\mu_n)\cong \mathbb{Z}/n\mathbb{Z}.$$

Constructible sheaves. A sheaf $\mathcal{F}$ on a curve U over a field k is *constructible* if

(a) $\mathcal{F}|V$ is locally constant for some nonempty open subset V of U;
(b) the stalks of $\mathcal{F}$ are finite.

Applying (8.17) and (6.16), we see that to give a constructible sheaf on U, we have to give

(a) a finite set S of closed points of U; let $V = U \setminus S$;
(b) a finite discrete $\pi_1(V, \bar{\eta})$-module M_V (here η is the generic point of U);
(c) a finite discrete $\mathrm{Gal}(k(v)^{\mathrm{sep}}/k(v))$-module M_v for each $v \in S$;
(d) a homomorphism $M_v \to M_V$ that is equivariant for the actions of the groups.

A sheaf of abelian groups $\mathcal{F}$ can be regarded as a sheaf of $\mathbb{Z}$-modules (here $\mathbb{Z}$ is the constant sheaf). To say that it is a sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules means simply that $n\mathcal{F} = 0$.

The Ext group. For any variety (or scheme) X, let $Sh(X_{et}, n)$ be the category of sheaves of $\mathbb{Z}/n\mathbb{Z}$-modules on X_{et}. Just as for the full category, $Sh(X_{et}, n)$ has enough injectives. For a sheaf $\mathcal{F}$ on $Sh(X_{et}, n)$, $H^r(X_{et}, \mathcal{F})$ will be the same whether computed using injective resolutions in $Sh(X_{et})$ or in $Sh(X_{et}, n)$ (see EC III 2.25). However, if $\mathcal{F}$ and $\mathcal{G}$ are two sheaves killed by n, then $\mathrm{Ext}^r(\mathcal{F}, \mathcal{G})$ will depend on which category we compute it in. This can be seen already when X is the spectrum of an algebraically closed field. Here $Sh(X_{et})$ is the category of $\mathbb{Z}$-modules, and $Sh(X_{et}, n)$ is the category of $\mathbb{Z}/n\mathbb{Z}$-modules. In the first category, $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \neq 0$, because

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n^2\mathbb{Z} \xrightarrow{n} \mathbb{Z}/n\mathbb{Z} \to 0$$

doesn't split, whereas in the second category $\mathrm{Ext}^1(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) = 0$ because $\mathbb{Z}/n\mathbb{Z}$ is injective (as a $\mathbb{Z}/n\mathbb{Z}$-module).

By $\mathrm{Ext}^r_{U,n}(\mathcal{F}, \mathcal{G})$ we mean the Ext group computed in the category $Sh(U_{et}, n)$.

LEMMA 14.21. *If $\mathcal{F}$ is locally constant, then* $\mathrm{Ext}^r_{U,n}(\mathcal{F}, \mu_n) \cong H^r(U_{et}, \check{\mathcal{F}}(1))$ *where* $\check{\mathcal{F}}(1)$ *is the sheaf*

$$V \mapsto \mathrm{Hom}_V(\mathcal{F}|V, \mu_n).$$

SKETCH OF PROOF. For sheaves $\mathcal{F}$ and $\mathcal{G}$ on an arbitrary variety (or scheme) Y, we define $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ to be the sheaf

$$V \mapsto \mathrm{Hom}_{V_{et}}(\mathcal{F}|V, \mathcal{G}|V)$$

— it is easy to check that $\underline{\mathrm{Hom}}(\mathcal{F}, \mathcal{G})$ is in fact a sheaf. Let $\mathcal{F}_0$ be a locally constant sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules with finite stalks. The functor

$$\mathcal{G} \mapsto \underline{\mathrm{Hom}}(\mathcal{F}_0, \mathcal{G})\colon Sh(Y_{et}, n) \to Sh(Y_{et}, n)$$

is left exact, and so we can form its right derived functors $\underline{\mathrm{Ext}}^r(\mathcal{F}_0, -)$. These are called the *local Ext groups.* They are related to the global Ext groups by a spectral sequence

$$H^r(Y_{et}, \underline{\mathrm{Ext}}^s(\mathcal{F}_0, \mathcal{G})) \Rightarrow \mathrm{Ext}^s(\mathcal{F}_0, \mathcal{G}).$$

Under the hypothesis on $\mathcal{F}_0$, the stalk

$$\underline{\mathrm{Ext}}^r(\mathcal{F}_0, \mathcal{G})_{\bar{y}} = \mathrm{Ext}^r((\mathcal{F}_0)_{\bar{y}}, \mathcal{G}_{\bar{y}})$$

for all geometric points $\bar{y}$. The Ext at right is formed in the category of $\mathbb{Z}/n\mathbb{Z}$-modules. Therefore, in the situation of the lemma

$$\underline{\mathrm{Ext}}^r(\mathcal{F}, \mu_n)_{\bar{y}} = \mathrm{Ext}^r(\mathcal{F}_{\bar{y}}, \mu_n) \approx \mathrm{Ext}^r(\mathcal{F}_{\bar{y}}, \mathbb{Z}/n\mathbb{Z}) = 0 \quad \text{for } r > 0,$$

and the spectral sequence gives isomorphisms

$$H^r(X_{et}, \underline{\mathrm{Hom}}(\mathcal{F}, \mu_n)) \cong \mathrm{Ext}^r_{X,n}(\mathcal{F}, \mu_n),$$

as required. For more details, see EC II 3.20, III 1.22, III 1.31. □

Thus, for a locally constant sheaf $\mathcal{F}$, the pairing in Theorem 14.20 takes the form

$$H^r_c(U,\mathcal{F})\times H^{2-r}(U,\check{\mathcal{F}}(1))\to H^2_c(U,\mu_n)\cong\mathbb{Z}/n\mathbb{Z}.$$

The pairing. There are various ways of defining the pairings, but (fortunately) they all agree, at least up to sign. Perhaps the simplest is to interpret the Ext groups as equivalence classes of extensions, and then define the pairing as an iterated boundary map.

In any abelian category, the elements of $\mathrm{Ext}^r(A,B)$, $r>0$, can be interpreted as equivalence classes of r-fold extensions

$$0\to B\to E_1\to\cdots\to E_r\to A\to 0.$$

There is a natural pairing

$$\mathrm{Ext}^r(A,B)\times\mathrm{Ext}^s(B,C)\to\mathrm{Ext}^{r+s}(A,C)$$

that can either be defined by splicing extensions, or as an iterated boundary map, i.e., break the extension

$$0\to C\to E_1\to\cdots\to E_s\to B\to 0$$

into short exact sequences

$$\begin{array}{ccccccccc}0&\to&K_s&\to&E_s&\to&B&\to&0\\0&\to&K_{s-1}&\to&E_{s-1}&\to&K_s&\to&0\\&&&&\cdots&&&&\end{array}$$

and form the boundary maps

$$\mathrm{Ext}^r(A,B)\to\mathrm{Ext}^{r+1}(A,K_s)\to\mathrm{Ext}^{r+2}(A,K_{s-1})\to\cdots\to\mathrm{Ext}^{r+s}(A,C).$$

See Mitchell, B., *Theory of Categories*, Academic Press, 1965, VII.3 or Bourbaki, N., Algèbre, Chap. X.

In our case, an element of $\mathrm{Ext}^s(\mathcal{F},\mu_n)$ is represented by an exact sequence

$$0\to\mu_n\to E_1\to\cdots\to E_s\to\mathcal{F}\to 0.$$

Apply $j_!$ to get an extension

$$0\to j_!\mu_n\to j_!E_1\to\cdots\to j_!E_s\to j_!\mathcal{F}\to 0,$$

which defines an iterated boundary map

$$H^r(X,j_!\mathcal{F})\to H^{r+s}(X,j_!\mu_n).$$

The proof of Theorem 14.20. For a constructible sheaf $\mathcal{F}$, $\underline{\mathrm{Ext}}^r_{X,n}(\mathcal{F},\Lambda)=0$ for $r\geq 2$ and $\underline{\mathrm{Ext}}^1_{X,n}(\mathcal{F},\Lambda)$ has support on a finite set (the complement of the open set V such that $\mathcal{F}|V$ is locally constant), and so the vanishing of $\mathrm{Ext}^r_{X,n}(\mathcal{F},\mu_n)$ follows from the first statement and the spectral sequence in the proof of (14.21). Theorem 14.7 implies Theorem 14.20 in the case that $\mathcal{F}$ is locally constant, and a five-lemma argument then allows one to obtain the full theorem (see EC p177, Step 2).

Rings of integers in number fields. As before, let K be a totally imaginary number field, and let S be a finite set of primes of K including the infinite primes. Let R_S be the ring of S-integers in K, and let n be an integer that is a unit in R_S.

THEOREM 14.22. *Let* $U = \operatorname{Spec} R_S$.

(a) *There is a canonical isomorphism* $H^3_c(U, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$.
(b) *For any constructible sheaf* $\mathcal{F}$ *of* $\mathbb{Z}/n\mathbb{Z}$*-modules on* U*, there is a canonical perfect pairing*

$$H^r_c(U, \mathcal{F}) \times \operatorname{Ext}^{3-r}_{U,n}(\mathcal{F}, \mu_n) \to H^3_c(U, \mu_n) \cong \mathbb{Q}/\mathbb{Z}$$

of finite groups.

For a proof of this theorem, see my *Arithmetic Duality Theorems*, II 3. Alternatively, Theorem 14.19 implies Theorem 14.22 in the case that $\mathcal{F}$ is locally constant, and it is easy to deduce from that the general case (ib. II 3.5).

15. Cohomological Dimension.

A variety (or scheme) X is said to have *cohomological dimension* c if c is the least integer (or ∞) such that $H^r(X_{et}, \mathcal{F}) = 0$ for $r > c$ and all torsion sheaves $\mathcal{F}$ on X_{et}. The cohomological dimension c of a profinite group is defined similarly: $H^r(G, M) = 0$ for $r > c$ and M a discrete torsion G-module. For a field K,

$$cd(K) \stackrel{\text{df}}{=} cd(\operatorname{Spec} K) = cd(\operatorname{Gal}(K^{\text{sep}}/K)).$$

For example, $cd(K) = 0$ if K is separably closed, $cd(K) = 1$ if K is finite (although $H^2(\operatorname{Gal}(K^{\text{al}}/K), \mathbb{Z}) = \mathbb{Q}/\mathbb{Z} \neq 0$), and $cd(\mathbb{R}) = \infty$ (because the cohomology of a cyclic group is periodic).

A standard result in algebraic topology says that, for a manifold M of real dimension d, $H^r(M, \mathcal{F}) = 0$ for $r > d$, and a standard result in complex analysis says that for a Stein manifold M of complex dimension d (hence real dimension $2d$), $H^r(M, \mathcal{F}) = 0$ for $r > d$. The analogues of these statements are true for the étale topology.

Theorem 15.1. *For a variety X over an algebraically closed field k,*

$$cd(X) \leq 2\dim(X).$$

If X is affine, then

$$cd(X) \leq \dim(X).$$

In fact, as we shall see later, $H^{2\dim X}(X, \mathbb{Z}/n\mathbb{Z}) \neq 0$ when X is complete. Throughout the proof, $m = \dim X$.

We say that a sheaf $\mathcal{F}$ has support in dimension d if $\mathcal{F} = \cup i_{Z*}\mathcal{F}_Z$ where the union is over the irreducible closed subvarieties Z of X of dimension d, i_Z is the inclusion $Z \hookrightarrow X$, and $\mathcal{F}_Z$ is a sheaf on Z. For such a sheaf

$$H^r(X, \mathcal{F}) = \varinjlim H^r(X, i_{Z*}\mathcal{F}_Z) = \varinjlim H^r(Z, \mathcal{F}_Z).$$

The cohomological dimension of fields. The starting point of the proof is the following result for fields (which was conjectured by Grothendieck and proved by Tate).

Theorem 15.2. *Let K be a field of transcendence degree d over its subfield k. Then*

$$cd(K) \leq cd(k) + d.$$

Proof. Tsen's Theorem is the case $d = 1$, k algebraically closed. The general case is proved by induction starting from this. See Shatz 1972, p119, Theorem 28. □

Proof of the first statement of Theorem 15.1. The theorem is proved by induction on the dimension of X. Thus, we may suppose that if $\mathcal{F}$ has support in dimension $d < m$, then

$$H^r(X, \mathcal{F}) = 0 \qquad \text{for } r > 2d.$$

We may assume X to be connected. Let $g\colon\ \eta \hookrightarrow X$ be the generic point of X. The map $\mathcal{F} \to g_*g^*\mathcal{F}$ corresponding by adjointness to the identity map on $g^*\mathcal{F}$ induces an isomorphism $\mathcal{F}_{\bar\eta} \to (g_*g^*\mathcal{F})_{\bar\eta}$ on the generic stalks, and so every section of the kernel and cokernel of the map has support on a proper closed subscheme. Thus, by induction, it suffices to prove the theorem for a sheaf of the form $g_*\mathcal{F}$.

Local and strictly local rings of subvarieties. Let X be an irreducible algebraic variety over a field k. To any irreducible closed subvariety Z of X we can attach a ring $\mathcal{O}_{X,Z}$. In terms of varieties, it is $\varinjlim \Gamma(U, \mathcal{O}_X)$ where U runs over the open subsets of X such that $U \cap Z \neq \emptyset$. In terms of schemes, it is the local ring at the generic point of Z. Its residue field is the field $k(Z)$ of rational functions on Z, which has transcendence degree $\dim Z$ over k. Moreover, $k(Z)$ is separable over k, i.e., there exist algebraically independent elements $T_1, \ldots, T_m$ in $k(Z)$ such that $k(Z)$ is a separable algebraic extension of $k(T_1, \ldots, T_m)$ (otherwise Z wouldn't be a variety).

Lemma 15.3. *Let A be a Henselian local ring containing a field k. Let K be the residue field of A, and assume that K is separable over k. Then A contains a field L that is mapped isomorphically onto K by the quotient map $\pi\colon\ A \to K$.*

Proof. Let $T_1, \ldots, T_m$ be elements of A such that $\pi(T_1), \ldots, \pi(T_m)$ are algebraically indpendent over k and K is a separable algebraic extension of $k(\pi(T_1), \ldots, \pi(T_m))$. Let L be a maximal subfield of A containing $k(T_1, \ldots, T_m)$ — such a field exists by Zorn's Lemma. Because $\mathfrak{m}_A \cap L = \{0\}$, π maps L isomorphically onto a subfield πL of K. Let $\alpha \in K$. Then α is a simple root of a polynomial $f(T) \in L[T]$. Because A is Henselian, there exists an $\alpha' \in A$ such that $\pi(\alpha') = \alpha$ and α' is a root of $f(T)$. Now $L[\alpha']$ is a subfield of A containing L. Because L is maximal, $\alpha' \in L$, which shows that $\alpha \in \pi L$. □

Lemma 15.4. *Let Z be a closed irreducible subvariety of a variety X, and A be the Henselization of $\mathcal{O}_{X,Z}$. The field of fractions K of A contains $k(Z)$, and has transcendence degree $\dim X - \dim Z$ over $k(Z)$.*

Proof. The preceding lemma shows that $K \supset k(Z)$, and so it remains to compute its transcendence degree. But K is an algebraic extension of $k(X)$, and so has transcendence degree $\dim X$ over k, and $k(Z)$ has transcendence degree $\dim Z$ over k. □

Lemma 15.5. *The sheaf $R^s g_*\mathcal{F}$ has support in dimension $\leq m - s$.*

Proof. Let Z be a closed irreducible subvariety of X, and let $z = \operatorname{Spec} k(Z)$ be its generic point. The choice of a separable closure $k(Z)^{\mathrm{sep}}$ of $k(Z)$ determines a geometric point $\bar{z} \to z \hookrightarrow X$ of X, and the strictly local ring at $\bar{z}$ is the maximal unramified extension of $\mathcal{O}^h_{X,Z}$. Thus, its residue field is $k(Z)^{\mathrm{sep}}$, and its field of fractions $K_{\bar{z}}$ is an algebraic extension of $k(X)$. The same argument as above shows that it has transcendence degree $\dim X - \dim Z$ over $k(Z)^{\mathrm{sep}}$. According to (8.5),

$$(R^s g_*\mathcal{F})_{\bar{z}} = H^s(K_{\bar{z}}, \mathcal{F}),$$

which is zero for $s > m - \dim Z$ by (15.2). Thus, $(R^s g_*\mathcal{F})_{\bar{z}} \neq 0 \Rightarrow s \leq m - \dim Z$, as claimed. □

From the induction hypothesis, we find that

$$H^r(X, R^s g_*\mathcal{F}) = 0 \text{ whenever } s \neq 0 \text{ and } r > 2(m - s).$$

In the spectral sequence

$$H^r(X, R^s g_*\mathcal{F}) \Rightarrow H^{r+s}(\eta, \mathcal{F}),$$

the final term $H^{r+s}(\eta, \mathcal{F}) = 0$ for $r + s > n$ (by 15.2 again). It follows that $H^r(X, g_*\mathcal{F}) = 0$ for $r > 2m$.

QUESTION 15.6. It also follows that the complex

$$E_2^{0,n} \to E_2^{2,n-1} \to \cdots \to E_2^{2n,0}$$

in the above spectral sequence must be exact. What is it?

Proof of the second statement. The proof of the second statement requires a complicated induction and limiting argument, starting from the case $m = 1$ (which was proved in 14.12). It requires the Proper Base Change Theorem §17.

16. Purity; the Gysin Sequence.

Fix an integer $n > 0$, let $\Lambda = \mathbb{Z}/n\mathbb{Z}$. For any ring R such that $n1$ is a unit in R, we define $\mu_n(R)$ to be the group of n^{th} roots of 1 in R, and we define

$$\mu_n(R)^{\otimes r} = \begin{cases} \mu_n(R) \otimes \cdots \otimes \mu_n(R), & r \text{ copies, } r > 0 \\ \Lambda, & r = 0 \\ \mathrm{Hom}_\Lambda(\mu_n(R)^{\otimes r}, \Lambda), & r < 0. \end{cases}$$

When R is an integral domain containing the n^{th} roots of 1, then each of these groups is a free module of rank 1 over Λ, and the choice of a primitive n^{th} root of 1 determines bases simultaneously for all of them.

Let X be a variety over a field k whose characteristic doesn't divide n (or a scheme such that $n\mathcal{O}_X = \mathcal{O}_X$). We define $\Lambda(r)$ to be the sheaf on X_{et} such that

$$\Gamma(U, \Lambda(r)) = \mu_n(\Gamma(U, \mathcal{O}_U))^{\otimes r}$$

for all $U \to X$ étale and affine. If k contains the n^{th} roots of 1, then each sheaf is isomorphic to the constant sheaf Λ, and the choice of a primitive n^{th} root of 1 in k determines isomorphisms $\Lambda(r) \xrightarrow{\approx} \Lambda$ for all r. In any case, each sheaf $\Lambda(r)$ is locally constant. For a sheaf $\mathcal{F}$ on X_{et} killed by n, let

$$\mathcal{F}(r) = \mathcal{F} \otimes \Lambda(r), \quad \text{all } r \in \mathbb{Z}.$$

Statement of the theorem; consequences. For simplicity, now assume that k is algebraically closed.

A *smooth pair* (Z, X) of k-varieties is a nonsingular k-variety X together with a nonsingular subvariety Z. We say that (Z, X) has *codimension* c if every connected component[26] of Z has codimension c in the corresponding component of X.

Theorem 16.1. *For any smooth pair of k-varieties (Z, X) of codimension c and locally constant sheaf $\mathcal{F}$ of Λ-modules on X, there are canonical isomorphisms*

$$H^{r-2c}(Z, \mathcal{F}(-c)) \to H^r_Z(X, \mathcal{F})$$

for all $r \geq 0$.

Corollary 16.2. *In the situation of the theorem, there are isomorphisms*

$$H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F}), \quad 0 \leq r < 2c - 1$$

*and an exact sequence (*the Gysin sequence)

$$0 \to H^{2c-1}(X, \mathcal{F}) \to H^{2c-1}(U, \mathcal{F}|U) \to \cdots \to H^{r-2c}(Z, \mathcal{F}(-c)) \to H^r(X, \mathcal{F}) \to H^r(U, \mathcal{F}) \to \cdots.$$

Proof. Use the theorem to replace the groups $H^r_Z(X, \mathcal{F})$ in the exact sequence of the pair $(X, X \setminus Z)$ with the groups $H^{r-2c}(Z, \mathcal{F}(-c))$. □

Example 16.3. Recall that, for any nonsingular affine curve U over k, $H^1(U, \mathbb{G}_m) = Pic(U)$, and that $H^r(U, \mathbb{G}_m) = 0$ for $r > 1$. Therefore $H^1(\mathbb{A}^1, \mathbb{G}_m) = 0$,

[26]Recall that every connected component of a nonsingular variety is irreducible.

because $k[T]$ is a principal ideal domain. The cohomology sequence of the Kummer sequence

$$\begin{array}{ccccccccc} 0 & \to & H^0(\mathbb{A}^1,\mu_n) & \to & H^0(\mathbb{A}^1,\mathbb{G}_m) & \xrightarrow{n} & H^0(\mathbb{A}^1,\mathbb{G}_m) & \to & H^1(\mathbb{A}^1,\mu_n) & \to & 0 \\ & & & & \| & & \| & & & & \\ & & & & k[T]^\times & \xrightarrow{n} & k[T]^\times & & & & \\ & & & & \| & & \| & & & & \\ & & & & k^\times & \xrightarrow{n} & k^\times & & & & \end{array}$$

shows that $H^r(\mathbb{A}^1,\mu_n)=0$ for $r>0$. The Künneth formula (§22) now implies that $H^r(\mathbb{A}^m,\Lambda)=0$ for $r>0$, i.e., $\mathbb{A}^m$ is "acyclic". Therefore, the Gysin sequence for $(\mathbb{P}^{m-1},\mathbb{P}^m)$ shows that

$$\begin{array}{rcl} H^0(\mathbb{P}^m,\Lambda) & \cong & H^0(\mathbb{A}^m,\Lambda)\cong\Lambda \\ H^1(\mathbb{P}^m,\Lambda) & \hookrightarrow & H^1(\mathbb{A}^m,\Lambda)=0 \end{array}$$

and that

$$H^{r-2}(\mathbb{P}^{m-1},\Lambda(-1))\cong H^r(\mathbb{P}^m,\Lambda).$$

An induction argument now shows that

$$\begin{array}{ccccccccc} H^r(\mathbb{P}^m,\Lambda) & = & \Lambda, & 0, & \Lambda(-1), & 0, & \Lambda(-2), & \ldots, & \Lambda(-m), \\ r & = & 0 & 1 & 2 & 3 & 4 & \ldots & 2m \end{array}$$

that is, that

$$H^r(\mathbb{P}^m,\Lambda)=\begin{cases} \Lambda(-\frac{r}{2}), & r\text{even}, \ \leq 2m; \\ 0, & \text{otherwise.}\end{cases}$$

EXAMPLE 16.4. Let X be a nonsingular hypersurface in $\mathbb{P}^{m+1}$, i.e., a closed subvariety of $\mathbb{P}^{m+1}$ whose homogeneous ideal $I(X)=(f)$ where f is a homogeneous polynomial in $k[T_0,\ldots,T_{m+1}]$ such that

$$f,\frac{\partial f}{\partial T_0},\ldots,\frac{\partial f}{\partial T_{m+1}}$$

have no common zero in $\mathbb{P}^{m+1}$. Then $U\overset{\mathrm{df}}{=}\mathbb{P}^{m+1}\setminus X$ is affine (AG 5.19), and so $H^r(U,\Lambda)=0$ for $r>m$. Therefore, the Gysin sequence provides us with maps

$$H^r(X,\Lambda)\to H^{r+2}(\mathbb{P}^{m+1},\Lambda(1))$$

that are isomorphisms for $r>m$ and a surjection for $r=m$. Hence, $H^r(X,\Lambda)\cong H^r(\mathbb{P}^m,\Lambda)$ for $r>m$, and $H^m(X,\Lambda)\approx H^m(\mathbb{P}^m,\Lambda)\oplus H^m(X,\Lambda)'$ where $H^m(X,\Lambda)'$ is the kernel of $H^m(X,\Lambda)\to H^{m+2}(\mathbb{P}^{m+1},\Lambda(1))$. Using Poincaré duality (see the next aside), we obtain a canonical decomposition

$$H^*(X,\Lambda)\cong H^*(\mathbb{P}^m,\Lambda)\oplus H^m(X,\Lambda)'$$

where $H^*(-,\Lambda)$ denotes the graded Λ-module $\oplus H^r(-,\Lambda)$.

More generally, a closed subvariety X in $\mathbb{P}^N$ of dimension m is said to be a *complete intersection* if its homogeneous ideal is generated by $N-m$ polynomials. Then there is a chain

$$\mathbb{P}^N\supset X_{N-1}\supset\cdots\supset X_m=X$$

with each $X_r = H_r \cap X_{r+1}$ (as schemes) where H_r is a hypersurface in $\mathbb{P}^m$. We say that X is a smooth complete intersection if there exists such a chain with all X_r nonsingular. For a smooth complete intersection X of dimension n, an induction argument using the Gysin sequence and Poincaré duality again shows that there is a canonical decomposition

$$H^*(X,\Lambda) \cong H^*(\mathbb{P}^m,\Lambda) \oplus H^m(X,\Lambda)'.$$

We shall see later that $H^m(X,\Lambda)'$ is a free Λ-module whose rank β'_m depends only on m and the degrees of the polynomials generating $I(X)$, and, in fact, that there are explicit formulas for β'_m. For example, if X is a nonsingular hypersurface of degree d and dimension m, then

$$\beta'_m = \frac{(d-1)^{m+2} + (-1)^m(d-1)}{d}.$$

Later, we shall see that these results imply that if X is smooth complete intersection of dimension m over a finite field $\mathbb{F}_q$, i.e., the $N-m$ polynomials generating $I(X)$ can be chosen to have coefficients in $\mathbb{F}_q$, then

$$|\#X(\mathbb{F}_q) - \#\mathbb{P}^m(\mathbb{F}_q)| \leq \beta'_m q^{\frac{m}{2}}.$$

Compared with $\#X(\mathbb{F}_q) \sim q^m$, the error term $\beta'_m q^{\frac{m}{2}}$ is *very small!*

Unfortunately, smooth complete intersections are very special varieties. For example, when we blow up a point P on a nonsingular variety X of dimension $m > 1$, P is replaced by the projective space associated with the tangent space at P, i.e., with a copy of $\mathbb{P}^{m-1}$. Thus, for the new variety Y,

$$\#Y(\mathbb{F}_q) = \#X(\mathbb{F}_q) + \#\mathbb{P}^{m-1}(\mathbb{F}_q) - 1 = \#X(\mathbb{F}_q) + q^{m-1} + \cdots + q.$$

If the above inequality holds for X, then it fails for Y.

Aside 16.5. Later we shall see that there is a Poincaré duality theorem of the following form: for a nonsingular variety U of dimension m over k, there is a canonical perfect pairing

$$H^r_c(U,\Lambda) \times H^{2m-r}(U,\Lambda(2m)) \to H^{2m}_c(U,\Lambda(2m)) \cong \Lambda.$$

Here H^r_c is a "cohomology group with compact support", which equals the usual cohomology group when X is complete.

Example 16.6. Let X be a connected nonsingular closed variety of dimension m in projective space. According to Bertini's Theorem (Hartshorne 1977, III.7.9.1) there is a sequence

$$X = X_m \supset X_{m-1} \supset \cdots \supset X_1$$

of connected nonsingular varieties with each X_r a hyperplane section of the preceding variety. From the Gysin sequence, we obtain maps

$$H^1(X_1,\Lambda) \xrightarrow{\text{onto}} H^3(X_2,\Lambda(1)) \xrightarrow{\approx} \cdots \xrightarrow{\approx} H^{2m-1}(X_m,\Lambda(m-1)).$$

Using Poincaré duality, we obtain an injection $(*)$

$$H^1(X,\Lambda(1)) \to H^1(X_1,\Lambda(1)).$$

The inclusion $X_1 \hookrightarrow X$ defines a map

$$Pic(X) \to Pic(X_1)$$

on Picard groups. The map $(*)$ can be identified with

$$Pic(X)_m \to Pic(X_1)_m.$$

The torsion points on an abelian variety over an algebraically closed field are Zariski dense, and so this shows that the map

$$PicVar(X) \to Jac(X_1)$$

induced by $X_1 \hookrightarrow X$ is injective. This result is known classically: for a "general" curve C on a nonsingular variety X, the Picard variety of X is a subvariety of the Jacobian variety of C.

Restatement of the theorem. We now allow k to be an arbitrary field.

Let Z be a closed subvariety of a variety X (or a closed subscheme of a scheme X), and let $i\colon\ Z \hookrightarrow X$ be the inclusion map. As we saw in 8.18, the functors i^* and i_* define an equivalence between the category of étale sheaves on X with support on Z and the category of étale sheaves on Z. Since i_* is exact and preserves injectives, $H^r(X, i_*\mathcal{F}) = H^r(Z, \mathcal{F})$, and it is sometimes permissible to confuse $\mathcal{F}$ with $i_*\mathcal{F}$.

As usual, we shall let $U = X \setminus Z$ and denote the inclusion $U \hookrightarrow X$ by j.

For a sheaf $\mathcal{F}$ on X, we define $\mathcal{F}^!$ to be the largest subsheaf of $\mathcal{F}$ with support on Z. Thus, for any étale $\varphi\colon\ V \to X$,

$$\mathcal{F}^!(V) = \Gamma_{\varphi^{-1}(Z)}(V, \mathcal{F}) = \mathrm{Ker}(\mathcal{F}(V) \to \mathcal{F}(\varphi^{-1}(U))).$$

It is easy to see that this does in fact define a sheaf on X, and that

$$\mathcal{F}^! = \mathrm{Ker}(\mathcal{F} \to j_*j^*\mathcal{F}).$$

Let $\mathcal{G}$ be a sheaf on X with support on Z. Then the image of any homomorphism $\alpha\colon\ \mathcal{G} \to \mathcal{F}$ lies in $\mathcal{F}^!$, and so

$$\mathrm{Hom}_X(\mathcal{G}, \mathcal{F}^!) \cong \mathrm{Hom}_X(\mathcal{G}, \mathcal{F}).$$

Now define $i^!\mathcal{F}$ to be $\mathcal{F}^!$ regarded as a sheaf on Z, i.e., $i^!\mathcal{F} = i^*\mathcal{F}^!$. Then $i^!$ is a functor $Sh(X_{et}) \to Sh(Z_{et})$, and the displayed isomorphism can be interpreted as a canonical isomorphism

$$\mathrm{Hom}_Z(\mathcal{G}, i^!\mathcal{F}) \cong \mathrm{Hom}_X(i_*\mathcal{G}, \mathcal{F}).$$

This shows that $i^!$ is the right adjoint of i_*. Because $i^!$ has a left adjoint, it is left exact, and because the left adjoint is exact, it preserves injectives.

THEOREM 16.7 (Cohomological Purity). *Let (Z, X) be a smooth pair of algebraic varieties of codimension c. For any locally constant sheaf of Λ-modules on X, $R^{2c}i^!\mathcal{F} \cong (i^*\mathcal{F})(-c)$, and $R^r i^!\mathcal{F} = 0$ for $r \neq 2c$.*

Why Theorem 16.7 implies Theorem 16.1. Consider the functors

$$Sh(X_{et}) \xrightarrow{i^!} Sh(Z_{et}) \xrightarrow{\Gamma(Z,-)} Ab.$$

These functors are left exact, $i^!$ preserves injectives, and the composite is $\Gamma_Z(X,-)$. Therefore (see 12.7), there is a spectral sequence

$$E_2^{r,s} = H^r(Z, R^s i^! \mathcal{F}) \Rightarrow H_Z^{r+s}(X, \mathcal{F}).$$

From Theorem 16.7, we know that $E_2^{r,s} = 0$ unless $s = 2c$. It follows that $E_\infty^{r,s} = E_2^{r,s}$, and hence

$$H_Z^r(X, \mathcal{F}) \cong H^{r-2c}(Z, R^{2c} i^! \mathcal{F}) \cong H^{r-2c}(Z, \mathcal{F}(-c)).$$

Why Theorem 16.7 is true. Once one defines the map, the problem is local for the étale topology on X. The next lemma shows that any smooth pair of codimension c is locally isomorphic (for the étale topology) to a standard smooth pair $(\mathbb{A}^{m-c}, \mathbb{A}^m)$.

LEMMA 16.8. *Let $Z \subset X$ be a smooth pair of codimension c, and let $P \in Z$. There exists an open neighbourhood V of P and an étale map $V \to \mathbb{A}^m$, $m = \dim X$, whose restriction to $V \cap Z$ is an étale map to $\mathbb{A}^{m-c}$.*

PROOF. By assumption, $Tgt_P(Z)$ is a subspace of $Tgt_P(X)$ of codimension c. Choose regular functions $f_1, \ldots, f_m$ defined on an open neighbourhood V of P (in X) such that $df_1, \ldots, df_{m-c}$ form a basis for the dual space of $Tgt_P(Z)$ and $df_1, \ldots, df_m$ form a basis for the dual space of $Tgt_P(X)$. Consider the map

$$\alpha\colon V \to \mathbb{A}^m, \quad Q \mapsto (f_1(Q), \ldots, f_m(Q)).$$

The map $d\alpha\colon\ Tgt_P(X) \to Tgt_{\alpha(P)}(\mathbb{A}^m)$ is $v \mapsto (df_1(v), \ldots, df_m(v))$, which is an isomorphism. Therefore, α is étale at P. Similarly, its restriction to $Z \to \mathbb{A}^{m-c}$ is étale at P (regarded as a point of Z). □

The proof of Theorem 16.7 is by induction, starting from the case $m = 1 = c$, which was proved in (14.3).

Generalization. Recall that a scheme X is said to be *regular* if the local rings $\mathcal{O}_{X,x}$ are regular for all $x \in X$. A morphism $\varphi\colon\ X \to S$ of finite-type is said to be *smooth* if

(a) it is flat, and
(b) for every algebraically closed geometric point $\bar{s} \to S$, the fibre $X_{\bar{s}}$ is regular.

The second condition means that, for any $s \in S$, the fibre $\varphi^{-1}(s)$ is a nonsingular variety. When $X \to S$ is smooth, we say that X is smooth over S.

For a variety X over a field k, we say that X is nonsingular when $X \to \operatorname{Spec} k$ is smooth. Thus, a nonsingular variety is regular, and a regular variety over an algebraically closed field is smooth. A variety over a nonperfect field can be regular without being nonsingular — for example, $Y^2 = X^p - t$ is such a variety if $t \notin k$ and $p = \operatorname{char}(k)$.

EXAMPLE 16.9. Let X_1 be a nonsingular complete curve over $\mathbb{Q}_p$. Then there exists a proper flat morphism $X \to \operatorname{Spec} \mathbb{Z}_p$ whose generic fibre is $X_1 \to \operatorname{Spec} \mathbb{Q}_p$. In fact, there always exists a regular such X, but there does not always exist an X

that is smooth over $\operatorname{Spec}\mathbb{Z}_p$. For elliptic curves, this is discussed in §9 of my notes on elliptic curves.

The following conjecture remains open in general.

Conjecture 16.10. *Let X be a regular scheme, and let $i\colon\ Z \hookrightarrow X$ be a regular closed subscheme of X such that Z has codimension c in X at each point. Then, for all integers n such that $n\mathcal{O}_X = \mathcal{O}_X$, $R^{2c}i^!\Lambda \cong \Lambda(-c)$ and $R^r i^!\Lambda = 0$ for $r \neq 2c$.*

The conjecture is proved in SGA4 when Z and X are both smooth over a base scheme S. Under some hypotheses on X and n, Thomason was able to deduce the conjecture from Quillen's Localization Theorem, which is a purity statement for K-groups (Thomason, R.W., Absolute Cohomological Purity, Bull. Soc. Math. France 112 (1984), 397–406).

Proposition 16.11. *Let (Z, X) be a pair of codimension c for which the purity conjecture (16.10) holds. Let $U = X \setminus Z$, and let i, j be the inclusion maps as usual. Then*

$$R^r j_*\Lambda \cong \begin{cases} \Lambda & r = 0 \\ i_*\Lambda(-c) & r = 2c-1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The exact sequence of the pair (X, U) is

$$\cdots \to H^r_Z(X, \Lambda(-c)) \to H^r(X, \Lambda) \to H^r(U, \Lambda) \to \cdots .$$

For any $\varphi\colon\ X' \to X$ étale we get a similar sequence with U and Z replaced with $U' = \varphi^{-1}(U)$ and $Z' = \varphi^{-1}(Z)$. When we vary X', this becomes an exact sequence of presheaves, which remains exact after we apply the "sheafification" functor a. Now the sheaf associated with $X' \mapsto H^r(X', \Lambda)$ is zero for $r > 0$ and Λ for $r = 0$, and the sheaf associated with $X' \mapsto H^r(U', \Lambda)$ is $R^r j_*\Lambda$. Finally, the sheaf associated with $X' \mapsto H^r_{Z'}(X', \Lambda)$ is $R^r i^!\Lambda$, which is $\Lambda(-c)_Z$ for $r = 2c$ and is zero otherwise. $\square$

17. The Proper Base Change Theorem.

The proper base change theorem in topology. In this subsection, all topological spaces are assumed to be Hausdorff.

Recall that the image of a compact space under a continuous map is compact, and hence is closed. Moreover, a topological space X is compact if and only if, for all topological spaces Y, the projection map $X \times Y \to Y$ is closed (Bourbaki, N., Topologie Générale, I, §10, Cor. 1 to Th. 1).

A continuous map $\pi\colon\ X \to S$ is said to be *proper* if it is universally closed, i.e., if $X \times_S T \to T$ is closed for all continuous maps $T \to S$. When S is locally compact, a continuous map $\pi\colon\ X \to S$ is proper if and only if the inverse image of every compact subset of S is compact, in which case X is also locally compact (ib. Proposition 7).

Let $\pi\colon\ X \to S$ be a continuous map, and let $\mathcal{F}$ be a sheaf on X. For any $s \in S$,

$$(R^r\pi_*\mathcal{F})_s = \varinjlim_V H^r(\pi^{-1}(V), \mathcal{F})$$

where the limit is over the open neighbourhoods V of s in S (cf. 12.2). If π is closed, then, for any open neighbourhood U of $\pi^{-1}(s)$, $\pi(X \smallsetminus U)$ is closed and $V \overset{\text{df}}{=} X \smallsetminus \pi(X \smallsetminus U)$ is an open neighbourhood of s. Since $\pi^{-1}(V) \subset U$, we see that

$$(R^r\pi_*\mathcal{F})_s = \varinjlim_U H^r(U, \mathcal{F})$$

where the limit is now over *all* open neighbourhoods U of $\pi^{-1}(s)$.

Lemma 17.1. *Let Z be a compact subset of a locally compact space X. Then, for any sheaf $\mathcal{F}$ on X, there is a canonical isomorphism*

$$\varinjlim H^r(U, \mathcal{F}) \cong H^r(Z, i^*\mathcal{F})$$

(limit over the open neighbourhoods U of Z in X).

Proof. Since both groups are zero on injective sheaves for $r > 0$, and transform short exact sequences of sheaves to long exact sequences, it suffices (by the uniqueness of derived functors) to verify the statement for $r = 0$. Here one uses the hypotheses on Z and X (see Iverson 1986, III 2.2 for the case $r = 0$ and III 6.3 for the generalization to all r). $\square$

Theorem 17.2. *Let $\pi\colon\ X \to S$ be a proper map where S is a locally compact topological space. For any sheaf $\mathcal{F}$ on X and $s \in S$, $(R^r\pi_*\mathcal{F})_s \cong H^r(X_s, \mathcal{F})$ where $X_s = \pi^{-1}(s)$.*

Proof. Combine the above statements. $\square$

Let $\pi\colon\ X \to S$ and $\mathcal{F}$ be as in the theorem, and let $f\colon\ T \to S$ be a continuous map. Form the fibre product diagram

$$\begin{array}{ccc} X & \xleftarrow{\;f'\;} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi'} \\ S & \xleftarrow{\;f\;} & T \end{array} \qquad \text{where} \quad X' = X \times_S T.$$

Let $t \in T$ map to $s \in S$. Then f' defines an isomorphism $X'_t \to X_s$ on the fibres, and so the theorem implies that

$$(R^r\pi_*\mathcal{F})_s \cong H^r(X_s, \mathcal{F}) \cong H^r(X'_t, f'^*\mathcal{F}) \cong (R^r\pi'_*(f'^*\mathcal{F}))_t.$$

THEOREM 17.3. *In the above situation, there is a canonical isomorphism of sheaves on T*

$$f^*(R^r\pi_*\mathcal{F}) \to R^r\pi'_*(f'^*\mathcal{F}).$$

PROOF. We first define the map. Since f^* and f_* are adjoint, if suffices to define a morphism of functors

$$R^r\pi_* \to f_* \circ R^r\pi'_* \circ f'^*.$$

This we take to be the composite of the morphisms

$$R^r\pi_* \to R^r\pi_* \circ f'_* \circ f'^*$$

(induced by $\mathrm{id} \to f'_* \circ f'^*$),

$$R^r\pi_* \circ f'_* \circ f'^* \to R^r(\pi \circ f')_* \circ f'^* = R^r(f \circ \pi)_* \circ f'^*,$$

and

$$R^r(f \circ \pi')_* \circ f'^* \to f_* \circ R^r\pi'_* \circ f'^*.$$

Thus, we have a canonical map of sheaves $f^*(R^r\pi_*\mathcal{F}) \to R^r\pi'_*(f'^*\mathcal{F})$, which is an isomorphism because it is so on stalks. □

REMARK 17.4. In fact, Theorem 17.2 is the special case of Theorem 17.3 in which $T \to S$ is taken to be $s \to S$.

The proper base change theorem in geometry. Recall (AG 5.25) that an algebraic variety X over k is said to be *complete* if, for all k-varieties T, the projection map $X \times T \to T$ is closed (i.e., sends Zariski closed sets to closed sets). For example, a projective variety over k is complete (AG 5.30), and an affine variety is complete if and only if it is finite.

More generally, a regular map $X \to S$ of varieties (or schemes) is said to be *proper* if, for all $T \to S$, the projection map $X \times_S T \to T$ is closed (and $X \to S$ is separated). For example, a finite morphism, for example, a closed immersion, is proper (ib. 6.24), and a k-variety X is complete if and only if the map $X \to P$ (the point) is proper.

Recall that a sheaf $\mathcal{F}$ on X_{et} is locally constant if there is a covering $(U_i \to X)_{i \in I}$ such that $\mathcal{F}|U_i$ is constant for all $i \in I$. In the case that $\mathcal{F}$ has finite stalks, this is equivalent to there existing a single finite surjective étale map $U \to X$ such that $\mathcal{F}|U$ is constant.

Unfortunately, the class of locally constant sheaves is not stable under the formation of direct images, even by proper maps (not even closed immersions), and so we shall need a larger class that is.

DEFINITION 17.5. A sheaf $\mathcal{F}$ on X_{et} is *constructible* if

(a) for every closed immersion $i\colon\ Z \hookrightarrow X$ with Z irreducible, there exists a nonempty open subset $U \subset Z$ such that $(i^*\mathcal{F})|U$ is locally constant;
(b) $\mathcal{F}$ has finite stalks.

REMARK 17.6. One can show that every torsion sheaf is the union of its constructible subsheaves.

THEOREM 17.7. *Let* $\pi\colon X \to S$ *be a proper morphism, and let* $\mathcal{F}$ *be a constructible sheaf on* X. *Then* $R^r\pi_*\mathcal{F}$ *is constructible for all* $r \geq 0$, *and* $(R^r\pi_*\mathcal{F})_{\bar{s}} = H^r(X_{\bar{s}}, \mathcal{F}|X_{\bar{s}})$ *for every geometric point* $\bar{s} \to S$ *of* S.

Here $X_{\bar{s}} \overset{\text{df}}{=} X \times_S \bar{s}$ is the geometric fibre of π over $\bar{s}$, and $\mathcal{F}|X_{\bar{s}}$ is the inverse image of $\mathcal{F}$ under the map $X_{\bar{s}} \to X$.

COROLLARY 17.8. *Let* X *be a complete variety over a separably closed field* k *and let* $\mathcal{F}$ *be a constructible sheaf on* X.

(a) *The groups* $H^r(X_{et}, \mathcal{F})$ *are finite for all* r.
(b) *For any separably closed field* $k' \supset k$, $H^r(X, \mathcal{F}) = H^r(X', \mathcal{F})$ *where* $X' = X_{k'}$.

PROOF. (a) Consider the map $\pi\colon X \to s$ from X to a point. The map $\mathcal{G} \mapsto \Gamma(s, \mathcal{G})$ identifies the category of constructible sheaves on s_{et} with the category of finite abelian groups, and $R^r\pi_*\mathcal{F}$ with $H^r(X_{et}, \mathcal{F})$. The statement now follows from the theorem, because π is proper.

(b) Let $\bar{s} = \operatorname{Spec} k$ and $\bar{s}' = \operatorname{Spec} k'$. The maps $\bar{s} \to s$ and $\bar{s}' \to s$ corresponding to the inclusions $k \hookrightarrow k$ (identity map) and $k \hookrightarrow k'$ are geometric points of s. The stalk of a sheaf at a geometric point $\bar{t}$ depends only on t, and so the theorem implies that,

$$H^r(X, \mathcal{F}) \overset{17.7}{\cong} (R^r\pi_*\mathcal{F})_{\bar{s}} \cong (R^r\pi_*\mathcal{F})_{\bar{s}'} \overset{17.7}{\cong} H^r(X_{k'}, \mathcal{F}).$$

□

REMARK 17.9. Statement (a) of the corollary is *false* without the condition that k be separably closed, even for the point P over $\mathbb{Q}$ and the sheaf $\mathbb{Z}/2\mathbb{Z}$, for in this case $H^1(P, \mathbb{Z}/2\mathbb{Z}) = H^1(P, \mu_2) \cong \mathbb{Q}^\times/\mathbb{Q}^{\times 2}$, which is an infinite-dimensional vector space over $\mathbb{F}_2$ generated by the classes of -1 and the prime numbers.

Statement (b) of the corollary is false without the condition that X be complete, even for the sheaf $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{A}^1$, because the inverse image of $\mathbb{Z}/p\mathbb{Z}$ on $\mathbb{A}^1_{k'}$ is again $\mathbb{Z}/p\mathbb{Z}$, and (see 7.9b)

$$H^1(\mathbb{A}^1, \mathbb{Z}/p\mathbb{Z}) = k[T]/(T^p - T) \neq k'[T]/(T^p - T) = H^1(\mathbb{A}^1_{k'}, \mathbb{Z}/p\mathbb{Z}).$$

THEOREM 17.10. *Let* $\pi\colon X \to S$ *be proper, and let* $X' = X \times_S T$ *for some morphism* $f\colon T \to S$:

$$\begin{array}{ccc} X & \xleftarrow[f']{} & X'. \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ S & \xleftarrow[f]{} & T \end{array}$$

For any torsion sheaf on X, *there is a canonical isomorphism*

$$f^*(R^r\pi_*\mathcal{F}) \to R^r\pi'_*(f'^*\mathcal{F}).$$

PROOF. This may deduced from the Theorem 17.7 exactly as Theorem 17.3 is deduced from Theorem 17.2: first one uses the adjointness of the functors to construct the map, and then one uses the theorem to verify that the map is an isomorphism on stalks. □

Theorem 17.10 is the Proper Base Change Theorem.

Remarks on the Proofs. Theorems 17.7 and 17.10 are the most difficult of the basic theorems in étale cohomology to prove. I explain why. Throughout, I consider only varieties over an algebraically closed field.

Let S be a variety, and let S_{Et} be the big étale site on S: the underlying category of S_{Et} is the category of all varieties over S, and the covering families are the surjective families $(T_i \to T)$ of étale maps of S-varieties. A sheaf on S_{Et} is a contravariant functor $\mathcal{F}\colon\ \mathrm{Cat}(S_{Et}) \to Ab$ that satisfies the sheaf condition (§5, S) for all coverings, i.e., such that the restriction of $\mathcal{F}$ to T_{et} is a sheaf for every regular map $T \to S$.

Let $f\colon\ S_{Et} \to S_{et}$ be the obvious continuous morphism. Consider the following conditions on a sheaf $\mathcal{F}$ on S_{Et}:

(a) $\mathcal{F} \xrightarrow{\approx} f^* f_* \mathcal{F}$;
(b) $f_* \mathcal{F}$ is constructible.

Note that $f_*\mathcal{F}$ is the restriction of $\mathcal{F}$ to S_{et}. Thus (a) says that $\mathcal{F}$ is determined (in a natural way) by its restriction to S_{et} and (b) says that the restriction of $\mathcal{F}$ to S_{et} is constructible.

PROPOSITION 17.11. *A sheaf $\mathcal{F}$ on S_{Et} satisfies (a) and (b) if and only if it is representable by a variety $F \to S$ quasi-finite over S, i.e., if and only $\mathcal{F}(T) = \mathrm{Hom}_S(T, F)$ for all $T \to S$.*

PROOF. For a discussion of the proof, see EC V.1. □

Now let $\pi\colon\ X \to S$ be a proper map, and let $\mathcal{F}$ be a constructible sheaf on X_{et}. Define $\mathcal{F}^r$ to be the sheaf on S_{Et} such that $\mathcal{F}^r|T = R^r\pi'_*(f'^*\mathcal{F})$ (notations as in the theorem). The restriction maps $\mathcal{F}^r(T) \to \mathcal{F}^r(T')$ when $T' \to T$ is not étale are given by the base change maps in the theorem. Alternatively, one can define $\mathcal{F}^r$ to be $R^r\pi_*(f'^*\mathcal{F})$ where $f'^*\mathcal{F}$ is inverse image of $\mathcal{F}$ on X_{Et} and $R^r\pi_*$ is computed on the big étale sites. The theorem asserts that $\mathcal{F}^r$ satisfies (a) and (b). According to the proposition, this is equivalent to asserting that $\mathcal{F}^r$ is representable by a variety quasi-finite over S.

Hence the difficulty: we have a functor defined on the category of all S-varieties and wish to prove that it is representable by a variety quasi-finite over S. Such statements are usually very difficult to prove.

The starting point of the proof in this case is the following theorem of Grothendieck: let $\pi\colon\ X \to S$ be a proper regular map; then $R^1\pi_*\mathbb{G}_m$ is representable on S_{Et} by the Picard scheme $Pic_{X/S}$ — this is an infinite union of varieties over S. From the Kummer sequence, we find that

$$R^1\pi_*\mu_n = \mathrm{Ker}(Pic_{X/S} \xrightarrow{n} Pic_{X/S}),$$

and hence is representable. This argument suffices to prove the theorem when the fibres of π are curves, and the general case is proved by induction on the relative dimension of X over S. See the last chapter of Artin 1973.

18. Cohomology Groups with Compact Support.

Heuristics from topology. It is important in étale cohomology, as it is topology, to define cohomology groups with compact support — we saw this already in the case of curves in §14. They should be dual to the ordinary cohomology groups.

The traditional definition (Greenberg 1967, p162) is that, for a manifold U,

$$H_c^r(U,\mathbb{Z}) = \varinjlim{}_Z H_Z^r(U,\mathbb{Z})$$

where Z runs over the compact subsets of U. More generally (Iverson 1986, III.1) when $\mathcal{F}$ is a sheaf on a locally compact topological space U, define

$$\Gamma_c(U,\mathcal{F}) = \varinjlim{}_Z \Gamma_Z(U,\mathcal{F})$$

where Z again runs over the compact subsets of U, and let $H_c^r(U,-) = R^r\Gamma_c(U,-)$.

For an algebraic variety U and a sheaf $\mathcal{F}$ on U_{et}, this suggests defining

$$\Gamma_c(U,\mathcal{F}) = \varinjlim{}_Z \Gamma_Z(U,\mathcal{F}),$$

where Z runs over the complete subvarieties Z of U, and setting $H_c^r(U,-) = R^r\Gamma_c(U,-)$. However, this definition leads to anomolous groups. For example, if U is an affine variety over an algebraically closed field, then the only complete subvarieties of U are the finite subvarieties (AG 5.28), and for a finite subvariety $Z \subset U$,

$$H_Z^r(U,\mathcal{F}) = \oplus_{z\in Z} H_z^r(U,\mathcal{F}).$$

Therefore, if U is smooth of dimension m and Λ is the constant sheaf $\mathbb{Z}/n\mathbb{Z}$, then

$$H_c^r(U,\Lambda) = \varinjlim H_Z^r(U,\Lambda) = \oplus_{z\in U} H_z^r(U,\Lambda) = \begin{cases} \oplus_{z\in U}\Lambda(-m) & \text{if } r = 2m, \\ 0 & \text{otherwise.} \end{cases}$$

These groups are not even finite. We need a different definition.

If $j\colon\ U \to X$ is a homeomorphism of the topological space U onto an open subset of a locally compact space X, then

$$H_c^r(U,\mathcal{F}) = H_c^r(X, j_!\mathcal{F})$$

(Iverson 1986, p184). In particular, when X is compact, then

$$H_c^r(U,\mathcal{F}) = H^r(X,\mathcal{F}).$$

We make this our definition.

Cohomology with compact support.

Definition 18.1. For any torsion sheaf $\mathcal{F}$ on a variety U, we define

$$H_c^r(U,\mathcal{F}) = H^r(X, j_!\mathcal{F}),$$

where X is any complete variety containing U as a dense open subvariety and j is the inclusion map.

An open immersion $j\colon\ U \hookrightarrow X$ from U into a complete variety X such that $j(U)$ is dense in X is called a *completion* (or *compactification*) of U. Following the terminology in topology, we call the $H_c^r(U,\mathcal{F})$ the *cohomology groups of* $\mathcal{F}$ *with compact support* (rather than the more logical complete support).

This definition[27] raises two questions:

(a) Does every variety admit a completion?
(b) Are the cohomology groups with compact support independent of the completion?

The first question was shown to have a positive answer by Nagata in 1962[28] More generally, he showed that, for any separated morphism $\pi\colon\ U \to S$ of finite type from one Noetherian scheme to a second, there is a proper morphism $\bar{\pi}\colon\ X \to S$ and an open immersion $j\colon\ U \to X$ such that $\pi = \bar{\pi} \circ j$:

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\bar{\pi}"] \\ & S & \end{tikzcd}$$

Nagata's original proof was in terms of valuation rings. A more modern, scheme-theoretic, proof can be found in (Lütkebohmert, W., On compactification of schemes, Manuscripta Math. 80 (1993), 95–111, MR 94h:14004).

Note, unlike the case of curves, in higher dimensions the embedding will not be unique: from any completion, we can construct others by blowing up and blowing down subvarieties of the boundary. Nevertheless, the next proposition shows that the answer to (b) is also positive — we need to require $\mathcal{F}$ to be torsion in order to be able to apply the proper base change theorem.

Proposition 18.2. *When $\mathcal{F}$ is a torsion sheaf, the groups $H^r(X, j_!\mathcal{F})$ are independent of the choice of the embedding $j\colon\ U \hookrightarrow X$.*

Proof. Let $j_1\colon\ U \hookrightarrow X_1$ and $j_2\colon\ U \hookrightarrow X_2$ be two completions of U. Consider the diagonal mapping into the product, $j\colon\ U \hookrightarrow X_1 \times X_2$, and let X be the closure of U in $X_1 \times X_2$. Then $j\colon\ U \hookrightarrow X$ is a completion of U, and the projections are proper maps $X \to X_1$, $X \to X_2$ inducing the identity map on U. It suffices to prove that $H^r_c(X_1, j_{1!}\mathcal{F}) \cong H^r_c(X, j_!\mathcal{F})$. Consider:

$$\begin{array}{ccc} U & \overset{j}{\hookrightarrow} & X \\ \| & & \downarrow \pi \\ U & \overset{j_1}{\hookrightarrow} & X_1. \end{array}$$

The Leray spectral sequence reads

$$H^r(X_1, R^s\pi_*(j_!\mathcal{F})) \Rightarrow H^{r+s}(X, j_{1!}\mathcal{F}).$$

According to Theorem 17.7, the stalks of $(R^r\pi_*)(j_!\mathcal{F})$ can be computed on the geometric fibres of X/X_1. But the fibre of π over $\bar{x}$ consists of a single point if $x \in U$, and $j_!\mathcal{F}$ is zero on the fibre if $x \notin U$. It follows that $R^r\pi_*(j_!\mathcal{F}) = j_!\mathcal{F}$ for $r = 0$, and is zero otherwise. □

[27] For disciples of the nineteenth century superstition, that all varieties come naturally endowed with embeddings into a projective space, neither problem will arise.

[28] Nagata, Masayoshi, Imbedding of an abstract variety in a complete variety, J. Math. Kyoto Univ. 2 (1962), 1–10.

Nagata, Masayoshi, A generalization of the imbedding problem of an abstract variety in a complete variety, J. Math. Kyoto Univ. 3 (1963), 89–102.

PROPOSITION 18.3. (a) *A short exact sequence of sheaves gives rise to a long exact sequence of cohomology groups with compact support.*

(b) *For any complete subvariety Z of U, there is a canonical map $H^r_Z(U,\mathcal{F}) \to H^r_c(U,\mathcal{F})$; for $r=0$, these maps induce an isomorphism*

$$H^0_c(U,\mathcal{F}) \cong \varinjlim H^0_Z(U,\mathcal{F}),$$

(limit over the complete subvarieties Z of U).

(c) *If $\mathcal{F}$ is constructible, then $H^r_c(U,\mathcal{F})$ is finite.*

PROOF. (a) As $j_!$ is exact (8.13), a short exact sequence of sheaves on U

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

gives rise to a short exact sequence

$$0 \to j_!\mathcal{F}' \to j_!\mathcal{F} \to j_!\mathcal{F}'' \to 0$$

of sheaves on X, and hence to an exact cohomology sequence

$$\cdots \to H^r_c(U,\mathcal{F}') \to H^r_c(U,\mathcal{F}) \to H^r_c(U,\mathcal{F}'') \to \cdots.$$

(b) Let $j\colon\ U \hookrightarrow X$ be a completion of U, and let $i\colon\ X \setminus U \hookrightarrow X$ be the complementary closed immersion. From the exact sequence

$$0 \to j_!\mathcal{F} \to j_*\mathcal{F} \to i_*i^*j_*\mathcal{F} \to 0$$

(see 8.15), we find that

$$H^0_c(U,\mathcal{F}) = \operatorname{Ker}(\Gamma(U,\mathcal{F}) \to \Gamma(X \setminus U, i^*j_*\mathcal{F}).$$

But, from the definitions of i^* and j_*, we see that[29]

$$\Gamma(X \setminus U, i^*j_*\mathcal{F}) = \varinjlim \Gamma(V \times_X U, \mathcal{F})$$

where the limit is over étale maps $\varphi\colon\ V \to X$ whose image contains $X \setminus U$. Therefore, $H^0_c(U,\mathcal{F})$ is the subgroup of $\Gamma(U,\mathcal{F})$ consisting of sections that vanish on $V \times_X U$ for some étale $V \to X$ whose image contains $X \setminus U$.

Let Z be a complete subvariety of U, and let $s \in \Gamma_Z(U,\mathcal{F})$. Thus $s \in \Gamma(U,\mathcal{F})$ and $s|U \setminus Z = 0$. Because Z is complete, it is closed in X. Now s vanishes on $V \cap U$ where V is the (open) complement of Z in X, which shows that $s \in H^0_c(U,\mathcal{F})$. We have shown that

$$\Gamma_Z(U,\mathcal{F}) \subset H^0_c(U,\mathcal{F}).$$

Conversely, let $s \in H^0_c(U,\mathcal{F})$. Then s vanishes on $V \times_X U$ for some étale $\varphi\colon\ V \to X$ whose image contains $X \setminus U$. Now $Z \overset{\text{df}}{=} X \setminus \varphi(V)$ is a complete subvariety of U, and $U \setminus Z = U \cap \varphi(V)$. Because $V \times_X U \to U \cap \varphi(V)$ is an étale covering and $s|V \times_X U = 0$, we have that $s|U \setminus Z = 0$. Therefore, $s \in \Gamma_Z(U,\mathcal{F})$, and so

$$\cup_Z \Gamma_Z(U,\mathcal{F}) = H^0_c(U,\mathcal{F}).$$

In the course of the above proof, we showed that, for any complete $Z \subset U$, $H^0_Z(U,\mathcal{F}) \subset H^0_c(U,\mathcal{F})$. A general result about δ-functors shows that the morphism

[29] In fact, this is not quite correct, since in forming the inverse image we need to sheafify (see §8).

$H^0_Z(U,-) \to H^0_c(U,-)$ extends uniquely to a morphism of δ-functors. Explicitly, suppose that the morphism has been extended to dimensions $\leq r$ in a way that is compatible with the connecting homomorphisms. Given $\mathcal{F}$, embed it into an injective sheaf, $\mathcal{F} \hookrightarrow \mathcal{I}$, and let $\mathcal{F}'$ be the quotient. There is a unique map $H^{r+1}_Z(U,\mathcal{F}) \to H^{r+1}_c(U,\mathcal{F})$ making the following diagram commute:

$$\begin{tikzcd}
\cdots \arrow[r] & H^r_Z(U,\mathcal{I}) \arrow[r] \arrow[d] & H^r_Z(U,\mathcal{F}') \arrow[r] \arrow[d] & H^{r+1}_Z(U,\mathcal{F}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^r_c(U,\mathcal{I}) \arrow[r] & H^r_c(U,\mathcal{F}') \arrow[r] & H^{r+1}_c(U,\mathcal{F}) \arrow[r] & H^{r+1}_c(U,\mathcal{I}) \arrow[r] & \cdots
\end{tikzcd}$$

(c) If $\mathcal{F}$ is constructible, then so also is $j_!\mathcal{F}$, and so this follows from Corollary 17.8. □

Note that $H^r_c(U,-)$ is not the r^{th} right derived functor of $H^0_c(U,-)$ (and hence it is not a derived functor). This is unfortunate: although $H^r_c(U,-)$ is independent of the choice of a completion of U, there seems to be no way of defining it without first choosing a completion: although $H^r_c(U,-)$ is intrinsic to U, there seems to be no purely intrinsic definition of it.

Higher direct images with proper support. By using the full strength of Nagata's theorem, it is possible to define higher direct images with proper support (here the terminology from topology coincides with that required by logic). Let $\pi\colon\ U \to S$ be a regular map of varieties (or a separated morphism of finite type of schemes), and let $\bar{\pi}\colon\ X \to S$ be a proper morphism for which there is an open immersion $j\colon\ U \to X$ such that $\pi = \bar{\pi} \circ j$ and $j(U)$ is dense in X. For any torsion sheaf $\mathcal{F}$ on U, define

$$R^r\pi_!\mathcal{F} = R^r\pi_*(j_!\mathcal{F}).$$

PROPOSITION 18.4. (a) *The sheaf $R^r\pi_!\mathcal{F}$ is independent of the choice of the factorization $\pi = \bar{\pi} \circ j$.*

(b) *A short exact sequence of sheaves gives rise to a long exact sequence of higher direct images with proper support.*

(c) *If $\mathcal{F}$ is constructible, then so also is $R^r\pi_!\mathcal{F}$.*

(d) *For any pair $U_1 \xrightarrow{\pi_1} U_2 \xrightarrow{\pi_2} S$ of regular maps, and torsion sheaf $\mathcal{F}$ on U_1, there is a spectral sequence*

$$R^r\pi_{2!}(R^s\pi_{1!}\mathcal{F}) \Rightarrow R^{r+s}(\pi_2 \circ \pi_1)_!(\mathcal{F}).$$

PROOF. The proofs of (a), (b), and (c) are similar to those of corresponding statements in Proposition 18.3. The proof of (d) is complicated by the need to construct a "compactification" of $\pi_2 \circ \pi_1$ lying over a "compactification" of π_1 — see EC p229. □

Again $R^r\pi_!$ is not the rth right derived functor of $R^0\pi_!$. It would be more accurate to denote it by $R^r_c\pi_*$.

19. Finiteness Theorems; Sheaves of $\mathbb{Z}_\ell$-modules.

Finiteness Theorems.

Theorem 19.1. *Let X be a variety over a separably closed field k, and let $\mathcal{F}$ a constructible sheaf on X_{et}. The groups $H^r(X_{et}, \mathcal{F})$ are finite in each of the following two cases.*

(a) *X is complete, or*
(b) *$\mathcal{F}$ has no p-torsion, where p is the characteristic of k.*

Proof. Case (a) is part of the proper base change theorem, discussed in Section 17.

For a nonsingular variety X and locally constant sheaf $\mathcal{F}$ on X_{et}, it is possible to prove that $H^r(X_{et}, \mathcal{F})$ is finite by using induction on the dimension of X and making use of the existence of elementary fibrations (§21) below — see SGA 4, VI.5.2.

Alternatively, in this case, it follows from the Poincaré duality theorem (see later), which shows that $H^r(X, \mathcal{F})$ is dual to $H_c^{2m-r}(X, \check{\mathcal{F}}(m))$, where $m = \dim X$.

The general case is more difficult — see SGA $4\frac{1}{2}$, p233–261. [But surely, the proof can be simplified by using de Jong's resolution theorem.] □

Sheaves of $\mathbb{Z}_\ell$-modules. So far, we have talked only of torsion sheaves. However, it will be important for us to have cohomology groups that are vector spaces over a field of characteristic zero in order, for example, to have a good Lefschetz fixed-point formula. However, the étale cohomology groups with coefficients in nontorsion sheaves are anomolous. For example

$$H^1(X_{et}, \mathbb{Z}) = \mathrm{Hom}_{\mathrm{conts}}(\pi_1(X, \bar{x}), \mathbb{Z}),$$

($\mathbb{Z}$ with the discrete topology), which is zero because a continuous homomorphism $f\colon \pi_1(X, \bar{x}) \to \mathbb{Z}$ must be zero on an open subgroup of $\pi_1(X, \bar{x})$, and such a subgroup is of finite index ($\pi_1(X, \bar{x})$ being compact). Similarly, with the obvious definition, $H^1(X_{et}, \mathbb{Z}_\ell) = 0$ (it consists of homomorphisms $\pi_1(X, \bar{x}) \to \mathbb{Z}_\ell$ that are continuous for the *discrete* topology on $\mathbb{Z}_\ell$). The solution is to define

$$H^r(X_{et}, \mathbb{Z}_\ell) = \varprojlim_n H^r(X_{et}, \mathbb{Z}/\ell^n\mathbb{Z})$$

— cohomology does not commute with inverse limits of sheaves. With this definition,

$$H^1(X_{et}, \mathbb{Z}_\ell) \overset{\mathrm{df}}{=} \varprojlim_n H^1(X_{et}, \mathbb{Z}/\ell^n\mathbb{Z}) \cong \varprojlim_n \mathrm{Hom}_{\mathrm{conts}}(\pi_1(X, \bar{x}), \mathbb{Z}/\ell^n\mathbb{Z}) \cong \mathrm{Hom}_{\mathrm{conts}}(\pi_1(X, \bar{x}), \mathbb{Z}_\ell)$$

where $\mathbb{Z}/\ell^n\mathbb{Z}$ has its discrete topology and $\mathbb{Z}_\ell$ its ℓ-*adic* topology (the reader should check the last $\cong$).

To give a finitely generated $\mathbb{Z}_\ell$-module M is the same as to give a family $(M_n, f_{n+1}\colon\ M_{n+1} \to M_n)_{n\in\mathbb{N}}$ such that

(a) for all n, M_n is a finite $\mathbb{Z}/\ell^n\mathbb{Z}$-module;
(b) for all n, the map $f_{n+1}\colon\ M_{n+1} \to M_n$ induces an isomorphism $M_{n+1}/\ell^n M_{n+1} \to M_n$.

Given M, we take $M_n = M/\ell^n M$ and f_{n+1} to be the quotient map. Conversely, given (M_n, f_{n+1}), we define $M = \varprojlim_n M_n$. In fact, the correspondence $M \leftrightarrow (M_n, f_{n+1})$ can be turned into an equivalence[30] of categories.

Let $(M_n, f_n)_{n\in\mathbb{N}}$ be a system satisfying (a,b). By induction, we obtain a canonical isomorphism $M_{n+s}/\ell^n M_{n+s} \cong M_n$. On tensoring

$$0 \to \mathbb{Z}/\ell^s\mathbb{Z} \xrightarrow{\text{“}\ell^n\text{”}} \mathbb{Z}/\ell^{n+s}\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z} \to 0$$

with M, we obtain a sequence

$$0 \to M_s \to M_{n+s} \to M_n \to 0,$$

which is exact if M is flat (equivalently, torsion-free).

The above discussion motivates the following definition: *a sheaf of* $\mathbb{Z}_\ell$*-modules* on X (or an ℓ-adic sheaf) is a family $(\mathcal{M}_n, f_{n+1}\colon \mathcal{M}_{n+1} \to \mathcal{M}_n)$ such that

(a) for each n, $\mathcal{M}_n$ is a constructible sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules;
(b) for each n, the map $f_{n+1}\colon \mathcal{M}_{n+1} \to \mathcal{M}_n$ induces an isomorphism $\mathcal{M}_{n+1}/\ell^n\mathcal{M}_{n+1} \to \mathcal{M}_n$.

Let $(\mathcal{M}_n, f_n)_{n\in\mathbb{N}}$ be a sheaf of $\mathbb{Z}_\ell$-modules on X. By induction, we obtain a canonical isomorphism $\mathcal{M}_{n+s}/\ell^n\mathcal{M}_{n+s} \cong \mathcal{M}_n$. On tensoring

$$0 \to \mathbb{Z}/\ell^s\mathbb{Z} \xrightarrow{\text{“}\ell^n\text{”}} \mathbb{Z}/\ell^{n+s}\mathbb{Z} \to \mathbb{Z}/\ell^n\mathbb{Z} \to 0$$

with $\mathcal{M}_{n+s}$, we obtain a sequence

$$0 \to \mathcal{M}_s \to \mathcal{M}_{n+s} \to \mathcal{M}_n \to 0.$$

We say that $\mathcal{M}$ is *flat* if this sequence is exact for all n and s.

For a sheaf $\mathcal{M} = (\mathcal{M}_n)$ of $\mathbb{Z}_\ell$-modules, we define

$$H^r(X_{et}, \mathcal{M}) = \varprojlim_n H^r(X_{et}, \mathcal{M}_n), \quad H^r_c(X_{et}, \mathcal{M}) = \varprojlim_n H^r_c(X_{et}, \mathcal{M}_n).$$

For example, if we let $\mathbb{Z}_\ell$ denote the sheaf of $\mathbb{Z}_\ell$-modules with $\mathcal{M}_n$ the constant sheaf $\mathbb{Z}/\ell^n\mathbb{Z}$ and the obvious f_n, then

$$H^r(X_{et}, \mathbb{Z}_\ell) \stackrel{\text{df}}{=} \varprojlim H^r(X_{et}, \mathbb{Z}/\ell^n\mathbb{Z}).$$

Theorem 19.2. *Let $\mathcal{M} = (\mathcal{M}_n)$ be a flat sheaf of $\mathbb{Z}_\ell$-modules on a variety X over a field k. Assume k is separably closed, and that either X is complete or that*

[30]Let

$$M_n = \mathbb{Z}/\ell^n\mathbb{Z} \times \mathbb{Z}/\ell^{n-1}\mathbb{Z} \times \cdots \times \mathbb{Z}/\ell\mathbb{Z}$$

and let $M_{n+1} \to M_n$ be the obvious quotient map. Then $M = \varprojlim M_n = \prod \mathbb{Z}_\ell$ (product of copies of $\mathbb{Z}_\ell$ indexed by the positive integers). This example shows that an inverse limit of finite $\mathbb{Z}_\ell$-modules needn't be a finitely generated $\mathbb{Z}_\ell$-module. However, this example fails the condition that $M_n = M_{n+1}/\ell^n M_{n+1}$, and so it doesn't contradict the statements in this paragraph. Nevertheless, perhaps one should add the condition that, for some n_0, the system $N_n \stackrel{\text{df}}{=} \ell^{n_0} M_{n+n_0}$ is flat. Any system $M/\ell^n M$ arising from a finitely generated $\mathbb{Z}_\ell$-module M satisfies this condition with n_0 chosen to be the number such that ℓ^{n_0} kills the torsion in M.

$\ell \neq char(k)$. Then each $H^r(X_{et}, \mathcal{M})$ is finitely generated, and there is an exact sequence of cohomology groups

$$\cdots \to H^r(X_{et}, \mathcal{M}) \xrightarrow{\ell^n} H^r(X_{et}, \mathcal{M}) \to H^r(X_{et}, \mathcal{M}_n) \to H^{r+1}(X_{et}, \mathcal{M}) \to \cdots.$$

PROOF. For each $s \geq 0$, we get an exact sequence

$$0 \to \mathcal{M}_s \to \mathcal{M}_{n+s} \to \mathcal{M}_n \to 0.$$

These are compatible in the sense that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}_{s+1} & \xrightarrow{\ell^n} & \mathcal{M}_{n+1+s} & \longrightarrow & \mathcal{M}_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{s+1}} & & \downarrow{\scriptstyle f_{n+1+s}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \mathcal{M}_s & \xrightarrow{\ell^n} & \mathcal{M}_{n+s} & \longrightarrow & \mathcal{M}_n & \longrightarrow & 0 \end{array}$$

commutes. On forming the cohomology sequence for each n and passing to the inverse limit over all n, we obtain an exact sequence

$$\cdots \to H^r(\mathcal{M}) \xrightarrow{\ell^n} H^r(\mathcal{M}) \to H^r(\mathcal{M}_n) \to H^{r+1}(\mathcal{M}) \to \cdots.$$

This gives an exact sequence

$$0 \to H^r(\mathcal{M})/\ell^n H^r(\mathcal{M}) \to H^r(\mathcal{M}_n) \to H^{r+1}(\mathcal{M})_{\ell^n} \to 0.$$

As $H^r(\mathcal{M})$ is an inverse limit of ℓ-power-torsion finite groups, no nonzero element of it is divisible by all powers of ℓ. Thus $\varprojlim H^{r+1}(\mathcal{M})_{\ell^n} = 0$ (the transition maps are $H^{r+1}(\mathcal{M})_{\ell^n} \xleftarrow{\ell} H^{r+1}(\mathcal{M})_{\ell^{n+1}}$) and $\varprojlim H^r(\mathcal{M})/\ell^n H^r(\mathcal{M}) \cong H^r(\mathcal{M})$. It follows that $H^r(\mathcal{M})$ is generated by any subset that generates it modulo ℓ. □

REMARK 19.3. A compact $\mathbb{Z}_\ell$-module need not be finitely generated — consider, for example, a product of an infinite number of copies of $\mathbb{F}_\ell$.

A sheaf $\mathcal{M} = (\mathcal{M}_n)$ of $\mathbb{Z}_\ell$-modules is said to be *locally constant* if each $\mathcal{M}_n$ is locally constant. To give a locally constant sheaf $\mathcal{M}$ on a connected variety X is to give a finitely generated $\mathbb{Z}_\ell$-module M together with a continuous action of $\pi_1(X, \bar{x})$ on M (ℓ-adic topology on M). Then

$$H^1(X_{et}, \mathcal{M}) = H^1_{\text{conts}}(\pi_1(X, \bar{x}), M)$$

where $H^1_{\text{conts}}(\pi_1(X, \bar{x}), M)$ consists of equivalence classes of crossed homomorphisms $f\colon\ \pi_1(X, \bar{x}) \to M$ that are continuous for the ℓ-adic topology on M.

Note that a locally constant sheaf of $\mathbb{Z}_\ell$-modules need not become trivial on any étale covering of X, i.e., it is not locally constant. In order for this to happen, the action of $\pi_1(X, \bar{x})$ would have to factor through a finite quotient. Thus the term "locally constant sheaf of $\mathbb{Z}_\ell$-modules" is an abuse of terminology. Usually they are called "constant-tordu", "lisse", or "smooth".

Sheaves of $\mathbb{Q}_\ell$-modules. A sheaf of $\mathbb{Q}_\ell$-vector spaces is just a $\mathbb{Z}_\ell$-sheaf $\mathcal{M} = (\mathcal{M}_n)$, except that we define

$$H^r(X_{et}, \mathcal{M}) = (\varprojlim H^r(X_{et}, \mathcal{M}_n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

For example,

$$H^r(X_{et}, \mathbb{Q}_\ell) = (\varprojlim H^r(X_{et}, \mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell = H^r(X_{et}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell.$$

For the remainder of the notes, $H^r(X, \mathbb{Z}_\ell)$ will denote $\varprojlim H^r(X_{et}, \mathbb{Z}/\ell^n\mathbb{Z})$ and $H^r(X_{et}, \mathbb{Q}_\ell)$ will denote $H^r(X_{et}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell$

20. The Smooth Base Change Theorem.

This is a brief summary only.

The proper-smooth base change theorem in topology. Let $\pi\colon\ X \to S$ be a proper map of manifolds. If π is smooth, i.e., the map $d\pi$ on tangent spaces is surjective at all points of X, then π is a fibration, and it follows that, for any locally constant sheaf $\mathcal{F}$ on X, $R^r\pi_*\mathcal{F}$ is locally constant.

The smooth base change theorem. In the remainder of this section, we consider only varieties over an algebraically closed field k.

Consider a diagram:

$$\begin{array}{ccc} X & \xleftarrow{f'} & X' \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi'} \\ S & \xleftarrow{f} & T \end{array} \qquad \text{where } X' = X \times_S T.$$

For any sheaf $\mathcal{F}$ on X, there is a canonical morphism of sheaves on T

$$f^*(R^r\pi_*\mathcal{F}) \to R^r\pi'_*(f'^*\mathcal{F}),$$

called the base change morphism (or map) — see the proof of 17.3.

Theorem 20.1 (Smooth base change). *If $f\colon\ T \to S$ is smooth, then the base change morphism is an isomorphism for all constructible sheaves $\mathcal{F}$ whose torsion is prime to the characteristic of k.*

The proper-smooth base change theorem. Recall that the finiteness part of the proper base change theorem says that, if $\pi\colon\ X \to S$ is proper and $\mathcal{F}$ is constructible, then $R^r\pi_*\mathcal{F}$ is constructible.

Theorem 20.2 (Proper-smooth base change). *If $\pi\colon\ X \to S$ is proper and smooth and $\mathcal{F}$ is locally constant with finite stalks, then $R^r\pi_*\mathcal{F}$ is locally constant with finite stalks, provided the torsion in $\mathcal{F}$ is prime to the characteristic of k.*

Let x_0 and x_1 be points of X regarded as a scheme (so x_0 and x_1 need not be closed). We say that x_0 is a *specialization* of x_1 if it is contained in the closure of x_1. Then every Zariski open subset of X containing x_0 also contains x_1. Choose geometric points $\bar{x}_0 \to x_0 \hookrightarrow X$ and $\bar{x}_1 \to x_1 \hookrightarrow X$. Then an étale neighbourhood (U, u) of $\bar{x}_0$ can be given the structure of an étale neighbourhood of $\bar{x}_1$ (the image of $U \to X$ contains x_1, and so we can choose a morphism $\bar{x}_1 \to U$ lifting $\bar{x}_1 \to X$). Once this has been done compatibly for every étale neighbourhood of $\bar{x}_0$, then we get a *cospecialization map*

$$\mathcal{F}_{\bar{x}_0} \to \mathcal{F}_{\bar{x}_1}$$

for every sheaf $\mathcal{F}$ on X_{et}. For example, taking $\mathcal{F} = \mathbb{G}_a$, we get a map on the strictly local rings

$$\mathcal{O}_{X,\bar{x}_0} \to \mathcal{O}_{X,\bar{x}_1}.$$

The definition of the cospecialization maps involve choices, but once we fix the map $\mathcal{O}_{X,\bar{x}_0} \to \mathcal{O}_{X,\bar{x}_1}$, then the map $\mathcal{F}_{\bar{x}_0} \to \mathcal{F}_{\bar{x}_1}$ becomes canonical in $\mathcal{F}$. In other words, there is no one canonical map $\mathcal{F}_{\bar{x}_0} \to \mathcal{F}_{\bar{x}_1}$ but rather, a distinguished class of maps, whose members we call cospecialization maps. If one in the class is an isomorphism, so are they all.

PROPOSITION 20.3. *A constructible sheaf $\mathcal{F}$ on X_{et} is locally constant if and only if the cospecialization maps $\mathcal{F}_{\bar{x}_0} \to \mathcal{F}_{\bar{x}_1}$ are all isomorphisms.*

Compare this with the following topological situation. Let $\mathcal{F}$ be a locally constant sheaf on the punctured disk U, and let $j\colon U \hookrightarrow X$ be the inclusion of U into the full disk. The sheaf $\mathcal{F}$ corresponds to a module M endowed with an action of $\pi_1(U, u)$, $u \in U$. The stalk of $j_*\mathcal{F}$ at u is M, and its stalk at o is $M^{\pi_1(U,u)}$. Thus, the special stalk $(j_*\mathcal{F})_o$ is isomorphic to the general stalk $(j_*\mathcal{F})_u$ (by a cospecialization map) if and only if π_1 acts trivially on M, which means that $j_*\mathcal{F}$ is constant.

Now we can restate the proper-smooth base change theorem as follows.

THEOREM 20.4. *Let $\pi\colon X \to S$ be proper and smooth, and let $\mathcal{F}$ be a locally constant sheaf on X with finite stalks whose torsion is prime to the characteristic of k. For any pair of geometric points $\bar{s}_0$ and $\bar{s}_1$ with s_0 a specialization of s_1, the cospecialization map*

$$H^r(X_{\bar{s}_0}, \mathcal{F}) \to H^r(X_{\bar{s}_1}, \mathcal{F})$$

is an isomorphism. Here $X_{\bar{s}} = X \times_S \bar{s}$, the geometric fibre of π over $\bar{s}$.

Applications. The above results also hold for schemes, and here we have a remarkable application. Let X be a complete nonsingular variety over an algebraically closed field k of characteristic $p \neq 0$. We say that X can be *lifted to characteristic zero* if

(a) there is a discrete valuation ring R with field of fractions K of characteristic zero and residue field k; and
(b) a scheme $\pi\colon \mathcal{X} \to S$, $S = \operatorname{Spec} R$, proper and smooth over S whose special fibre is X.

For example, a subvariety X of $\mathbb{P}^n$ can be lifted to characteristic zero if there exist homogeneous polynomials $f_i(T_0, \dots, T_n) \in R[T_0, \dots, T_n]$ such that

(a) modulo $\mathfrak{m}_R$, the f_i generate the homogeneous ideal of X embedded in $\mathbb{P}^n$;
(b) when regarded as polynomials in $K[T_0, \dots, T_n]$, the f_i define a variety X_1 over K with the same dimension as that of X.

Clearly, any nonsingular hypersurface in $\mathbb{P}^n$ can be lifted — just lift the single polynomial defining X from $k[T_0, \dots, T_n]$ to $R[T_0, \dots, T_n]$. Similarly, any smooth complete intersection in $\mathbb{P}^n$ can be lifted to characteristic zero. Curves and abelian varieties can be lifted to characteristic zero, but otherwise little is known. Certainly, many varieties can not be lifted. The problem is the following: suppose X has codimension r in $\mathbb{P}^n$, but its homogeneous ideal $I(X)$ needs $s > r$ generators; when the generators of $I(X)$ are lifted to $R[T_0, \dots, T_n]$, in general, they will define a variety X_1 in characteristic zero of dimension less than that of X — all one can say in general is that

$$n - s \leq \dim X_1 \leq \dim X = n - r.$$

THEOREM 20.5. *Suppose that a variety X_0 over an algebraically closed field k of characteristic $p \neq 0$ can be lifted to a variety X_1 over a field K of characteristic zero. For any finite abelian group Λ,*

$$H^r(X_0, \Lambda) \approx H^r(X_{1,K^{al}}, \Lambda).$$

In particular, the Betti numbers of X_0 are equal to the Betti numbers of X_1. In the next section, we shall show that the cohomology groups of X_1 equal those of the topological space $X_1(\mathbb{C})$ (assuming K can be embedded in $\mathbb{C}$).

21. The Comparison Theorem.

Let X be a nonsingular variety. Then X can be endowed in a natural way with the structure of a complex manifold. I write X^{an} for X regarded as a complex manifold and X_{cx} for X regarded as a topological space with the complex topology (thus X^{an} is X_{cx} together with a sheaf of rings).

Theorem 21.1. *Let X be a nonsingular variety over $\mathbb{C}$. For any finite abelian group Λ and $r \geq 0$, $H^r(X_{et}, \Lambda) \cong H^r(X_{cx}, \Lambda)$.*

Remark 21.2. The theorem holds also for singular varieties, but then it becomes a little more difficult to state (one needs to know about analytic varieties), and the proof is a little longer. The theorem holds also for all constructible sheaves.

In both topologies, $H^0(X, \Lambda) = \Lambda^{\pi_0(X)}$, where $\pi_0(X)$ is the set of connected components of X. Thus, for $r = 0$, the theorem simply asserts that the set of connected components of X with respect to the Zariski topology is the same as the set of connected components of X_{cx} with respect to the complex topology, or, equivalently, that if X is connected for the Zariski topology, then it is connected for the complex topology.

This is a slightly surprising result. Let $X = \mathbb{A}^1$. Then, certainly, it is connected for both the Zariski topology (that for which the nonempty open subsets are those that omit only finitely many points) and the complex topology (that for which X is homeomorphic to $\mathbb{R}^2$). When we remove a circle from X, it becomes disconnected for the complex topology, but remains connected for the Zariski topology. This doesn't contradict the theorem, because $\mathbb{A}^1_{\mathbb{C}}$ with a circle removed is not an algebraic variety.

Let X be a connected nonsingular (hence irreducible) curve. We prove that it is connected for the complex topology. Removing or adding a finite number of points to X will not change whether it is connected for the complex topology, and so we can assume that X is projective. Suppose X is the disjoint union of two nonempty open (hence closed) sets X_1 and X_2. According to the Riemann-Roch theorem, there exists a nonconstant rational function f on X having poles only in X_1. Therefore, its restriction to X_2 is holomorphic. Because X_2 is compact, f is constant on each connected component of X_2 (Cartan, H., *Elementary Theory of Analytic Functions of One or Several Variables*, Addison-Wesley, 1963, VI.4.5) say, $f(z) = a$ on some infinite connected component. Then $f(z) - a$ has infinitely many zeros, which contradicts the fact that it is a rational function.

A connected nonsingular variety X can be shown to be connected for the complex topology by using induction on the dimension — see Remark 21.9 below (also Shafarevich, I., *Basic Algebraic Geometry*, 1994, VII.2).

For $r = 1$, the theorem asserts that there is a natural one-to-one correspondence between the finite étale coverings of X and X^{an}.

Theorem 21.3 (Riemann Existence Theorem). *For any nonsingular algebraic variety X over $\mathbb{C}$, the functor $Y \mapsto Y^{an}$ defines an equivalence between the categories of finite étale coverings of X and X^{an}.*

Proof. Apparently, this was proved for Riemann surfaces by Riemann. The general case is due to Grauert and Remmert. The proof can be shortened by using resolution of singularities. I sketch the proof from (SGA 1, XII).

Step 1. *For any projective nonsingular algebraic variety X, the functor $Y \mapsto Y^{an}$ defines an equivalence from the category of finite coverings of X to the category of finite coverings of X^{an}.*

In fact, for a complete variety X, the functor $\mathcal{M} \mapsto \mathcal{M}^{an}$ is an equivalence from the category of coherent $\mathcal{O}_X$-modules on X_{zar} to the category of coherent $\mathcal{O}_{X^{an}}$-modules on X^{an} (see Serre, GAGA[31], 1956). To give a finite covering of X is to give a coherent $\mathcal{O}_X$-module together with an $\mathcal{O}_X$-algebra structure. Since the same is true for X^{an}, the statement is obvious.

Step 2. *The functor $Y \mapsto Y^{an}$ from the category of finite coverings of X to the category of finite coverings of X^{an} is fully faithful.*

We have to prove that

$$\mathrm{Hom}_X(Y, Y') \xrightarrow{\approx} \mathrm{Hom}_{X^{an}}(Y^{an}, Y'^{an})$$

for any finite étale coverings Y and Y' of X. We may suppose that X is connected. To give an X-morphism $Y \to Y'$ is to give a section to $Y \times_X Y' \to Y$, which is the same as to give a connected component Γ of $Y \times_X Y'$ such that the morphism $\Gamma \to X$ induced by the projection $Y \times_X Y' \to Y$ is an isomorphism (see 2.15). But, as we have just noted, the connected components of $Y \times_X Y'$ coincide with the connected components of $Y^{an} \times_{X^{an}} Y'^{an}$, and if Γ is a connected component of $Y \times_X Y'$, then the projection $\Gamma \to X$ is an isomorphism if and only if $\Gamma^{an} \to Y^{an}$ is an isomorphism.

It remains to prove that the functor $Y \mapsto Y^{an}$ is essentially surjective.

Step 3. *The problem is local for the Zariski topology on X.*

By this we mean that if the functor is essentially surjective for all X_i in some Zariski open covering of X, then it is essentially surjective for X itself. This follows immediately from Step 2: from a finite étale covering $Y \to X^{an}$, we obtain finite étale coverings $Y_i \to X_i^{an}$ plus patching data $Y_{ij} \to Y_{ji}$; if each $Y_i \to X_i^{an}$ is algebraic, then (because of Step 2), the patching data will also be algebraic, and so will give an algebraic étale covering of X.

We may now suppose that X is affine. According to resolution of singularities (Hironaka), there exists a nonsingular projective variety $\bar{X}$ and an open immersion $X \hookrightarrow \bar{X}$ identifying X with a dense open subset of $\bar{X}$ and such that $\bar{X} \setminus X$ is a divisor with normal crossings. Under these hypotheses, one can show that every finite étale covering $Y \to X^{an}$ extends to a finite covering of $\bar{X}^{an}$ (SGA 1, XII 5.3) to which one can apply Step 1. □

Example 21.4. The hypotheses that Y is étale over X^{an} is needed in the last paragraph. Let X be the unit disk in the complex plane, U the complement of the origin in X, and U' the covering of U defined by the equation

$$T^2 = \sin \frac{1}{z}$$

where z is the coordinate function on U. Then U' doesn't extend to a finite covering of X, for suppose it did; then the set of points where $X' \to X$ is not étale is a closed analytic set containing all the points z such that $\sin \frac{1}{z} = 0$, which is absurd.

[31]Géométrie algébrique et géométrie analytique. Ann. Inst. Fourier, Grenoble 6 (1955–1956), 1–42.

Assume from now on that X is connected. Theorem 21.3 implies that, for any $x \in X$, $\pi_1(X_{cx}, x)$ and $\pi_1(X_{et}, x)$ have the same finite quotients. Therefore, there is a natural one-to-one correspondence $\mathcal{F} \leftrightarrow \mathcal{F}^{cx}$ between the locally constant sheaves $\mathcal{F}$ on X_{et} with finite stalks and the locally constant sheaves on X_{cx} with finite stalks. We now restate the theorem in stronger form as:

Theorem 21.5. *Let X be a connected nonsingular variety over $\mathbb{C}$. For any locally constant sheaf $\mathcal{F}$ on X_{et} with finite stalks, $H^r(X_{et}, \mathcal{F}) \cong H^r(X_{cx}, \mathcal{F}^{cx})$ for all $r \geq 0$.*

From now on, I'll drop the superscript on $\mathcal{F}^{cx}$.

Consider the following situation: X is a set with two topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$ (in the conventional sense), and assume that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. Let X_i denote X endowed with the topology $\mathcal{T}_i$. Because $\mathcal{T}_2$ is finer than $\mathcal{T}_1$, the identity map $f\colon X_2 \to X_1$ is continuous. We ask the question: for a sheaf $\mathcal{F}$ on X_2, when is $H^r(X_1, f_*\mathcal{F}) \cong H^r(X_2, \mathcal{F})$ for all $r \geq 0$? Note that $f_*\mathcal{F}$ is simply the restriction of $\mathcal{F}$ to X_1. An answer is given by the Leray spectral sequence

$$H^r(X_1, R^s f_*\mathcal{F}) \Rightarrow H^{r+s}(X_2, \mathcal{F}).$$

Namely, the cohomology groups agree if $R^s f_*\mathcal{F} = 0$ for $s > 0$. But $R^s f_*\mathcal{F}$ is the sheaf associated with the presheaf $U \mapsto H^s(U_2, \mathcal{F})$ (here U is an open subset of X_1, and U_2 denotes the same set endowed with the $\mathcal{T}_2$-topology). Thus we obtain the following criterion for $H^r(X_1, f_*\mathcal{F}) \cong H^r(X_2, \mathcal{F})$:

> for any open $U \subset X_1$ and any $t \in H^s(U_2, \mathcal{F})$, $s > 0$, there exists a covering $U = \cup U(i)$ (for the $\mathcal{T}_1$-topology) such that t maps to zero in $H^s(U(i)_2, \mathcal{F})$ for all i.

Loosely speaking, we can say that the topology $\mathcal{T}_1$ is sufficiently fine to compute the $\mathcal{T}_2$-cohomology of $\mathcal{F}$ if it is sufficiently fine to kill $\mathcal{T}_2$-cohomology classes.

A similar remark applies to continuous maps of sites. Let X_{ecx} denote X endowed with the Grothendieck topology for which the coverings are surjective families of étale maps $(U_i \to U)$ of complex manifolds over X. There are continuous morphisms (of sites)

$$X_{cx} \leftarrow X_{ecx} \rightarrow X_{et}.$$

The inverse mapping theorem shows that every complex-étale covering $(U_i \to U)$ of a complex manifold U has a refinement that is an open covering (in the usual sense). Therefore, the left hand arrow gives isomorphisms on cohomology. It remains to prove that the right hand arrow does also, and for this, the above discussion shows that it suffices to prove the following statement:

Lemma 21.6. *Let U be a connected nonsingular variety, and let $\mathcal{F}$ be a locally constant sheaf on U_{ecx} with finite stalks. For any $t \in H^s(U_{cx}, \mathcal{F})$, $s > 0$, there exists an étale covering $U_i \to U$ (in the algebraic sense), such that t maps to zero in $H^s(U_{i,cx}, \mathcal{F})$ for each i.*

The idea of the proof of the lemma is as follows. We use induction on the dimension of U. Clearly the problem is local on U for the étale topology, i.e., it suffices to prove the statement for the image t_i of t in $H^r(U_i, \mathcal{F})$ for each U_i in some étale covering $(U_i \to U)_{i \in I}$ of U. Thus we can assume that $\mathcal{F}$ is constant, and that U has been replaced by some "Zariski-small" set U. We then find a morphism $f\colon\ U \to S$ from U to a nonsingular variety S of lower dimension such that $R^s f_* \mathcal{F}$ is zero for $s \neq 0, 1$ and is locally constant for $s = 0, 1$ (direct images for the complex topology). The Leray spectral sequence for f gives an exact sequence

$$\cdots \to H^s(S_{cx}, f_*\mathcal{F}) \to H^s(U_{cx}, \mathcal{F}) \to H^{s-1}(S_{cx}, R^1 f_*\mathcal{F}) \to \cdots .$$

Let t map to t'' in $H^{s-1}(S_{cx}, R^1 f_*\mathcal{F})$. If $s - 1 > 0$, then, by induction, there is an étale covering $S_i \to S$ (algebraic sense) such that t'' restricts to zero in each $H^{s-1}(S_{icx}, R^1 f_*\mathcal{F})$. After replacing U with $U \times_S S_i$, we may assume that t is the image of an element $t' \in H^r(S_{cx}, f_*\mathcal{F})$, and apply induction again. This completes the proof for $s > 1$. The case $s = 1$ follows from the Riemann Existence Theorem.

Definition 21.7. An *elementary fibration* is a regular map of varieties $f\colon\ U \to S$ that can be embedded into a commutative diagram

$$\begin{array}{ccccc} U & \xrightarrow{j} & Y & \xleftarrow{i} & Z \\ & \searrow f & \downarrow h & \swarrow g & \\ & & S & & \end{array}$$

in which:

(a) j is an open immersion, $j(U)$ is dense in every fibre of h, and $Y = i(Z) \cup j(U)$;
(b) h is smooth and projective, with geometrically irreducible fibres of dimension 1;
(c) g is finite and étale, and each fibre of g is nonempty.

Proposition 21.8. *Let X be a nonsingular variety over an algebraically closed field k. For any closed point x of X, there is an elementary fibration $U \to S$ with U an open neighbourhood of x and S nonsingular.*

Proof. The idea is to find an embedding $U \hookrightarrow \mathbb{P}^r$ such that the closure $\bar{U}$ of U is normal, and for which there is a particularly good projection map $\mathbb{P}^r \setminus E \to \mathbb{P}^{m-1}$, $m = \dim U$. After blowing up U at the centre of the projection map, one obtains an elementary fibration $U \to S$. See SGA 4, XI (maybe I'll put this in AG one day). □

Remark 21.9. Using the proposition, we may easily complete the proof of Theorem 21.1 for $r = 0$. Since removing or adding a component of real codimension ≥ 2 will not connect or disconnect a manifold, we may replace X by an open subset U as in the proposition, and then by Y. Thus, we have to prove that Y is connected for the complex topology (assuming that S is). Suppose $Y = Y_1 \amalg Y_2$ with Y_1 and Y_2 both open and closed. Because the fibres of h are connected, each of Y_1 and Y_2 must be a union of fibres, i.e., $Y_i = h^{-1}h(Y_i)$. Because h is proper, $h(Y_1)$ and $h(Y_2)$ are closed, and therefore they will disconnect S unless one is empty.

We now prove Lemma 21.6. Because the statement is local for the étale topology on U, we may assume that U admits an elementary fibration and that $\mathcal{F}$ is constant, $\mathcal{F} = \Lambda$. We have an exact (Gysin) sequence

$$\cdots \to H^{r-2}(Z_{cx}, \Lambda(-1)) \to H^r(Y_{cx}, \Lambda) \to H^r(U_{cx}, \Lambda) \to \cdots .$$

For any complex-open subset $S' \subset S$, we obtain a similar sequence with Y, Z, U replaced with $h^{-1}(S')$, $g^{-1}(S')$, $f^{-1}(U)$. In this way, we obtain an exact sequence of presheaves on S_{cx}, and the associated sequence of sheaves is

$$\cdots \to R^{r-2}g_*\Lambda(-1) \to R^r h_*\Lambda \to R^r f_*\Lambda \to \cdots$$

(higher direct images for the complex topology). From this, it follows that the sheaves $R^r f_*\Lambda$ are locally constant, and that for all $s \in S$, the natural map $(R^r f_*\Lambda)_s \to H^r(U_s, \Lambda)$ is an isomorphism (apply the topological proper base change theorem to g and h, and use the five-lemma). Hence $R^r f_*\Lambda = 0$ for $r > 1$. The discussion preceding (21.7) shows that this completes the proof of Theorem 21.5.

Remark 21.10. We now know that, for any complete nonsingular variety X_0 over an algebraically closed field k of characteristic $p \neq 0$ that is liftable to a complete nonsingular variety X_1 in characteristic zero, $H^r(X_0, \Lambda) \approx H^r(X_{1cx}, \Lambda)$ for all r.

22. The Künneth Formula.

This is a brief summary only. Formally, the theory of the Künneth formula is the same for the étale topology as for topological spaces — one only has to replace the easy topological proper base change theorems with the much harder étale versions. See Iverson 1986, VII.2, for the topological version.

Cup-products. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Then there are cup-product maps

$$H^r(X_{et}, \mathcal{F}) \times H^s(X_{et}, \mathcal{G}) \to H^{r+s}(X_{et}, \mathcal{F} \otimes \mathcal{G}).$$

The easiest way to define them is to identify the groups with Čech cohomology groups and set

$$(f \cup g)_{i_0 \cdots i_{r+s}} = f_{i_0 \cdots i_r} \otimes g_{i_r \cdots i_{s+1}}.$$

The Künneth Formula. Let Λ be a finite ring. A map $A^\bullet \to B^\bullet$ of complexes of Λ-modules is called a *quasi-isomorphism* if it induces isomorphisms on the cohomology groups of the complexes.

Let X and Y be algebraic varieties over an algebraically closed field k, and let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X and Y respectively. Consider

$$\begin{array}{ccccc} & & X \times Y & & \\ & \swarrow p & & \searrow q & \\ X & & & & Y. \end{array}$$

On combining the restriction maps

$$\begin{array}{rcl} H^r(X, \mathcal{F}) & \to & H^r(X \times Y, p^*\mathcal{F}) \\ H^s(Y, \mathcal{G}) & \to & H^s(X \times Y, q^*\mathcal{G}) \end{array}$$

with the cup-product map

$$H^r(X \times Y, p^*\mathcal{F}) \times H^s(X \times Y, q^*\mathcal{F}) \to H^{r+s}(X \times Y, p^*\mathcal{F} \otimes q^*\mathcal{G})$$

we obtain a map

$$H^r(X, \mathcal{F}) \times H^s(Y, \mathcal{G}) \to H^{r+s}(X \times Y, p^*\mathcal{F} \otimes q^*\mathcal{G}).$$

The Künneth formula studies the extent to which the map

$$\oplus_{r+s=n} H^r(X, \mathcal{F}) \otimes H^s(Y, \mathcal{G}) \to H^n(X \times Y, p^*\mathcal{F} \otimes q^*\mathcal{G})$$

is an isomorphism.

Theorem 22.1. *Let X and Y be complete varieties; there exists an quasi-isomorphism*

$$H(X, \Lambda) \otimes_\Lambda H(Y, \Lambda) \to H(X \times Y, \Lambda)$$

where $H(X, \Lambda)$, $H(Y, \Lambda)$, and $H(X \times Y, \Lambda)$ are complexes of Λ-modules such that

(a) $H^r(H(X, \Lambda)) \cong H^r(X_{et}, \Lambda)$, *and similarly for the other two;*
(b) $H(X, \Lambda)$ *is a complex of flat Λ-modules.*

COROLLARY 22.2. *There is a spectral sequence*

$$\sum_{i+j=s} Tor^{\Lambda}_{-r}(H^i(X_{et},\Lambda),H^j(Y_{et},\Lambda)) \Rightarrow H^{r+s}(X\times Y,\Lambda).$$

PROOF. That the quasi-isomorphism in the theorem yields such a spectral sequence is a standard result in homological algebra. □

REMARK 22.3. (a) Unfortunately, in general the spectral sequence in the corollary is infinite. For example, let $\Lambda=\mathbb{Z}/\ell^2\mathbb{Z}$. There are exact sequences of any length

$$0\to\mathbb{Z}/\ell\mathbb{Z}\to\mathbb{Z}/\ell^2\mathbb{Z}\to\cdots\to\mathbb{Z}/\ell^2\mathbb{Z}\xrightarrow{\ell}\mathbb{Z}/\ell^2\mathbb{Z}\to\mathbb{Z}/\ell\mathbb{Z}\to 0.$$

This shows that $\mathrm{Tor}_r^{\mathbb{Z}/\ell^2\mathbb{Z}}(\mathbb{Z}/\ell\mathbb{Z},\mathbb{Z}/\ell\mathbb{Z})$ is nonzero for infinitely many r.

(b) The proof of the theorem (see below) uses the proper base change theorem. It also holds noncomplete varieties and cohomology with compact support, and for ordinary cohomology if the varieties are smooth or one assumes a theorem of Deligne (24.3) below).

(c) To state, and prove, the theorem, it is most natural to use derived categories.

(d) The theorem holds with constructible sheaves of Λ modules $\mathcal{F}$ and $\mathcal{G}$, provided at least one of them is a flat sheaf of Λ-modules (i.e., $\mathcal{F}\otimes_\Lambda -$ is exact).

For $\Lambda=\mathbb{Z}/\ell^n\mathbb{Z}$, it is possible to define the complexes and the quasi-isomorphism compatibly for varying n, so that we can pass to the inverse limit, and obtain a quasi-isomorphism

$$H(X,\mathbb{Z}_\ell)\otimes_{\mathbb{Z}_\ell}H(Y,\mathbb{Z}_\ell)\to H(X\times Y,\mathbb{Z}_\ell).$$

Because $\mathbb{Z}_\ell$ is a principal ideal domain, this yields a Künneth formula in its usual form.

THEOREM 22.4. *For varieties X and Y, there is a canonical exact sequence*

$$0\to\sum_{r+s=m}H^r(X,\mathbb{Z}_\ell)\otimes H^s(Y,\mathbb{Z}_\ell)\to H^m(X\times Y,\mathbb{Z}_\ell)\to\sum_{r+s=m+1}Tor_1^{\mathbb{Z}_\ell}(H^r(X,\mathbb{Z}_\ell),H^s(Y,\mathbb{Z}_\ell))\to 0$$

(and similarly for cohomology with compact support).

The Proof. The first step is to prove a projection formula.

PROPOSITION 22.5 (Projection formula). *Consider a regular map $f\colon X\to S$. Then, for any flat sheaf $\mathcal{F}$ of Λ-modules on X and bounded above complex of sheaves $\mathcal{G}^\bullet$ on S, there is a quasi-isomorphism*

$$(Rf_*\mathcal{F})\otimes\mathcal{G}^\bullet\xrightarrow{\sim}Rf_*(\mathcal{F}\otimes f^*\mathcal{G}^\bullet).$$

PROOF. To prove this, one reduces the general case to the case $\mathcal{G}^\bullet=\Lambda$, which is obvious. □

Next, consider

$$\begin{tikzcd} & X \times_S Y \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[dd, "h"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd} .$$

Let $\mathcal{F}$ be a flat constructible sheaf of Λ-modules on X, and let $\mathcal{G}$ be a constructible sheaf on Λ-modules on Y. Then, in the language of derived categories,

$$\begin{array}{cc} Rf_*\mathcal{F} \otimes Rg_*\mathcal{G} & \\ \| & \text{(projection formula)} \\ Rf_*(\mathcal{F} \otimes f^* Rg_*\mathcal{G}) & \\ \| & \text{(base change)} \\ Rf_*(\mathcal{F} \otimes Rp_*(q^*\mathcal{G})) & \\ \| & \text{(projection formula)} \\ Rf_*(Rp_*(p^*\mathcal{F} \otimes q^*\mathcal{G})) & \\ \| & (Rf_* \circ Rp_* = R(f \circ p)_*) \\ Rh_*(p^*\mathcal{F} \otimes q^*\mathcal{G}). & \end{array}$$

In order to be able to apply the base change theorem, we need that

(a) g is proper (use the proper base change theorem), or
(b) f is smooth (use the smooth base change theorem), or
(c) S is the spectrum of a field (use Deligne's theorem).

23. The Cycle Map; Chern Classes

We want to associate a cohomology class with an algebraic cycle on a variety. The direct definition is quite short, but, unfortunately, it is difficult to derive the properties one wants from it. Another definition, using Chern classes, yields all the general properties, but is not very explicit. The best approach is to give both definitions, and to verify that they coincide.

Throughout this section, all varieties will over an algebraically closed field k, and $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$ with $\ell \neq char(k)$. We also allow Λ to be $\mathbb{Z}_\ell$, or $\mathbb{Q}_\ell$, although this requires minor modifications to the exposition. We set $H^*(X) = \oplus_{r\geq 0} H^{2r}(X, \Lambda(r))$ (notation as at the start of §16) — it becomes a ring under cup-product.

For reference, we note that given a variety and open subvarieties $X \supset U \supset V$, we get an exact sequence (of the triple) of étale cohomology groups

$$\cdots \to H^r_{X-U}(X) \to H^r_{X-V}(X) \to H^r_{U-V}(U) \to \cdots .$$

When V is empty, this is the exact sequence of the pair (X, U).

Preliminaries. Let X be a nonsingular variety over k. A *prime cycle* on X is an irreducible closed subvariety. Let $C^r(X)$ be the free abelian group generated by the prime cycles of codimension r — its elements are called the *algebraic cycles of codimension* r on X. Thus $C^1(X) = \mathrm{Div}(X)$. We let $C^*(X) = \oplus_{r\geq 0} C^r(X)$.

We refer the reader to Hartshorne 1977, Appendix A, for the notion of two algebraic cycles being rationally equivalent and for the intersection product of two algebraic cycles.

The quotient $CH^*(X)$ of $C^*(X)$ by rational equivalence becomes a ring relative to intersection product. It is called the *Chow ring.*

For example, $CH^1(X) = Pic(X)$. Recall that $Pic(X) = H^1(X_{et}, \mathbb{G}_m)$, and so the cohomology sequence of the Kummer sequence gives a homorphism

$$\mathrm{Pic}(X) \to H^2(X_{et}, \Lambda(1)).$$

Our object in this section is to define a canonical homomorphism of graded rings

$$cl^*_X \colon CH^*(X) \to H^*(X)$$

such that $cl^1_X\colon\ CH^1(X) \to H^2(X_{et}, \Lambda(1))$ is the map just defined.

Direct definition of the cycle map. Again X is nonsingular. When Z is nonsingular, we let $cl_X(Z)$ be the image of 1 under the Gysin map

$$\Lambda = H^0(Z, \Lambda) \to H^{2r}_Z(X, \Lambda(r)) \to H^{2r}(X, \Lambda(r)).$$

To extend this definition to nonsingular prime cycles, we need the following lemma.

Lemma 23.1 (Semi-purity). *For any closed subvariety Z of codimension c in X, $H^r_Z(X, \Lambda) = 0$ for $r < 2c$.*

Proof. When Z is nonsingular, $H^s_Z(X, \Lambda) \cong H^{s-2c}(Z, \Lambda(-c))$, which is 0 for $s - 2c < 0$ (see 16.1). We prove the lemma for a general Z by induction on the dimension of Z. If Z has dimension 0, it is nonsingular, and so the statement is true. Let Z be a closed subvariety of codimension c. Its singular locus, Y, is a closed subvariety of dimension less than that of Z, and the exact sequence of the triple $(X, X \setminus Y, X \setminus Z)$ is

$$\cdots \to H^r_Y(X, \Lambda) \to H^r_Z(X, \Lambda) \to H^r_{Z \setminus Y}(X \setminus Y, \Lambda) \to \cdots .$$

Now $H^r_{Z\setminus Y}(X \setminus Y, \Lambda) = 0$ for $r < 2c$ because $Z \setminus Y$ is nonsingular, and $H^r_Y(X, \Lambda) = 0$ for $r < 2c + 2$ by induction, and this implies that $H^r_Z(X, \Lambda) = 0$ for $r < 2c$. □

Now let Z be a prime cycle in X of codimension c, and let Y be the singular locus of Z. From the exact sequence of the triple $(X, X \setminus Y, X \setminus Z)$ we obtain (using the lemma) an isomorphism

$$H^{2c}_Z(X, \Lambda) \cong H^{2c}_{Z\setminus Y}(X \setminus Y, \Lambda).$$

We define $cl_X(Z)$ to be the image of 1 under the composite of the maps

$$\Lambda \cong H^0(Z \setminus Y, \Lambda(c)) \cong H^{2c}_{Z\setminus Y}(X \setminus Y, \Lambda(c)) \cong H^{2c}_Z(X, \Lambda(c)) \to H^{2c}(X, \Lambda(c)).$$

We extend this by linearity to a homomorphism

$$cl^r \colon C^r(X) \to H^{2r}(X, \Lambda(r)).$$

For $r = 1$, this is the map defined by the Kummer sequence.

Chern classes. We now take X to be a nonsingular projective variety.

Let E be a vector space of dimension $m + 1$ over k. We define $\mathbb{P}(E)$ to be the projective space of lines in E. The choice of a basis for E determines an isomorphism $\mathbb{P}(E) \to \mathbb{P}^m$.

More generally, let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank $m + 1$ on a nonsingular variety X (for the Zariski topology). Then (see Hartshorne 1977, p162) there is a *projective space bundle* $\mathbb{P}(\mathcal{E})$ associated with $\mathcal{E}$. This is a nonsingular variety equipped with a regular map $\pi\colon\ \mathbb{P}(\mathcal{E}) \to X$ and a canonical invertible sheaf $\mathcal{O}(1)$. The fibre $\pi^{-1}(x) = \mathbb{P}(\mathcal{E}_x)$ where $\mathcal{E}_x$ is the fibre of $\mathcal{E}$ at x (a k-vector space of dimension $m + 1$). If $\mathcal{E}$ is free over a Zariski open subset U of X, then the choice of an isomorphism $\mathcal{O}^{m+1}_U \to \mathcal{E}|U$ determines an isomorphism $\mathbb{P}^m \to \mathbb{P}(\mathcal{E})$

THEOREM 23.2. *Let $\mathcal{E}$ be a locally free sheaf rank $m{+}1$ on X_{zar}, and let $\pi\colon\ \mathbb{P}(\mathcal{E}) \to X$ be the associated projective bundle. Let ξ be the class of $\mathcal{O}(1)$ in $H^2(\mathbb{P}(\mathcal{E}), \Lambda(1))$. Then π^* makes $H^*(\mathbb{P}(\mathcal{E}))$ into a free $H^*(X)$-module with basis $1, \xi, \ldots, \xi^m$.*

PROOF. There is an isomorphism of graded rings

$$\Lambda[T]/(T^{m+1}) \xrightarrow{\approx} H^*(\mathbb{P}^m)$$

sending T to the class ξ of the hyperplane section; in particular, $H^*(\mathbb{P}^m)$ is a free Λ-module with basis $1, \ldots, \xi^m$ (see §16).

If $\mathcal{E}$ is free, the choice of an isomorphism $\alpha\colon \mathcal{E} \approx \mathcal{O}^{m+1}_X$ determines an isomorphism $\mathbb{P}(\mathcal{E}) \approx X \times \mathbb{P}^m_k$, and the Künneth formula shows that $H^*(X \times \mathbb{P}^m_k) \cong H^*(X) \otimes H^*(\mathbb{P}^m)$, which is a free module over $H^*(X)$ with basis $1, T, \ldots, T^m$. The isomorphism $H^*(\mathbb{P}(\mathcal{E})) \approx H^*(X \times \mathbb{P}^m_k)$ is an $H^*(X)$-isomorphism and sends ξ to T. Thereore $H^*(\mathbb{P}(\mathcal{E}))$ is a free module over $H^*(X)$ with basis $1, \xi, \ldots, \xi^m$.

Next, if $U = X_0 \cup X_1$ and we know the proposition for X_0, X_1, and $X_0 \cap X_1$, then the Mayer-Vietoris sequence (Theorem 10.7) allows us to prove it for U. Since we know the proposition for any Zariski-open subset of X over which $\mathcal{E}$ is trivial, this argument allows us to obtain it for X step-by-step. □

The proposition shows that, when we regard $H^*(\mathbb{P}(\mathcal{E}))$ as an $H^*(X)$-module, there is a linear relation between $1, \xi, \ldots, \xi^{m+1}$, which is unique if we normalize the coefficient of ξ^{m+1} to be 1. In other words, there are unique elements $ch_r(\mathcal{E}) \in H^{2r}(X, \Lambda(r))$ such that

$$\begin{cases} \sum_{r=0}^{m+1} ch_r(\mathcal{E}) \cdot \xi^{m+1-r} &= 0 \\ ch_0(\mathcal{E}) &= 1. \end{cases}$$

Then $ch_r(\mathcal{E})$ is called the *rth Chern class* of $\mathcal{E}$, $ch(\mathcal{E}) = \sum ch_r(\mathcal{E})$ the *total Chern class* of $\mathcal{E}$, and

$$ch_t(\mathcal{E}) = 1 + ch_1(\mathcal{E})t + \cdots$$

the *Chern polynomial* of $\mathcal{E}$.

THEOREM 23.3. *The Chern classes have the following properties:*

(a) *(Functoriality). If $\pi\colon\ Y \to X$ is morphism of smooth varieties and $\mathcal{E}$ is a vector bundle on X, then $ch_r(\pi^{-1}(\mathcal{E})) = \pi^*(ch_r(\mathcal{E}))$.*
(b) *(Normalization). If $\mathcal{E}$ is an invertible sheaf on X, then $ch_1(\mathcal{E})$ is the image of $\mathcal{E}$ under the map $Pic(X) \to H^2(X, \Lambda(1))$ given by the Kummer sequence.*
(c) *(Additivity). If*

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$$

is an exact sequence of vector bundles on a smooth variety X, then

$$ch_t(\mathcal{E}) = ch_t(\mathcal{E}') \cdot ch_t(\mathcal{E}'').$$

PROOF. See Grothendieck, A., La théorie des classes de Chern, Bull. Soc. Math. France 86 (1958), 137-158. □

The above reference may appear to be an anachronism, but Grothendieck proves an abstract theorem. He assumes only that there is

(i) a functor from the category of nonsingular projective varieties X to graded anti-commutative rings,

(ii) a functorial homomorphism $Pic(X) \to H^2(X)$, and

(iii) for every closed immersion $Z \hookrightarrow X$ of (smooth projective) varieties, a "Gysin map" $H^*(Z) \to H^*(X)$,

satisfying certain natural axioms. He then shows that there is a theory of Chern classes satisfying conditions (a,b,c) of the Theorem.

Let $K(X)$ be the Grothendieck group of locally free sheaves of finite rank on X. Part (c) of the theorem shows that $\mathcal{E} \mapsto ch(\mathcal{E})$ factors through $K(X)$. Because X is smooth, $K(X)$ is also the Grothendieck group of coherent $\mathcal{O}_X$-modules (standard result). This allows us to define a map $\gamma\colon\ C^*(X) \to K(X)$: given a prime cycle Z, resolve $\mathcal{O}_Z$ by free $\mathcal{O}_X$-modules of finite rank,

$$0 \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to \cdots \to \mathcal{E}_0 \to \mathcal{O}_Z \to 0$$

and set $\gamma(Z) = \sum(-1)^i ch(\mathcal{E}_i)$ (here $\mathcal{O}_Z$ is the structure sheaf of Z, regarded as an $\mathcal{O}_X$-module with support on Z). In sum: we have maps

$$C^*(X) \to K(X) \to H^*(X).$$

In order to get homorphisms of *graded* groups, we need to replace $K(X)$ with a graded group.

There is a filtration on $K(X)$. Define $K^r(X)$ to be the subgroup of $K(X)$ generated by coherent $\mathcal{O}_X$-modules with support in codimension $\geq r$. The groups K^r define a filtration of $K(X)$. Let

$$GK^*(X) = gr(K(X)) \overset{\text{df}}{=} \oplus K^r(X)/K^{r+1}(X).$$

Then $GK^*(X)$ becomes a group under the product law

$$[\mathcal{M}][\mathcal{N}] = \sum(-1)^r[\underline{Tor}_r^{\mathcal{O}}(\mathcal{M},\mathcal{N})].$$

Here $[*]$ denotes the class of $*$ in the Grothendieck group. The map $C^*(X) \to K(X)$ is clearly compatible with the filtration, and so defines a homomorphism of graded modules $\gamma\colon\ C^*(X) \to GK^*(X)$.

Set

$$CH^*(X) = C^*(X)/(\text{rational equivalence}).$$

It becomes a ring under intersection product. Cycles rationally equivalent to zero map to zero under γ, and so γ defines a map $\gamma\colon\ CH^*(X) \to GK^*(X)$. Serre's description of intersection products shows that this is compatible with intersection products. See Hartshorne 1977, Appendix A.

The map $ch\colon\ K^*(X) \to H^*(X)$ induces a map $GK^*(X) \to H^*(X)$, but unfortunately, this isn't quite a ring homomorphism: let $H^*(X)' = H^*(X)$ as an abelian group, but give it the multiplicative structure

$$x_r * x_s = \frac{-(r+s-1)!}{(r-1)!(s-1)!} x_r x_s,\ x_r \in H^r(X),\ x_s \in H^s(X).$$

Then $ch\colon\ GK^*(X) \to H^*(X)'$ is a homomorphism.

When $(2\dim X - 1)!$ is invertible in Λ, the map

$$x_r \mapsto x_r/(-1)^{r-1}(r-1)!\colon H^*(X)' \to H^*(X)$$

is an isomorphism. On composing, the maps

$$CH^*(X) \xrightarrow{\gamma} GK^*(X) \xrightarrow{ch} H^*(X)' \to H^*(X),$$

we obtain a homomorphism

$$cl_X\colon CH^*(X) \to H^*(X).$$

This has the following properties:

(a) it is a homomorphism of graded rings (doubling degrees); in particular, intersection products map to cup-products;
(b) it is functorial in X.

Theorem 23.4. *This chern-class cycle map agrees the directly-defined cycle map.*

Proof. A correct proof is quite long. □

Exercise 23.5. Find the error in the proof of EC VI 10.6.

In the above, we have taken $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$. For different n, the cycle maps are compatible, and so we can pass to the inverse limit to get a homomorphism of graded rings

$$CH^*(X) \to \oplus H^{2r}(X, \mathbb{Z}_\ell(r))$$

provided $\ell \geq 2\dim X$. On tensoring with $\mathbb{Q}$, we get a homomorphism

$$CH^*(X) \to \oplus H^{2r}(X, \mathbb{Q}_\ell(r))$$

(no longer need $\ell \geq 2\dim X$).

Application. Assume X to be projective of dimension d. The group $C^d(X)$ consists of finite (formal) sums $\sum n_P P$, $n_P \in \mathbb{Z}$, P a closed point of X. The degree map

$$\sum n_P P \mapsto \sum n_P \colon C^d(X) \to \mathbb{Z}$$

factors through $CH^d(X)$. Therefore, we have a pairing

$$Z, Y \mapsto (Z \cdot Y) \colon CH^r(X) \times CH^{d-r}(X) \to \mathbb{Z}.$$

An algebraic cycle Z is said to be *numerically equivalent* to zero if $(Z \cdot Y) = 0$ for all algebraic cycles Y of complementary codimension. We let $N^r(X)$ be the quotient of $CH^r(X)$ by numerical equivalence. We now have the following remarkable theorem.

Theorem 23.6. *The groups $N^r(X)$ are finitely generated.*

Proof. By definition, we have a nondegenerate pairing

$$N^r(X) \times N^{d-r}(X) \to \mathbb{Z}.$$

Hence $N^r(X)$ is torsion-free (obviously), and so it suffices to show that $N^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell$ is finite-dimensional. Consider

$$\begin{array}{ccccc}
N^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \times & N^{d-r}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \to & \mathbb{Q}_\ell \\
\uparrow & & \uparrow & & \| \\
CH^r(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \times & CH^{d-r}(X) \otimes_{\mathbb{Z}} \mathbb{Q}_\ell & \to & \mathbb{Q}_\ell \\
\downarrow & & \downarrow & & \| \\
H^{2r}(X, \mathbb{Q}_\ell(r)) & \times & H^{2d-2r}(X, \mathbb{Q}_\ell(d-r)) & \to & \mathbb{Q}_\ell
\end{array}$$

This diagram commutes, and the top pairing is nondegenerate. Therefore, the kernel K of $cl_X \colon CH^r(X) \otimes \mathbb{Q}_\ell \to H^{2r}(X, \mathbb{Q}_\ell(r))$ is contained in the kernel of $CH^r(X) \otimes \mathbb{Q}_\ell \to N^r(X) \otimes \mathbb{Q}_\ell$. We have

$$H^{2r}(X, \mathbb{Q}_\ell(r)) \leftarrow CH^r(X) \otimes \mathbb{Q}_\ell / K \xrightarrow{\text{onto}} N^r(X) \otimes \mathbb{Q}_\ell.$$

As $H^{2r}(X, \mathbb{Q}_\ell(r))$ is finite-dimensional, this proves that $N^r(X) \otimes \mathbb{Q}_\ell$ is finite-dimensional. $\square$

Remark 23.7. To show $N^r(X)$ finitely generated, it would not be sufficient to prove that there is an injection $N^r \hookrightarrow H^{2r}(X, \mathbb{Q}_\ell(r))$. For example, there is an injection $\mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$, but $\mathbb{Q}$ is not a finitely generated abelian group.

Let X be a variety over a separably closed field k, and let Z be an irreducible closed subvariety of X. Then, because X and Z are defined by finitely many polynomials, each with finitely many coefficients, there will exist models of X and Z defined over a field k_0 that is finitely generated over the prime field. Therefore the cohomology class of Z will be "defined over" k_0. Tate conjectures that this can be used to characterize algebraic classes. More precisely, for each model X_0 of X over a field k_0 finitely generated over the prime field, let

$$T^r(X_0/k_0) = H^{2r}(X_{et}, \mathbb{Q}_\ell(r))^{\mathrm{Gal}(k_0^{\mathrm{sep}}/k_0)}.$$

As $H^{2r}(X,-) = H^{2r}(X_{0,k_0^{\mathrm{sep}}},-)$ (see 17.8), this makes sense. Then Tate conjectures that

$$T^r(X/k) \stackrel{\mathrm{df}}{=} \cup T^r(X_0/k_0)$$

is the $\mathbb{Q}_\ell$-subspace of $H^{2r}(X, \mathbb{Q}_\ell(r))$ generated by the classes of algebraic cycles. Although we all hope that the conjecture is true, there is no real reason for believing that it is. For abelian varieties and $r = 1$, it has been proved (Tate, Zarhin, Faltings). It is known that, for abelian varieties over $\mathbb{C}$, the Tate conjecture implies the Hodge conjecture (Piatetski-Shapiro, Deligne), and it is known that the Hodge conjecture for abelian varieties of CM-type over $\mathbb{C}$ implies the Tate conjecture for all abelian varieties over finite fields (all r) (Milne). However, although abelian varieties of CM-type are very special — they correspond to a set of dimension zero in the moduli space — there seems to be little hope that the complex analysts/classical algebraic geometers will prove it any time soon.

Poincaré duality for topological spaces. The classical Poincaré duality theorem (Greenberg, M., *Lectures on Algebraic Topology*, Benjamin, 1967, 26.6) says that, for an oriented connected m-dimensional manifold U, there is a canonical isomorphism $H^r_c(U, \mathbb{Z}/n\mathbb{Z}) \to H_{m-r}(U, \mathbb{Z}/n\mathbb{Z})$. Using the duality between H_s and H^s, we can rewrite this as a perfect pairing of finite groups

$$H^r_c(U, \mathbb{Z}/n\mathbb{Z}) \times H^{m-r}(U, \mathbb{Z}/n\mathbb{Z}) \to H^m_c(U, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}.$$

A better approach (e.g., Iverson 1986, V.3) avoids the choice of an orientation. Instead, one introduces an "orientation sheaf" o for which there is a canonical isomorphism $H^m_c(U, o) \cong \mathbb{Z}/n\mathbb{Z}$. The Poincaré duality theorem then becomes a perfect pairing

$$H^r_c(U, o) \times H^{m-r}(U, \mathbb{Z}/n\mathbb{Z}) \to H^m_c(U, o) \cong \mathbb{Z}/n\mathbb{Z}.$$

The manifold is orientable if and only if there is an isomorphism $\mathbb{Z}/n\mathbb{Z} \to o$, and the choice of such an isomorphism is an orientation of U.

Define $\mathbb{C}$ to be the algebraic closure of $\mathbb{R}$. To give an orientation of $\mathbb{C}$ regarded as a real manifold is the same as to give a choice of $\sqrt{-1}$ (and hence a choice $e^{2\pi\sqrt{-1}/n}$ of a primitive n^{th} root of 1 for all n). Once an orientation of $\mathbb{C}$ has been chosen, one obtains an orientation of any complex manifold of dimension 1 (conformal mappings preserve orientation), and, indeed, of a complex manifold of any dimension. Thus, for a connected complex manifold U of complex dimension m, the classical Poincaré duality theorem takes the form

$$H^r_c(U, \mathbb{Z}/n\mathbb{Z}) \times H^{2m-r}(U, \mathbb{Z}/n\mathbb{Z}) \to H^{2m}_c(U, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$$

once one has chosen a primitive 4^{th} root of 1.

More generally, there is fancy duality theorem, usually called Verdier duality, for any locally compact space of finite dimension.

Poincaré duality for nonsingular algebraic varieties. Let k be an algebraically closed field, and let $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for some n prime to the characteristic of k. For a sheaf of Λ-modules $\mathcal{F}$, we set $\check{\mathcal{F}}(m) = \underline{\mathrm{Hom}}(\mathcal{F}, \Lambda(m))$. Here $\Lambda(m) = \mu_n^{\otimes m}$.

Let X be a nonsingular variety of dimension d. For any closed point $P \in X$, the Gysin map is an isomorphism

$$H^0(P, \Lambda) \to H^{2d}_P(X, \Lambda(d)).$$

There is a canonical map $H^{2d}_P(X, \Lambda(d)) \to H^{2d}_c(X, \Lambda(d))$, and we let $cl(P)$ be the image of 1 under the composite of these maps.

Theorem 24.1. *Let X be a nonsingular variety of dimension d over an algebraically closed field k.*

(a) *There is a unique map $\eta(X)\colon\ H^{2d}_c(X, \Lambda(d)) \to \Lambda$ sending $cl(P)$ to 1 for any closed point P on X; it is an isomorphism (η is called the* trace map.*)*
(b) *For any locally constant sheaf $\mathcal{F}$ of Λ-modules, there are canonical pairings*

$$H^r_c(X, \mathcal{F}) \times H^{2d-r}(X, \mathcal{F}^\vee(d)) \to H^{2d}_c(X, \Lambda(d)) \cong \Lambda,$$

which are perfect pairings of finite groups.

One way to define the pairings is to identify $H^{2d-r}(X, \check{\mathcal{F}}(d))$ with $\mathrm{Ext}_X^{2d-r}(F, \Lambda(d))$, which can be regarded as a group of extensions of length $2d-r$, and then to use repeated coboundary maps (see the discussion following Theorem 14.20). When X is quasi-projective, the groups can be identified with the Čech groups, and the pairing can be defined by the usual cup-product formula.

The Gysin map. Let $\pi\colon\ Y \to X$ be a proper map of smooth separated varieties over an algebraically closed field k; let $a = \dim\ X$, $d = \dim\ Y$, and $c = d - a$. There is a restriction map

$$\pi^*\colon H_c^{2d-r}(X, \Lambda(d)) \to H_c^{2d-r}(Y, \Lambda(d)).$$

By duality, we get a map

$$\pi_*\colon H^r(Y, \Lambda) \to H^{r-2c}(X, \Lambda(-c)).$$

Remark 24.2. (a) The map π_* is uniquely determined by the equation:

$$\eta_X(\pi_*(y) \cup x) = \eta_Y(y \cup \pi^*(x)),\ x \in H_c^{2d-r}(X, \Lambda(d)),\ y \in H^r(Y, \Lambda).$$

This is the definition.

(b) If π is closed immersion $Y \hookrightarrow Z$, π_* is the Gysin map defined in §16 — this is a consequence of the proof of the duality theorem. Note that in this case $-c$ is the codimension of Y in X. In particular,

$$\pi_*(1_Y) = cl_X(Y),$$

where 1_Y is the identity element of $H^0(Y, \Lambda) = \Lambda$.

(c) For a composite of mappings

$$\pi_{1*} \circ \pi_{2*} = (\pi_1 \circ \pi_2)_*$$

This follows directly from the definition, because $(\pi_1 \circ \pi_2)^* = \pi_2^* \circ \pi_1^*$.

(d) If Y and X are complete, then

$$\eta_X(\pi_*(y)) = \eta_Y(y), \text{ for } \ y \in H^{2d}(Y, \Lambda(d)),$$

because $\eta_X(\pi_*(y)) = \eta_X(\pi_*(y) \cup 1_X) \overset{(a)}{=} \eta_Y(y \cup \pi^*(1_X)) = \eta_Y(y \cup 1_Y) = \eta_Y(y)$.

(e) (Projection formula) If Y and X are complete, then

$$\pi_*(y \cup \pi^*(x)) = \pi_*(y) \cup x \text{for } \ x \in H^r(X) \text{and } \ y \in H^s(Y).$$

To prove this, apply η_X to $\pi_*(y \cup \pi^*(x)) \cup x')$ for $x' \in H^{2d-r-s}(X)$.

(f) If $\pi\colon\ Y \to X$ is a finite map of degree δ, then $\pi_* \circ \pi^* = \delta$. [Exercise] Since, π^* acts as the identity on $H^0(X)$, π_* acts as the identity on $H^{2\dim X}(X)$. Moreover, π^* acts as multiplication by δ on $H^{2\dim X}(X)$ and π_* acts as multiplication by δ on $H^0(X)$.

Sometimes, π_* is also called the *Gysin map.*

Application to base change theorems.

THEOREM 24.3. *Let* $\pi\colon\ Y \to X$ *be a regular map of varieties over a field* k *(not necessarily algebraically closed). Let* $\mathcal{F}$ *be a constructible sheaf on* Y*. Then* $R^r\pi_*\mathcal{F}$ *is constructible for all* r *and zero for all but finitely many* r*. Moreover, the formation of* $R^r\pi_*$ *commutes with all base changes* $T \to \operatorname{Specm} k$*.*

The proof uses the proper base change theorem and Poincaré duality. See Deligne, P., Théorèmes de finitude en cohomologie ℓ-adique, SGA $4\frac{1}{2}$, pp 233–251.

Sketch of the proof of Poincaré duality. Omitted for the present.

Verdier duality. Omitted for the present. A recent reference on fancy duality theorems is:

Neeman, A., The Grothendieck duality theorem via Bousfield's techniques and Brown representability, Jour. Amer. Math. Soc. 9 (1996), 205–236.

25. Lefschetz Fixed-Point Formula.

Here we show that the existence of a Lefschetz fixed-point formula is a formal consequence of the existence of a cycle map with good properties, the Künneth formula, and Poincaré duality. Throughout this section, X is a nonsingular variety (usually complete) over an algebraically closed field k.

Let V be a vector space, and let $\varphi\colon\ V \to V$ be a linear map. If (a_{ij}) is the matrix of φ with respect to a basis (e_i) of V, then the trace of φ, denoted $\mathrm{Tr}(\varphi|V)$, is $\sum a_{ii}$. It is independent of the choice of the basis. If (f_i) is the dual basis of the dual vector space $\check{V}$, so that $e_i \cdot f_j = \delta_{ij}$, then

$$\sum_i \varphi(e_i)\cdot f_i = \sum_i (\sum_j a_{ji}e_j)\cdot f_i = \sum a_{ii} = \mathrm{Tr}(\varphi|V).$$

Theorem 25.1 (Lefschetz fixed-point formula). *Let X be a complete nonsingular variety over an algebraically closed field k, and let $\varphi\colon\ X \to X$ be a regular map. Then*

$$(\Gamma_\varphi \cdot \Delta) = \sum (-1)^r \,\mathrm{Tr}(\varphi|H^r(X,\mathbb{Q}_\ell)).$$

Here Γ_φ is the graph of φ, and Δ is the diagonal in $X\times X$. Thus $(\Gamma_\varphi\cdot\Delta)$ is the number of fixed points of φ counted with multiplicities.

Example 25.2. Consider the map

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1, \quad (x_0\colon x_1) \mapsto (x_0+x_1\colon x_1).$$

On the affine piece where $x_1 \neq 0$, φ is $x \mapsto x+1$, and on the affine piece where $x_0 \neq 0$ it is $x \mapsto \frac{x}{1+x}$. Since φ acts as 1 on $H^0(\mathbb{P}^1)$ and on $H^2(\mathbb{P}^1)$ (because it has degree 1), the right hand side of the Lefschetz fixed-point formula is 2. The only fixed point of φ is $\infty = (1\colon\ 0)$ (the origin in the affine piece where $x_0 \neq 0$). To compute $(\Gamma_\varphi\cdot\Delta)_\infty$, we have to compute the intersection number of the curves

$$y(1+x) = x, \quad y = x$$

at $(0,0)$. It is the dimension of

$$k[x,y]/(y-x, y(1+x)-x) = k[x]/(x+x^2-x) = k[x]/(x^2),$$

which is 2, as Theorem 25.1 predicts.

In order to simplify the exposition, we fix an isomorphism $\mathbb{Q}_\ell \to \mathbb{Q}_\ell(1)$; this amounts to choosing compatible isomorphisms $\mathbb{Z}/\ell^n\mathbb{Z} \to \mu_{\ell^n}(k)$ for all n.

Write $H^*(X) = \oplus H^r(X,\mathbb{Q}_\ell)$ — it is a $\mathbb{Q}_\ell$-algebra. The Künneth formula allows us to identify $H^*(X\times X)$ with $H^*(X)\otimes H^*(X)$ by identifying $p^*(a)\cup q^*(b)$ with $a\otimes b$. Here p and q are the projection maps $X\times X \rightrightarrows X$.

Poincaré duality gives a nondegenerate pairing

$$H^*(X)\times H^*(X) \to H^{2d}(X,\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell.$$

Write e^{2d} for the canonical generator of $H^{2d}(X)$ (the class of any point P).

The proof of Theorem 25.1 that follows was copied almost word for word from a topology book.

Let $\varphi\colon\ X \to Y$ be a regular map. The next lemma shows that the map $\varphi^*\colon\ H^*(Y) \to H^*(X)$ is equal to that defined by the correspondence Γ_φ on $X \times Y$.

LEMMA 25.3. *For any regular map $\varphi\colon\ X \to Y$ and any $b \in H^*(Y)$,*

$$p_*(cl_{X\times Y}(\Gamma_\varphi) \cup q^*y) = \varphi^*(y).$$

PROOF. We compute:

$$\begin{aligned} p_*(cl_{X\times Y}(\Gamma_\varphi) \cup q^*y) &= p_*((1,\varphi)_*(1) \cup q^*y) && (24.2b)\\ &= p_*(1,\varphi)_*(1 \cup (1,\varphi)^*q^*y) && (24.2e)\\ &= (p\circ(1,\varphi))_*(1 \cup (q\circ(1,\varphi))^*y) && (24.2c)\\ &= id_*(1_X \cup \varphi^*y)\\ &= \varphi^*(y). \end{aligned}$$

□

LEMMA 25.4. *Let (e_i) be a basis for $H^*(X)$, and let (f_i) be the basis of $H^*(X)$ that is dual relative to cup-product, so that $e_i \cup f_j = \delta_{ij}e^{2d}$ (δ_{ij} =Kronecker delta). For any regular map $\varphi\colon\ X \to X$,*

$$cl_{X\times X}(\Gamma_\varphi) = \sum\ \varphi^*(e_i) \otimes f_i$$

PROOF. As the f_i form a basis for $H^*(X)$ as a $\mathbb{Q}_\ell$-vector space, they also form a basis for $H^*(X \times X) = H^*(X) \otimes_{\mathbb{Q}_\ell} H^*(X)$ as an $H^*(X)$-module. Therefore,

$$cl_{X\times X}(\Gamma_\varphi) = \sum\ a_i \otimes f_i$$

for unique elements $a_i \in H^*(X)$. According to Lemma 25.3,

$$\varphi^*(e_j) = p_*((\sum_i a_i \otimes f_i) \cup (1 \otimes e_j)) = p_*(a_j \otimes e^{2d}) = a_j.$$

□

PROOF OF THE THEOREM. Let e_i^r be a basis for H^r, and let f_i^{2d-r} be the dual basis for H^{2d-r}. Then

$$\begin{aligned} cl(\Gamma_\varphi) &= \sum_{r,i} \varphi^*(e_i^r) \otimes f_i^{2d-r}, \text{and}\\ cl(\Delta) &= \sum_{r,i} e_i^r \otimes f_i^{2d-r} = \sum_{r,i} (-1)^{r(2d-r)} f_i^{2d-r} \otimes e_i^r = \sum_{r,i} (-1)^r f_i^{2d-r} \otimes e_i^r. \end{aligned}$$

On taking the products of these two expressions we find that

$$cl_{X\times X}(\Gamma_\varphi \cdot \Delta) = \sum_{r,i} (-1)^r\ \varphi^*(e_i^r) f_i^{2d-r} \otimes e_{2d} = \sum_r (-1)^r Tr(\varphi^*|H^r)(e^{2d} \otimes e^{2d}).$$

Now apply $\eta_{X\times X}$ both sides. □

REMARK 25.5. Although in the above discussion, we have identified $\mathbb{Q}_\ell$ with $\mathbb{Q}_\ell(1)$, the above theorem holds as stated without this identification. The point is that

$$H^r(X, \mathbb{Q}_\ell(s)) = H^r(X, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(s)$$

and φ acts through $H^r(X, \mathbb{Q}_\ell)$. Tensoring with the one-dimensional $\mathbb{Q}_\ell$-vector space $\mathbb{Q}_\ell(s)$ doesn't change the trace.

It will be useful to have a criterion for when $(\Gamma_\varphi \cdot \Delta)_P = 1$ for a fixed point P of φ.

Let Y and Z be closed subvarieties of a nonsingular variety X, and suppose that the point P is an irreducible component of $Y \cap Z$. Then $(Y \cdot Z)_P = 1$ if

(a) Y and Z are nonsingular at P,
(b) $Tgt_P(Y) \cap Tgt_P(Z) = 0$, and
(c) $\dim Y + \dim Z = \dim X$.

Condition (b) means that Y and Z cross transversally at P, and condition (c) mean that Y and Z intersect properly at P (i.e., $\mathrm{codim} P = \mathrm{codim} Y + \mathrm{codim} Z$).

LEMMA 25.6. *Let $\varphi\colon\ X \to X$ be a regular map, and let $P \in X$ be a fixed point of φ. Then $(\Gamma_\varphi \cdot \Delta)_P = 1$ if 1 is not an eigenvalue of $(d\varphi)_P\colon\ Tgt_P(X) \to Tgt_P(X)$.*

PROOF. We apply the preceding remark to the point (P, P) on $\Gamma_\varphi \cap \Delta$. Because Γ_φ and Δ are both isomorphic to X, conditions (a) and (c) hold. Because $Tgt_{(P,P}(\Gamma_\varphi)$ is the graph of $(d\varphi)_P\colon\ Tgt_P(X) \to Tgt_P(X)$ and $Tgt_{(P,P}(\Delta)$ is the graph of the identity map $Tgt_P(X) \to Tgt_P(X)$, condition (b) holds if and only if 1 is not an eigenvalue of $(d\varphi)_P$. $\square$

Part II: Proof of the Weil Conjectures.

Throughout, $\mathbb{F}_q$ is a field with $q = p^a$ elements and $\mathbb{F}$ is an algebraic closure of $\mathbb{F}_q$. For a variety X_0 over $\mathbb{F}_q$, X denotes X_0 regarded as a variety over $\mathbb{F}$. We shall now assume that varieties are absolutely irreducible (unless stated otherwise) — for a nonsingular variety X_0 over F_q, this means that X is connected. Unless stated otherwise, ℓ is a prime $\neq p$.

26. The Weil Conjectures

Let X_0 be a nonsingular projective variety over $\mathbb{F}_q$. For each m, we let N_m be the number of points on X_0 with coordinates in $\mathbb{F}_{q^m}$, and we define the zeta function of X_0 to be

$$\begin{aligned} Z(X_0,t) &= \exp(\sum_{m\geq 1} N_m \frac{t^m}{m}) \\ &= 1+\sum_{m\geq 1} N_m\frac{t^m}{m}+\frac{1}{2!}(\sum_{m\geq 1} N_m\frac{t^m}{m})^2+\cdots \end{aligned}$$

It is a formal power series with coefficients in $\mathbb{Q}$, i.e., $Z(X_0,t)\in\mathbb{Q}[[t]]$. Note that

$$\frac{d}{dt}\log Z(X_0,t)=\sum_{m\geq 1} N_m t^{m-1}.$$

Thus $\frac{d}{dt}\log Z(X_0,t)$ is essentially the generating function for the sequence $N_1,N_2,N_3,\ldots$.

Apart from minor changes of notation, the following is quoted verbatim from Weil, *Numbers of solutions of equations in finite fields*, Bull. AMS 55 (1949), 497–508.

> ... This, and other examples which we cannot discuss here, seem to lend some support to the following conjectural statements, which are known to be true for curves, but which I have not so far been able to prove for varieties of higher dimension.
>
> Let X_0 be a variety[32] without singular points, of dimension d, defined over a finite field $\mathbb{F}_q$ with q elements. Let N_m be the number of rational points on X_0 over the extension $\mathbb{F}_{q^m}$ of $\mathbb{F}_q$ degree m. Then we have
>
> $$\sum_1^\infty N_m t^{m-1}=\frac{d}{dt}\log Z(t),$$
>
> where $Z(t)$ is a rational function of t, satisfying a functional equation
>
> $$Z(\frac{1}{q^d t})=\pm q^{d\chi/2}\cdot t^\chi\cdot Z(t)$$
>
> with χ equal to the Euler-Poincaré characteristic of X (intersection number of the diagonal with itself in the product $X\times X$).
>
> Furthermore, we have:
>
> $$Z(t)=\frac{P_1(t)P_3(t)\cdots P_{2d-1}(t)}{P_0(t)P_2(t)\quad\cdots\quad P_{2d}(t)}$$
>
> with $P_0(t)=1-t$, $P_{2d}(t)=1-q^dt$, and, for $1\leq r\leq 2d-1$:
>
> $$P_r(t)=\prod_{i=1}^{\beta_r}(1-\alpha_{r,i}t)$$
>
> where the $\alpha_{r,i}$ are algebraic integers of absolute value $q^{r/2}$.
>
> Finally, let us call the degrees β_r of the polynomials $P_r(t)$ the *Betti numbers* of the variety X; the Euler-Poincaré characteristic χ is then expressed

[32]In the paper, Weil seems to be using "variety" to mean "projective variety".

> by the usual formula $\chi = \sum_r (-1)^r \beta_r$. The evidence at hand seems to suggest that, if Y is a variety without singular points, defined over a field K of algebraic numbers, the Betti numbers of the varieties $Y(\mathfrak{p})$, derived from Y by reduction modulo a prime ideal $\mathfrak{p}$ in K, are equal to the Betti numbers of Y (considered as a variety over the complex numbers) in the sense of combinatorial topology, for all except at most a finite number of prime ideals $\mathfrak{p}$.

The remainder of the course is devoted to the proof of these remarkable conjectures. The condition $|\alpha_{r,i}| = q^{r/2}$ is usually called the *Riemann hypothesis.*

Although Weil doesn't explicitly say so in his 1949 paper, it is clear that the form of his conjectures was suggested in part by the formalism of algebraic topology (see Weil, Œuvres, I, p568). In particular, the functional equation is suggested by Poincaré duality, and the form of zeta function as a rational function is suggested by the Lefschetz fixed-point formula.

Remark 26.1. Let Y be a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$. The residue field at a closed point y of Y is finite — let $\mathbb{N}(y)$ be its order. The *zeta function* of Y is defined to be

$$\zeta(Y,s) = \prod_y \frac{1}{1-\mathbb{N}(y)^{-s}}$$

where y runs over the closed points of Y. The product converges, and defines $\zeta(Y,s)$ as a holomorphic function, for $\Re(s) > \dim Y$. For example, if $Y = \operatorname{Spec}\mathbb{Z}$, then $\zeta(Y,s)$ is the original zeta function of Riemann. A variety X_0 over $\mathbb{F}_q$ can be regarded as a scheme of finite type over $\operatorname{Spec}\mathbb{Z}$ by means of

$$X_0 \to \operatorname{Spec}\mathbb{F}_q \to \operatorname{Spec}\mathbb{Z}$$

(the second map is defined by $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{F}_q$), and we shall see in the next section (27.9.1) that

$$Z(X_0,t) = \prod_x \frac{1}{1-t^{\deg x}},$$

and so

$$\zeta(X_0,s) = Z(X_0,q^{-s}).$$

Therefore, the Riemann hypothesis for X_0 says that $\zeta(X_0,s)$ has its poles on the lines $\Re(s) = 0,1,2,\ldots,\dim X$ and its zeros on the lines $\Re(s) = \frac{1}{2},\frac{3}{2},\ldots,\frac{\dim X-1}{2}$. The analogy with the original Riemann hypothesis is evident.

27. Proof of the Weil Conjectures, except for the Riemann Hypothesis

The Frobenius map. Let A_0 be an affine $\mathbb{F}_q$-algebra. Then $a \mapsto a^q$ is a homomorphism $f_0\colon\ A_0 \to A_0$ of $\mathbb{F}_q$-algebras. By extension of scalars, we get a homomorphism $f\colon\ A \to A$ of $\mathbb{F}$-algebras, where $A = A_0 \otimes_{\mathbb{F}_q} \mathbb{F}$. The corresponding regular map $F\colon\ \operatorname{Specm} A \to \operatorname{Specm} A$ is called the *Frobenius map.*

For a variety X_0 over $\mathbb{F}_q$, the *Frobenius map* $F\colon\ X \to X$ is the unique regular map such that, for every open affine $U_0 \subset X_0$, $F(U) \subset U$ and $F|U$ is the Frobenius map on U.

One checks easily:

(a) the Frobenius map $F\colon\ \mathbb{A}^n \to \mathbb{A}^n$ is $(t_1, \ldots, t_n) \mapsto (t_1^q, \ldots, t_n^q)$;
(b) the Frobenius map $F\colon\ \mathbb{P}^n \to \mathbb{P}^n$ is $(t_0\colon\ \ldots\colon\ t_n) \mapsto (t_0^q\colon\ \ldots\colon\ t_n^q)$;
(c) for any regular map $\varphi_0\colon\ Y_0 \to X_0$ over $\mathbb{F}_q$, the diagram

$$\begin{tikzcd} Y \arrow[r, "\varphi"] \arrow[d, "F"] & X \arrow[d, "F"] \\ Y \arrow[r, "\varphi"] & X \end{tikzcd}$$

commutes.

On combining these statements, we see that the Frobenius map acts on any subvariety of $\mathbb{A}^n$ as $(t_1, \ldots, t_n) \mapsto (t_1^q, \ldots, t_n^q)$, and on any subvariety of $\mathbb{P}^n$ as $(t_0\colon\ \ldots\colon\ t_n) \mapsto (t_0^q\colon\ \ldots\colon\ t_n^q)$.

Lemma 27.1. *The Frobenius map $F\colon\ X \to X$ has degree $q^{\dim X}$ (in fact, it is finite of this degee).*

Proof. On $\mathbb{A}^n$, F corresponds to the homomorphism of $\mathbb{F}$-algebras

$$T_i \mapsto T_i^q\colon \mathbb{F}[T_1, \ldots, T_n] \to \mathbb{F}[T_1, \ldots, T_n].$$

The image of this homomorphism is $\mathbb{F}[T_1^q, \ldots, T_n^q]$, and $\mathbb{F}[T_1, \ldots, T_n]$ is free of rank q^n over $\mathbb{F}[T_1^q, \ldots, T_n^q]$ (with basis the elements $T_1^{i_1} \cdots T_n^{i_n}$, $0 \le i_j \le q-1$). Similarly, $\mathbb{F}(T_1, \ldots, T_n)$ has degree q^n over its subfield $\mathbb{F}(T_1^q, \ldots, T_n^q)$. This shows that $F\colon\ \mathbb{A}^n \to \mathbb{A}^n$ is finite of degree q^n.

In the general case, we choose a transcendence basis $T_1, \ldots, T_n$ for the function field $\mathbb{F}_q(X_0)$ of X_0. Let $f\colon\ \mathbb{F}(X) \to \mathbb{F}(X)$ be the homomorphism defined by $F\colon\ X \to X$. Then $f\mathbb{F}(X) \cap \mathbb{F}(T_1, \ldots) = f\mathbb{F}(T_1, \ldots)$ and $\mathbb{F}(X) = f\mathbb{F}(X) \cdot \mathbb{F}(T_1, \ldots)$ and so

$$[\mathbb{F}(X)\colon f\mathbb{F}(X)] = [\mathbb{F}(T_1, \ldots)\colon f\mathbb{F}(T_1, \ldots)] = q^n,$$

which shows that F has degree q^n. $\square$

Lemma 27.2. *The fixed points of F on X are the points of X_0 with coordinates in $\mathbb{F}_q$. Each occurs with multiplicity 1 in $(\Gamma_F \cdot \Delta)$.*

Proof. An element a of $\mathbb{F}$ lies in $\mathbb{F}_q$ if and only if $a^q = a$, and so it is clear from the description of F in terms of the coordinates of points that the $X^F = X(\mathbb{F}_q)$.

I claim that $(dF)_P = 0$ at any fixed point P of F, and so F satisfies the conditions of Lemma 25.4. In proving this, we can replace X_0 with an affine neighbourhood U_0

of P, say $U_0 = \operatorname{Specm} A_0$, $A_0 = \mathbb{F}_q[t_1, \ldots, t_n] = \mathbb{F}_q[T_1, \ldots, T_n]/\mathfrak{a}$. Then $t_i \circ F = t_i^q$, and so $(dt_i)_P \circ (dF)_P = (dt_i^q)_P = qt_i^{q-1}(dt_i)_P = 0$, as claimed. □

An expression of N_m as a trace.

PROPOSITION 27.3. *Let X_0 be a complete nonsingular variety over $\mathbb{F}_q$. For any m,*

$$N_m = \sum_r (-1)^r \operatorname{Tr}(F^m | H^r(X, \mathbb{Q}_\ell)).$$

PROOF. From the Lemma 27.2, we see that $(\Gamma_F \cdot \Delta)$ is the number N_1 of points of X_0 with coordinates in $\mathbb{F}_q$, which, according to the Lefschetz Fixed Point Formula (25.1), equals $\sum_r (-1)^r \operatorname{Tr}(F|H^r(X, \mathbb{Q}_\ell))$. To obtain the general case, note that F^m is the Frobenius map of X relative to $X_{0,\mathbb{F}_{q^m}}$, and so $X^{F^m} = X(\mathbb{F}_{q^m})$. □

REMARK 27.4. (a) Let X_0 be a complete nonsingular variety over $\mathbb{F}_q$. Then $F\colon X \to X$ defines maps $F^*\colon H^r(X, \mathbb{Q}_\ell) \to H^r(X, \mathbb{Q}_\ell)$ and (see §24) maps $F_*\colon H^r(X, \mathbb{Q}_\ell) \to H^r(X, \mathbb{Q}_\ell)$. Because F is finite of degree $q^{\dim X}$, the composite $F_* \circ F^* = q^d$ (24.2f).

(b) Let $\varphi\colon Y_0 \to X_0$ be a regular map of complete nonsingular varieties over $\mathbb{F}_q$. Then

$$F^* \varphi_* = q^{\dim X - \dim Y} \varphi_* F^*.$$

To prove this, we apply F_* to each side. On the left we get

$$F_* F^* \varphi_* = q^{\dim X} \varphi_*$$

and on the right we get

$$q^{\dim X - \dim Y} F_* \varphi_* F^* = q^{\dim X - \dim Y} \varphi_* F_* F^* = q^{\dim X} \varphi_*.$$

Note that this argument also works on the cohomology groups with coefficients in $\mathbb{Z}/\ell^n\mathbb{Z}$, $\ell \neq p$.

Rationality. We need an elementary lemma.

LEMMA 27.5. *Define the characteristic polynomial of an endomorphism $\varphi\colon V \to V$ of a vector space over a field k to be*

$$P_\varphi(t) = \det(1 - \varphi t | V).$$

If $P_\varphi(t) = \prod (1 - c_i t)$, then $\operatorname{Tr}(\varphi^m | V) = \sum c_i^m$. Therefore

$$\log \frac{1}{P_\varphi(t)} = \sum_{m=1}^{\infty} \operatorname{Tr}(\varphi^m | V) \frac{t^m}{m}$$

(equality of elements of $k[[t]]$).

PROOF. After possibly extending k, we may assume that there exists a basis relative to which the matrix of φ is upper triangular $\begin{pmatrix} c_1 & & * \\ & \ddots & \\ 0 & & c_d \end{pmatrix}$. Relative to this basis, φ^m has matrix $\begin{pmatrix} c_1^m & & * \\ & \ddots & \\ 0 & & c_d^m \end{pmatrix}$, from which the statement is obvious.

On summing both sides of

$$\log\frac{1}{1-c_i t}=\sum_{m=1}^{\infty} c_i^m \frac{t^m}{m}.$$

over i, we obtain the second formula. □

THEOREM 27.6. *For any complete nonsingular variety X_0 of dimension d over $\mathbb{F}_q$,*

$$Z(X_0,t)=\frac{P_1(X_0,t)\cdots P_{2d-1}(X_0,t)}{P_0(X_0,t)\quad\cdots\quad P_{2d}(X_0,t)}$$

where

$$P_r(X,t)=\det(1-Ft|H^r(X,\mathbb{Q}_\ell)).$$

PROOF.

$$\begin{aligned}
Z(X_0,t) &= \exp(\textstyle\sum_m N_m \frac{t^m}{m}) && \text{(definition)}\\
&= \exp(\textstyle\sum_m (\sum_{r=0}^{2d}(-1)^r \operatorname{Tr}(F^m|H^r))\frac{t^m}{m}) && (27.3)\\
&= \textstyle\prod_{r=0}^{2d}\left(\exp(\sum_m \operatorname{Tr}(F^m|H^r)\frac{t^m}{m}\right)^{(-1)^r} && \text{(move the inner sum outside)}\\
&= \textstyle\prod_{r=0}^{2d} P_r(t)^{(-1)^{r+1}} && (27.5).
\end{aligned}$$

□

REMARK 27.7. Because F acts as 1 on $H^0(X,\mathbb{Q}_\ell)$, $P_0(X_0,t)=1-t$, and because F acts as q^d on $H^{2d}(X,\mathbb{Q}_\ell)$, $P_{2d}(X_0,t)=1-q^d t$. In general, $P_r(X_0,t)=1+\cdots\in\mathbb{Q}_\ell[t]$.

COROLLARY 27.8. *The power series $Z(X_0,t)$ is a rational function with coefficients in $\mathbb{Q}$, i.e., it lies in $\mathbb{Q}(t)$.*

PROOF. Note that

$$Z(X_0,t)\overset{\text{df}}{=}\exp(\sum_{m\geq 1} N_m t^m/m)$$

is a power series with coefficients in $\mathbb{Q}$. Theorem 27.6 shows that it is a rational function with coefficients in $\mathbb{Q}_\ell$, which, according to the next lemma, implies that it is a rational function with coefficients in $\mathbb{Q}$. □

LEMMA 27.9. *Let $k\subset K$ be fields, and let $f(t)\in k[[t]]$; if $f(t)\in K(t)$, then $f(t)\in k(t)$.*

PROOF. From Bourbaki, Algèbre, IV.5, Exercise 3:

Let $f(t)=\sum_{n=0}^{\infty} a_i t^i$ be a formal power series over a field k.

(a) The power series $f(t)$ lies in $k(t)$ if and only if there exists a finite sequence $\lambda_1,\ldots,\lambda_r$ of elements of k, not all zero, and an integer d such that, for all $n\geq d$,

$$\lambda_1 a_n+\lambda_2 a_{n+1}+\cdots+\lambda_r a_{n+r-1}=0.$$

(b) Let

$$H_n^{(k)}=\begin{vmatrix} a_n & a_{n+1} & \ldots & a_{n+k-1}\\ a_{n+1} & a_{n+2} & \ldots & a_{n+k}\\ \vdots & \vdots & & \vdots\\ a_{n+k-1} & a_{n+k} & \ldots & a_{n+2k-2}\end{vmatrix}$$

(Hankel determinant). If, for some d and all $j \geq 0$, $H_{d+j}^{(r+1)} = 0$ and $H_{d+j}^{(r)} \neq 0$, then $f(t) \in k(t)$.

(c) Show that

$$H_n^{(k)} H_{n+2}^{(k)} - H_n^{(k+1)} H_{n+2}^{(k-1)}$$

is a power of $H_{n+1}^{(k)}$. Deduce that if $H_{m+j}^{(k+1)} = 0$ for $0 \leq j \leq s-1$, then the s determinants $H_{m+j}^{(k)}$, $1 \leq j \leq s$, are all zero or all nonzero.

(d) Deduce from (b) and (c) that $f(X) \in k(t)$ if and only if there exist two integers d and r such that $H_{d+j}^{(r+1)} = 0$ for all integers $j \geq 1$.

Obviously, if the condition in (d) is satisfied in K, then it is satisfied in k. □

The corollary doesn't imply that the polynomials $P_r(X_0, t)$ have rational coefficients (much less that they are independent of ℓ). It says that, once any common factors have been removed, the numerator and denominator of the expression in (27.6) will be polynomials with coefficients in $\mathbb{Q}$, and will be independent of ℓ.

Integrality. Let x be a closed point of X_0. The residue field $\kappa(x) \overset{\text{df}}{=} \mathcal{O}_{X_0,x}/\mathfrak{m}_x$ is a finite extension of $\mathbb{F}_q$; we set $\deg x = [\kappa(x)\colon \mathbb{F}_q]$.

A point of X_0 with coordinates in $\mathbb{F}_{q^m}$ is a map $\operatorname{Specm} \mathbb{F}_{q^m} \to X_0$. To give such a map with image x is the same as to give an $\mathbb{F}_q$-homomorphism $\kappa(x) \to \mathbb{F}_{q^m}$. The contribution $N_m(x)$ of x to N_m is the number of such homomorphisms. From the theory of finite fields (FT §4.6), we see that

$$N_m(x) = \begin{cases} \deg x & \text{if } \deg x | m \\ 0 & \text{otherwise.} \end{cases}$$

Recall, that $\log(\frac{1}{1-s}) = \sum_{m \geq 1} \frac{s^n}{n}$, and so

$$\log \frac{1}{1 - t^{\deg x}} = \sum_{n \geq 1} \frac{t^{n \cdot \deg x}}{n}.$$

The coefficient of t^m/m in this sum is 0 unless $\deg x | m$, in which case it is $\deg x$. Therefore

$$\log \frac{1}{1 - t^{\deg x}} = \sum_m N_m(x) \frac{t^m}{m}.$$

On summing over all the closed points of X_0 and taking exponentials, we find that

$$Z(X_0, t) = \prod_{x \in X_0} \frac{1}{1 - t^{\deg x}} \qquad (27.9.1).$$

Hence

$$Z(X_0, t) \in 1 + t\mathbb{Z}[[t]].$$

In the next lemma, ℓ is any prime number (e.g., p).

Lemma 27.10. *Let* $f(t) = g(t)/h(t)$ *where*

$$\begin{aligned} f(t) &\in 1 + t \cdot \mathbb{Z}_\ell[[t]] \\ g(t), h(t) &\in 1 + t \cdot \mathbb{Q}_\ell[t]. \end{aligned}$$

If g and h are relatively prime, then they have coefficients in $\mathbb{Z}_\ell$.

Proof. We have to show that the coefficients of g and h have ℓ-adic absolute values ≤ 1. After possibly replacing $\mathbb{Q}_\ell$ with a finite extension field, we may assume $h(t)$ splits, say $h(t) = \prod(1 - c_i t)$. If $|c_i|_\ell > 1$, then $|c_i^{-1}|_\ell < 1$, and the power series $f(c_i^{-1})$ converges. But then

$$f(t) \cdot h(t) = g(t) \Rightarrow f(c_i^{-1}) \cdot h(c_i^{-1}) = g(c_i^{-1}).$$

Since $h(c_i^{-1}) = 0$ but $g(c_i^{-1}) \neq 0$, this is impossible. Therefore $|c_i|_\ell < 1$. As this is true for all i, $h(t) \in \mathbb{Z}_\ell[t]$.

Because $f(t)^{-1} \in 1 + t \cdot \mathbb{Z}_\ell[[t]]$, the same argument applied to $f(t)^{-1}$ shows that $g(t) \in \mathbb{Z}_\ell[t]$. □

Proposition 27.11. *Let*

$$Z(X_0, t) = \frac{P(t)}{Q(t)}$$

where $P(t)$, $Q(t) \in \mathbb{Q}[t]$ are relatively prime (P and Q exist by 27.8). When $P(X_0, t)$ and $Q(X_0, t)$ are chosen to have constant terms 1, they have coefficients in $\mathbb{Z}$.

Proof. The hypotheses of the preceding lemma hold for all primes ℓ (including p). Therefore, the coefficients of $P(X_0, t)$ and $Q(X_0, t)$ are ℓ-adic integers for all ℓ, which implies that they are integers. □

Functional equation.

Theorem 27.12. *For any complete nonsingular variety X_0 over $\mathbb{F}_q$,*

$$Z(X_0, 1/q^d t) = \pm q^{d\chi/2} t^\chi Z(X_0, t),$$

where $\chi = \sum (-1)^r \beta_r = (\Delta \cdot \Delta)$.

Proof. Consider the pairing

$$H^{2d-r}(X, \mathbb{Q}_\ell) \times H^r(X, \mathbb{Q}_\ell(d)) \to H^{2d}(X, \mathbb{Q}_\ell) \xrightarrow{\eta_X} \mathbb{Q}_\ell, \quad d = \dim X.$$

By definition of F_*,

$$\eta_X(F_*(x) \cup x') = \eta_X(x \cup F^*(x')) \quad x \in H^{2d-r}(X), \quad x' \in H^r(X).$$

Therefore, the eigenvalues of F^* acting on $H^r(X)$ are the same as the eigenvalues of F_* acting on $H^{2d-r}(X)$. But $F^* = q^d/F_*$ (see 27.4a), and so if $\alpha_1, ..., \alpha_s$ are the eigenvalues of F^* acting on $H^r(X, \mathbb{Q}_\ell)$, then $q^d/\alpha_1, \ldots, q^d/\alpha_s$ are the eigenvalues of F^* acting on $H^{2d-r}(X, \mathbb{Q}_\ell)$. This implies the statement. □

Remark 27.13. The sign is $+$ if d is odd or $q^{d/2}$ occurs an even number of times as an eigenvalue of F acting on $H^d(X, \mathbb{Q}_\ell)$, and is $-$ otherwise.

Summary.

27.14. Let X_0 be a complete nonsingular variety over $\mathbb{F}_q$.

(a) Then $Z(X_0, t) \in \mathbb{Q}(t)$, and satisfies the functional equation

$$Z(\frac{1}{q^d t}) = \pm q^{d\chi/2} \cdot t^\chi \cdot Z(t)$$

with $\chi = (\Delta \cdot \Delta)$.

(b) Furthermore, for each $\ell \neq p$, we have an expression,

$$Z(t) = \frac{P_{1,\ell}(t)P_{3,\ell}(t)\cdots P_{2d-1,\ell}(t)}{P_{0,\ell}(t)P_{2,\ell}(t) \quad \cdots \quad P_{2d,\ell}(t)}$$

with $P_{0,\ell}(t) = 1 - t$, $P_{2d,\ell}(t) = 1 - q^d t$.

(c) If, for a fixed ℓ, the $P_{r,\ell}$ are relatively prime in pairs, then, for $1 \leq r \leq 2d-1$,

$$P_{r,\ell}(t) = 1 + \sum a_{i,r} t^i \in \mathbb{Z}[t].$$

(d) If, for all $\ell \neq p$, the inverse roots of $P_{r,\ell}$ have absolute value $q^{r/2}$, then the $P_{r,\ell}(t) \in 1 + t\mathbb{Z}[t]$ and are independent of ℓ.

(e) Let $\beta_r = \deg P_r(t)$. Then $\chi = \sum_r (-1)^r \beta_r$, and if X_0 lifts to a variety X_1 in characteristic zero, then the β_r are the Betti numbers of X_1 considered as a variety over the complex numbers.

Statement (a) was proved in (27.8), (27.12), and (b) in (27.6). If the $P_{r,\ell}(t)$ are relatively prime in pairs, then (27.11) shows that

$$\prod_{r \text{ odd}} P_{r,\ell}(t) \in 1 + t\mathbb{Z}[t], \qquad \prod_{r \text{ even}} P_{r,\ell}(t) \in 1 + t\mathbb{Z}[t].$$

Therefore the inverse roots of $P_{r,\ell}(t)$ are algebraic integers, which implies that $P_{r,\ell}(t) \in 1 + t\mathbb{Z}[t]$, whence (c). The hypothesis of (d) implies that of (c), and so the $P_{r,\ell}(t)$ have integer coefficients; moreover, $P_{r,\ell}(t)$ is characterized independently of ℓ as the factor of the numerator or denominator of $Z(X_0, t)$ whose roots have absolute value $q^{r/2}$. Finally, (e) follows from 25.1 and 20.5.

The statement 27.15 below implies that the hypotheses of (c) and (d) hold, and it completes the proof of the Weil conjectures.

An element α of some field containing $\mathbb{Q}$ (e.g., $\mathbb{Q}_\ell$) will be called an *algebraic number* if it is the root of a polynomial $P(T) \in \mathbb{Q}[T]$. We can choose $P(T)$ to be monic and irreducible; then the roots of $P(T)$ in $\mathbb{C}$ will be called the *complex conjugates* of α. Equivalently, α is algebraic if it generates a finite extension $\mathbb{Q}[\alpha]$ of $\mathbb{Q}$; the complex conjugates of α are its images under the various homomorphisms $\mathbb{Q}[\alpha] \to \mathbb{C}$.

THEOREM 27.15. *Let X_0 be a nonsingular projective variety over $\mathbb{F}_q$. Then the eigenvalues of F acting on $H^r(X, \mathbb{Q}_\ell)$ are algebraic numbers, all of whose complex conjugates have absolute value $q^{r/2}$.*

It is highly unusual for an algebraic number to have all its complex conjugates with the same absolute value. For example, $1 + \sqrt{2}$ doesn't have this property.

The rest of the course is devoted to Deligne's proof of Theorem 27.15.

The notes of this part of the course are based on Deligne's original article:

Deligne, Pierre, La conjecture de Weil. I, Inst. Hautes Études Sci. Publ. Math. No. 43 (1974), 273–307; MR 49 #5013

and

Katz, N., Deligne's Proof of the Weil-Riemann Conjectures. Handwritten notes (by S. Bloch) of a course at Princeton University, 1973–74.

28. Preliminary Reductions

We shall show that it suffices to prove Theorem 27.15 for the middle cohomology groups of varieties of even dimension, and even for those groups, that it suffices to prove an approximate result.

Lemma 28.1. *It suffices to prove Theorem 27.15 after $\mathbb{F}_q$ has been replaced by $\mathbb{F}_{q^m}$.*

Proof. The Frobenius map $F_m\colon\ X \to X$ defined relative to the field $\mathbb{F}_{q^m}$ is F^m. Therefore, if $\alpha_1, \alpha_2, \ldots$ are the eigenvalues of F on $H^r(X, \mathbb{Q}_\ell)$, then $\alpha_1^m, \alpha_2^m, \ldots$ are the eigenvalues of F_m on $H^r(X, \mathbb{Q}_\ell)$. If α^m satisfies the condition of 27.15 relative to q^m, then α satisfies the condition relative to q. □

Thus, Theorem 27.15 is really a statement about $X/\mathbb{F}$: if it is true for one model of X over a finite field, then it is true for all.

Exercise 28.2. Let X_0 be a cubic surface over $\mathbb{F}_q$. Use (28.1) to prove the Riemann hypothesis for X_0. (Hint: It is known that X is a rational surface. Hence its Albanese variety is zero, and so $H^1(X, \mathbb{Q}_\ell) = 0$. Moreover, $H^2(X, \mathbb{Q}_\ell(1))$ is generated by the classes of algebraic cycles on X; in fact, it has as basis the classes of any 6 skew lines on X together with any nonsingular hyperplane intersection.)

Proposition 28.3. *Assume that for all nonsingular projective varieties X_0 of even dimension d over $\mathbb{F}_q$, every eigenvalue α of F on $H^d(X, \mathbb{Q}_\ell)$ is an algebraic number such that*

$$q^{\frac{d}{2}-\frac{1}{2}} < |\alpha'| < q^{\frac{d}{2}+\frac{1}{2}}$$

for all complex conjugates α' of α. Then Theorem 27.15 holds for all nonsingular projective varieties.

Proof. Let X_0 be a smooth projective variety of dimension d (not necessarily even) over $\mathbb{F}_q$, and let α be an eigenvalue of F on $H^d(X, \mathbb{Q}_\ell)$. The Künneth formula shows that α^m occurs among the eigenvalues of F acting on $H^{dm}(X^m, \mathbb{Q}_\ell)$. The statement in the lemma applied to an even power of X_0 shows that

$$q^{\frac{md}{2}-\frac{1}{2}} < |\alpha'|^m < q^{\frac{md}{2}+\frac{1}{2}}.$$

On taking the mth root, and let m tend to ∞ over even integers, we find that

$$|\alpha'| = q^{\frac{d}{2}}.$$

We now prove (27.15) by induction on the dimension of X_0 (under the assumption of the proposition). For $\dim X_0 = 0$, it is obvious, and for $d = 1$ only case $r = 1$ isn't obvious, and this we have just proved. Thus we may assume that $d \geq 2$.

Recall from the proof of (27.12), that the Poincaré duality theorem implies that if α is an eigenvalue of F on $H^r(X, \mathbb{Q}_\ell)$, then q^d/α is an eigenvalue of F on $H^{2d-r}(X, \mathbb{Q}_\ell)$. Thus it suffices to prove the theorem for $r > d$. Bertini's Theorem (Hartshorne, II.8.18) shows that there is a hyperplane H in $\mathbb{P}^m$ such that $Z \stackrel{\text{df}}{=} H \cap X$ is a nonsingular variety. Lemma 28.1 allows us to assume that H (and hence Z) is defined over $\mathbb{F}_q$. Then the Gysin sequence (16.2) reads

$$\cdots \to H^{r-2}(Z, \mathbb{Q}_\ell(-1)) \to H^r(X, \mathbb{Q}_\ell) \to H^r(X \setminus Z, \mathbb{Q}_\ell) \to \cdots.$$

Because $X \setminus Z$ is affine, $H^r(X \setminus Z, \mathbb{Q}_\ell) = 0$ for $r > d$ (weak Lefschetz theorem, I.12.5). Thus the Gysin map

$$i_*\colon H^{r-2}(Z, \mathbb{Q}_\ell(-1)) \to H^r(X, \mathbb{Q}_\ell)$$

is surjective for $r > d$. By induction that the eigenvalues of F on $H^{r-2}(Z, \mathbb{Q}_\ell)$ are algebraic numbers whose conjugates have absolute value $q^{(r-2)/2}$. Since $F \circ i_* = q(i_* \circ F)$ (28.1), the eigenvalues of F acting on $H^r(X, \mathbb{Q}_\ell)$ are algebraic numbers whose conjugates have absolute value $q^{r/2}$. $\square$

The rest of the proof. In §29, we prove a Lefschetz formula for nonconstant sheaves on affine curves, and apply it in §30 to prove the "Main Lemma", which gives a criterion on a locally constant sheaf $\mathcal{E}$ on an open affine subset U_0 of $\mathbb{P}^1_{/\mathbb{F}_q}$ for the eigenvalues α of the Frobenius map on $H^1(\mathbb{P}^1, j_*\mathcal{E})$ to satisfy the inequalities

$$q^{\frac{n}{2}} < |\alpha| < q^{\frac{n}{2}+1}.$$

In §31, we study how to fibre a variety X with a pencil $(X_t)_{t \in \mathbb{P}^1}$ of hypersurface sections having especially good properties (it is a Lefschetz pencil). After blowing up X at $\cap X_t$, we obtain a variety X^* and a map $\pi\colon X^* \to \mathbb{P}^1$. In §32 we study the higher direct images of $\mathbb{Q}_\ell$ under π.

In §33, we combine these themes to complete the proof of the Weil conjectures.

29. The Lefschetz Fixed Point Formula for Nonconstant Sheaves.

We shall need a Lefschetz fixed point formula for noncomplete varieties, nonconstant sheaves, and for sheaves of modules over a finite ring. Each of these generalizations cause problems, which we now discuss.

Noncomplete varieties. Let U be an open subset of a complete nonsingular variety X over an algebraically closed field k, and assume that the complement Z of U in X is nonsingular. Let $\Lambda = \mathbb{Q}_\ell$. From the exact sequence (8.15)

$$0 \to j_!\Lambda \to \Lambda \to i_*\Lambda \to 0$$

we obtain a long exact sequence

$$\cdots \to H^r_c(U, \Lambda) \to H^r(X, \Lambda) \to H^r(Z, \Lambda) \to \cdots .$$

Let $\varphi\colon\ X \to X$ be a finite regular map. If φ preserves the decomposition $X = U \cup Z$, then it acts on the complex, and so

$$\sum_r (-1)^r \operatorname{Tr}(\varphi|H^r(X)) = \sum_r (-1)^r \operatorname{Tr}(\varphi|H^r_c(U)) + \sum_r (-1)^r \operatorname{Tr}(\varphi|H^r(Z)).$$

Let $-^\varphi$ denote the set of closed fixed points of φ. If

$$\#X^\varphi = \#U^\varphi + \#Z^\varphi,$$

then we obtain a fixed-point formula

$$\#U^\varphi = \sum (-1)^r \operatorname{Tr}(\varphi|H^r_c(U, \Lambda))$$

as the difference of the fixed-point formulas for X and Z. However, as the next example shows, this argument can be misleading.

Example 29.1. Consider the map

$$\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1, \quad (x_0\colon x_1) \mapsto (x_0 + x_1\colon x_1)$$

of (25.2). Let $U = \{(x_0\colon\ x_1) \mid x_1 \neq 0\} \cong \mathbb{A}^1$, and $Z = \{(1\colon\ 0)\} = \{\infty\}$. Then $H^r_c(U, \mathbb{Q}_\ell) = 0$ for $r \neq 2$, and φ acts on $H^2_c(U, \mathbb{Q}_\ell) \cong H^2(\mathbb{P}^1, \mathbb{Q}_\ell)$ as the identity map. Hence $\sum(-1)^r \operatorname{Tr}(\varphi|H^r_c(U, \mathbb{Q}_\ell)) = 1$ despite the fact that $\varphi|U$ has no fixed point (φ acts on U as $x \mapsto x + 1$) — the Lefschetz fixed point formula fails for U and $\varphi|U$.

The problem is that ∞ has multiplicity 1 as a fixed point of $\varphi|Z$ but multiplicity 2 as a fixed point fo φ. Therefore, when we count multiplicities, the equation

$$\#X^\varphi \neq \#U^\varphi + \#Z^\varphi.$$

Thus, we should only expect to have a Lefschetz fixed point formula for a noncomplete variety when the map extends to a map on a completion of the variety and has only simple fixed points on the complement of variety — in fact, the above argument does correctly show that we get a fixed-point formula (with constant coefficients) in this case.

Nonconstant sheaves. Let X be a complete variety over an algebraically closed field k, and let $\mathcal{E}$ be a locally constant sheaf of Λ-modules on X. A regular map $\varphi\colon\ X\to X$ defines a map $H^r(X,\mathcal{E})\to H^r(X,\varphi^*\mathcal{E})$ — this is a map between different vector spaces, and so its trace is not defined. In order to have a trace, we need also a homomorphism $\varphi_{\mathcal{E}}\colon\ \varphi^*\mathcal{E}\to\mathcal{E}$. Such a pair $(\varphi,\varphi_{\mathcal{E}})$ defines a maps

$$H^r(X,\mathcal{E})\to H^r(X,\varphi^*\mathcal{E})\to H^r(X,\mathcal{E})$$

whose composite we denote $(\varphi,\varphi_{\mathcal{E}})^*$. For each closed point x of X, $\varphi_{\mathcal{E}}$ defines a map on stalks:

$$\begin{array}{cccc}\varphi_x\colon & (\varphi^*\mathcal{E})_x & \to & \mathcal{E}_x\\ & \| & & \\ & \mathcal{E}_{\varphi(x)}. & & \end{array}$$

When x is a fixed point of φ, this becomes

$$\varphi_x\colon \mathcal{E}_x\to\mathcal{E}_x.$$

We may hope that, under suitable hypotheses, there is a formula,

$$\sum_{x\in X^\varphi}\operatorname{Tr}(\varphi_x|\mathcal{E}_x)=\sum(-1)^r\operatorname{Tr}((\varphi,\varphi')^*|H^r(X,\mathcal{E}))$$

where X^φ is the set of closed fixed points of φ.

When $\mathcal{E}$ is constant, say defined by a group E, then $\varphi^*\mathcal{E}$ is also the constant sheaf defined by E. Therefore, in this case the map $\varphi_{\mathcal{E}}$ is the identity. For example, if $\mathcal{E}=\mathbb{Q}_\ell$, the formula becomes

$$\#X^\varphi=\sum(-1)^r\operatorname{Tr}(\varphi|H^r(X,\mathbb{Q}_\ell))$$

where $\#X^\varphi$ is the number of fixed points not counting multiplicities.

Remark 29.2. Let X be a finite set, and regard X as a discrete topological space. To give a sheaf $\mathcal{E}$ of finite-dimensional $\mathbb{Q}$-vector spaces on X amounts to giving a family of finite-dimensional vector spaces $(\mathcal{E}_x)_{x\in X}$ indexed by the elements of X. A pair of maps $(\varphi,\varphi_{\mathcal{E}})$ as above is a map $\varphi\colon\ X\to X$ of sets and a family of maps $\varphi_x\colon\ \mathcal{E}_{\varphi(x)}\to\mathcal{E}_x$ indexed by the elements of X. The map $(\varphi,\varphi_{\mathcal{E}})^*\colon\ H^0(X,\mathcal{E})\to H^0(X,\mathcal{E})$ is the direct sum of the maps $\varphi_x\colon\ \mathcal{E}_{\varphi(x)}\to\mathcal{E}_x$. Clearly, φ_x does not contribute to $\operatorname{Tr}((\varphi,\varphi_{\mathcal{E}})^*|H^0(X,\mathcal{E}))$ unless $\varphi(x)=x$, in which case it contributes $\operatorname{Tr}(\varphi_x)$. Thus the formula is true in this case.

Coefficient ring finite. If Λ is not a field, then $H^r(X,\mathcal{E})$ may not be free a free Λ, in which case the trace of an endomorphism is not defined. Since this problem doesn't arise until the proof of the Theorem 29.4, we defer discussion of it.

Statement of the Theorem.

Lemma 29.3. *Let X_0 be a variety over $\mathbb{F}_q$, and let $\mathcal{E}_0$ be a sheaf on X_{0et}. Let $\mathcal{E}$ be the inverse image of $\mathcal{E}_0$ on X. Then there is a canonical homomorphism $F_{\mathcal{E}}\colon\ F^*\mathcal{E}\to\mathcal{E}$ of sheaves on X.*

Proof. (Sketch) We describe $F_{\mathcal{E}}$ only in the case that $\mathcal{E}_0$ is the sheaf defined by a variety $\pi_0\colon\ E_0 \to X_0$ over X_0. Then $\mathcal{E}$ is the sheaf defined by $\pi\colon\ E \to X$. Consider the diagram:

$$\begin{tikzcd} E \arrow[r, "F"] \arrow[d, shift right, "\pi"'] & E \arrow[d, shift right, "\pi"'] \\ X \arrow[r, "F"] \arrow[u, shift right, "s''"'] \arrow[ur, "s'"'] & X \arrow[u, shift right, "s"'] \end{tikzcd}$$

To give an element of $\Gamma(X,\mathcal{E})$ is to give a section $s\colon\ X \to E$ such that $\pi \circ s = \mathrm{id}$; to give an element of $\Gamma(X, F^*\mathcal{E})$ is to give a map $s'\colon\ X \to E$ such that $\pi \circ s' = F$. The map $\Gamma(X,\mathcal{E}) \to \Gamma(X, F^*\mathcal{E})$ sends $s \mapsto s' \stackrel{\mathrm{df}}{=} s \circ F$ and $F_{\mathcal{E}}\colon\ \Gamma(X, F^*\mathcal{E}) \to \Gamma(X,\mathcal{E})$ sends s' to the unique s'' such that $F \circ s'' = s'$. Since all constructible sheaves are representable by algebraic spaces, the argument applies to all such sheaves. $\square$

A nonsingular curve U_0 over $\mathbb{F}_q$ can be embedded (essentially uniquely) into a complete nonsingular curve X_0 over $\mathbb{F}_q$. The Frobenius map $F\colon\ X \to X$ preserves U and its fixed points in $X \setminus U$ have multiplicity 1. Therefore, the following theorem is at least plausible.

Theorem 29.4. *Let U_0 be a nonsingular curve over $\mathbb{F}_q$, and let $\mathcal{E}_0$ be a locally constant sheaf of $\mathbb{Q}_\ell$-vector spaces on U_0. Then*

$$\sum_{x \in U^F} \mathrm{Tr}(F_x|\mathcal{E}_x) = \sum (-1)^r \,\mathrm{Tr}(F|H^r_c(U,\mathcal{E})).$$

The sum at left is over the closed points of U fixed by F (which are in natural one-to-one correspondence with the elements of $U_0(\mathbb{F}_q)$), $\mathcal{E}_x$ is the stalk of $\mathcal{E}$ at x (regarded as a geometric point), and F_x is the map on stalks induced by $F_{\mathcal{E}}$. On the right, F is the map induced by $(F, F_{\mathcal{E}})$.

Example 29.5. Let $\pi\colon\ Y_0 \to U_0$ be a family of complete nonsingular curves over U_0, and let $\mathcal{E}_0 = R^1\pi_*\mathbb{Q}_\ell$. For $x \in U_0(\mathbb{F}_q)$, let $Y_x = \pi^{-1}(x)$. It is a curve over $\mathbb{F}_q$, and $\mathrm{Tr}(F_x|\mathcal{E}_x) = \mathrm{Tr}(F|H^1(Y_x,\mathbb{Q}_\ell))$.

Before explaining the proof of Theorem 29.4, we discuss some applications, and we re-interprete the theorem in terms of $\pi_1(U_0)$-modules.

The zeta function of a locally constant sheaf. Let U_0 be a nonsingular curve over $\mathbb{F}_q$, and let $\mathcal{E}_0$ be a locally constant sheaf of $\mathbb{Q}_\ell$-vector spaces on U_0. Define $Z(U_0,\mathcal{E}_0,t)$ by

$$\log Z(U_0,\mathcal{E}_0,t) = \sum_{m>0} \left(\sum_{x \in U^{F^m}} \mathrm{Tr}(F^m_x|\mathcal{E}_x) \right) \frac{t^m}{m}.$$

For example,

$$Z(U_0,\mathbb{Q}_\ell,t) = \exp\left(\sum_{m>0} N_m \frac{t^m}{m} \right) = Z(U_0,t).$$

The same argument as in §27 (see 27.9.1) shows that

$$Z(U_0,\mathcal{E}_0,t)=\prod_{x\in U_0}\frac{1}{\det(1-F_x t^{\deg x}|\mathcal{E}_x)}.$$

THEOREM 29.6. *With the above notations*

$$Z(U_0,\mathcal{E}_0,t)=\frac{\det(1-Ft|H^1_c(U,\mathcal{E}))}{\det(1-Ft|H^0_c(U,\mathcal{E}))\cdot\det(1-Ft|H^2_c(U,\mathcal{E}))}.$$

The deduction of this theorem from 29.4 is the same as the deduction of Theorem 27.6 from Theorem 25.1. In fact, Theorem 29.6 is often called the multiplicative form of the fixed-point formula.

REMARK 29.7. When U_0 is affine, $H^0_c(U,\mathcal{E})=0$ (no section of a locally constant sheaf on U has support on a complete subvariety of U), and so the equation becomes

$$Z(U_0,\mathcal{E}_0,t)=\frac{\det(1-Ft|H^1_c(U,\mathcal{E}))}{\det(1-Ft|H^2_c(U,\mathcal{E}))}.$$

The zeta function of an arbitrary variety. We define the zeta function $Z(X_0,t)$ of an arbitrary (possibly singular and not complete) variety over $\mathbb{F}_q$ as in the good[33] case:

$$Z(X_0,t)=\exp(\sum_{m\geq 1}N_m\frac{t^m}{m})=\prod_{x\in X_0}\frac{1}{1-t^{\deg x}}$$

where N_m is the number of points on X_0 with coordinates in $\mathbb{F}_{q^m}$ and the product is over the closed points of X_0.

THEOREM 29.8. *Let X_0 be a variety over $\mathbb{F}_q$. Then*

$$Z(X_0,t)=\frac{P_1(X_0,t)\cdots P_{2d-1}(X_0,t)}{P_0(X_0,t)\quad\cdots\quad P_{2d}(X_0,t)}$$

where

$$P_r(X,t)=\det(1-Ft|H^r_c(X,\mathbb{Q}_\ell)).$$

For the proof that Theorem 29.8 follows from Theorem 29.4, see EC pp 289–298.

The Theorem shows that $Z(X_0,t)\in\mathbb{Q}(t)$, but at present it is not known whether the $P_r(X,t)$ are independent of ℓ, or even whether they have coefficients in $\mathbb{Q}$.

Re-interpretation in terms of $\pi_1(U)$-modules; Frobenius elements. We now assume that U_0 is an affine nonsingular curve over $\mathbb{F}_q$, and we let K be the function field $\mathbb{F}_q(U_0)$ of U. Fix an algebraic closure Ω of K. Then $U_0=\operatorname{Specm}A$ with $A=\Gamma(U_0,\mathcal{O}_{U_0})\subset K$ a Dedekind domain, and we identify $\pi_1(U_0)$ with $\operatorname{Gal}(K^{\text{un}}/K)$ where K^{un} is the union of the subfields of Ω unramified over K at all the primes of A. Note that $K\cdot\mathbb{F}\subset K^{\text{un}}$; we identify $\pi_1(U)$ with $\operatorname{Gal}(K^{\text{un}}/K\cdot\mathbb{F})$.

Let $\Lambda=\mathbb{Z}/\ell^n\mathbb{Z}$ or $\mathbb{Q}_\ell$.

[33] The study of Shimura varieties suggests that this, in fact, is not the correct definition for noncomplete varieties.

PROPOSITION 29.9. *Let $\mathcal{E}$ be a locally constant sheaf of Λ-modules on an affine curve U over an algebraically closed field k, and let E be the corresponding $\pi_1(U)$-module. Then*

$$\begin{array}{rclcrcl} H^0(U,\mathcal{E}) & \cong & E^{\pi_1(U)} & \qquad & H^0_c(U,\mathcal{E}) & = & 0 \\ H^1(U,\mathcal{E}) & \cong & H^1(\pi_1(U),E) & & H^1_c(U,\mathcal{E}) & \cong & H^1(\pi_1(U),\check{E}(1))^\vee \\ H^2(U,\mathcal{E}) & = & 0 & & H^2_c(U,\mathcal{E}) & \cong & E_{\pi_1(U)}(-1). \end{array}$$

Here $E^{\pi_1(U)}$ is the largest submodule of E on which $\pi_1(U)$ acts trivially, and $E_{\pi_1(U)}$ is the largest quotient module of E on which $\pi_1(U)$ acts trivially.

PROOF. For the statements concerning $H^r(U,\mathcal{E})$, see §14. The statements concerning $H^r_c(U,\mathcal{E})$ follow by duality. For example

$$H^2_c(U,\mathcal{E}) = H^0(U,\check{\mathcal{E}}(1))^\vee = (\check{E}(1)^{\pi_1})^\vee = (\check{E}(1)^\vee)_{\pi_1} = (E(-1))_{\pi_1} = E_{\pi_1}(-1).$$

□

An isomorphism $\alpha\colon\ k \to k'$ of fields defines a one-to-one correspondence $X \leftrightarrow X'$ between k-varieties and k'-varieties under which étale maps correspond to étale maps and étale coverings to étale coverings. It therefore also defines a one-to-one correspondence $\mathcal{E} \leftrightarrow \mathcal{E}'$ between sheaves on X_{et} and sheaves on X'_{et} under which $H^r(X,\mathcal{E}) \cong H^r(X',\mathcal{E}')$.

We apply this remark to $\varphi\colon\ \mathbb{F} \to \mathbb{F}$ (the Frobenius automorphism $x \mapsto x^q$). If X and $\mathcal{E}$ arise from objects X_0 and $\mathcal{E}_0$ over $\mathbb{F}_q$, then $X' = X$ and $\mathcal{E}' = \mathcal{E}$. Therefore, for any variety X_0 over $\mathbb{F}_q$ and sheaf $\mathcal{E}_0$ on X_0, φ defines an automorphism of $H^r(X,\mathcal{E})$ (and similarly for cohomology with compact support).

PROPOSITION 29.10. *For any variety X_0 over $\mathbb{F}_q$ and sheaf $\mathcal{E}_0$ on X_0, the endomorphism of $H^r(X,\mathcal{E})$ defined by $(F,F_{\mathcal{E}})$ is inverse to that defined by φ (and similarly for cohomology with compact support).*

The proof is omitted, but I recommend that the reader verify the proposition for the sheaf $\mathbb{G}_a$ on $\mathbb{A}^1$. For this, one must verify that the maps

$$H^0(\mathbb{A}^1,\mathbb{G}_a) \to H^0(\mathbb{A}^1,F^*\mathbb{G}_a) \xrightarrow{F_{\mathbb{G}_a}} H^0(\mathbb{A}^1,\mathbb{G}_a)$$

send

$$P(T) \mapsto P(T^q) \mapsto P(T^q)^{\frac{1}{q}} = (\varphi^{-1}P)(T).$$

Here $P(T) \in \mathbb{F}[T]$ regarded as the ring of regular functions on $\mathbb{A}^1$ and φ^{-1} acts on the coefficients of P.

REMARK 29.11. Recall the following conventions.

When a group G acts on the left on a vector space V, then it acts on the dual $\check{V}$ of V by the rule

$$(gf)(v) = f(g^{-1}v), \quad f \in \check{V}, \quad g \in G, \quad v \in V.$$

This is the only natural way of defining a left action on G on $\check{V}$. This representation of G on $\check{V}$ is called the *contragredient* of the original representation.

When a k-algebra R acts on the left on k-vector space V, then R acts on $\check{V}$ by the rule

$$(fr)(v) = f(rv), \quad f \in \check{V}, \quad r \in R, \quad v \in V.$$

The left action of R becomes a right action on $\check{V}$ — in general, there will be no natural way to define a left action of R on $\check{V}$.

Note that, if $\gamma \in G$ has eigenvalues $a, b, \ldots$ on V, then it will have eigenvalues $a^{-1}, b^{-1}, \ldots$ on $\check{V}$. However, if $\gamma \in R$ has eigenvalues $a, b, \ldots$ on V, then it will have eigenvalues $a, b, \ldots$ on $\check{V}$. When γ can be considered both as an element of a ring and of a group, this can lead to confusion.

Consider an elliptic curve E_0 over $\mathbb{F}_q$. The number theorists define the "eigenvalues of the Frobenius" to be the eigenvalues of $\varphi \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acting on $T_\ell E$. But $T_\ell E$ is the dual of $H^1(E, \mathbb{Z}_\ell)$, and so these are also the eigenvalues of φ^{-1} acting on $H^1(E, \mathbb{Z}_\ell)$. According to Proposition 29.10, they are also the eigenvalues of F acting on $H^1(E, \mathbb{Z}_\ell)$. Thus, happily, the number theorists agree with the geometers.

The endomorphism ring of E also contains a Frobenius element π. How does it act? By definition, $\pi = F$, and so it acts on $H^1(E, \mathbb{Z}_\ell)$ as F (hence, with eigenvalues with absolute value $q^{\frac{1}{2}}$). And it acts on $T_\ell E$ with the *same* eigenvalues. Because H^1 is a contravariant functor, $\mathrm{End}(E)$ acts on it on the right; because T_ℓ is a covariant functor, it acts on it on the left. In fact, $\pi \in \mathrm{End}(E)$ will be a root of a polynomial with integer coefficients and with roots that have absolute value $q^{\frac{1}{2}}$. Thus, when $\mathrm{End}(E)$ acts on a space, π will always have eigenvalues of absolute value $q^{\frac{1}{2}}$.

PROPOSITION 29.12. *Let $\mathcal{E}_0$ be the locally constant sheaf on U_0 corresponding to a $\pi_1(U_0)$-module E. For any $x \in U^F$, there is a isomorphism $(\mathcal{E}_x, F_x) \approx (E, \varphi_x^{-1})$. Here φ_x is a Frobenius element in* $\mathrm{Gal}(K^{un}/K)$ *corresponding to x (φ_x is well-defined up to conjugation).*

The isomorphism in the proposition is noncanonical, necessarily so because φ_x is only defined up to conjugation, but its existence implies that

$$\mathrm{Tr}(F_x|\mathcal{E}_x) = \mathrm{Tr}(\varphi_x^{-1}|E).$$

Thus, the theorem can be rewritten as:

THEOREM 29.13. *Let E be a finite-dimensional $\mathbb{Q}_\ell$-vector space on which $\pi_1(U_0)$ acts continuously. Then*

$$\sum_{x \in U^F} \mathrm{Tr}(\varphi_x^{-1}|E) = -\mathrm{Tr}(\varphi^{-1}|H^1(\pi_1(U), \check{E}(1))^\vee) + \mathrm{Tr}(\varphi^{-1}|E_{\pi_1(U)}(-1)).$$

EXAMPLE 29.14. Let K be a function field in one variable over $\mathbb{F}_q$, and let L be a finite Galois extension of K with Galois group G. Let $\rho\colon\ G \to \mathrm{GL}(V)$ be a representation of G on a finite-dimensional vector space over $\mathbb{Q}$ (or a finite extension of $\mathbb{Q}$). Let A be a Dedekind domain in K with field of fractions K whose integral closure in L is unramified over A. Define

$$L^{\mathrm{Artin}}(s, \rho) = \prod_{\mathfrak{p} \subset A} \frac{1}{\det(1 - \rho(\varphi_x)\mathbb{N}\mathfrak{p}^{-s}|V)}, \quad s \in \mathbb{C}.$$

Here the product is over the nonzero prime ideals of A, and $\mathbb{N}\mathfrak{p} = (A\colon\ \mathfrak{p})$. This is the Artin L-series of ρ, except that I've omitted some factors.

Let $U_0 = \operatorname{Specm} A$. Then G is a quotient of $\pi_1(U_0)$, and we let $\mathcal{E}_0$ be the sheaf of $\mathbb{Q}_\ell$-vector spaces on U_0 corresponding to the $\pi_1(U_0)$-module $\check{V} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$. Then

$$L^{\text{Artin}}(s,\rho) = L(U_0, \mathcal{E}, q^{-s}).$$

It follows from the multiplicative form of Theorem 29.13 that $L^{\text{Artin}}(s,\rho) = \frac{P(q^{-s})}{Q(q^{-s})}$ where $P(t), Q(t) \in \mathbb{Q}[t]$. Moreover, $Q(t) = \det(1 - \varphi t | \check{V}_{\bar{G}}(-1))$, where $\bar{G}$ is the image of G in $\pi_1(U)$. Therefore, $L^{\text{Artin}}(s,\rho)$ is a meromorphic function of the complex variable s, which is even holomorphic when $V^{\bar{G}} = 0$.

Restatement of the theorem for finite sheaves. Roughly speaking, the idea of the proof of Theorem 29.4 is to pass to a finite étale covering $V_0 \to U_0$ where $\mathcal{E}$ becomes constant, and then apply the usual Lefschetz fixed-point formula on V_0. The problem is that, in general, a locally constant sheaf of $\mathbb{Q}_\ell$-vector spaces will not become constant on any finite covering (the action of $\pi_1(U_0)$ on a stalk of the sheaf need not factor through a finite quotient). Thus we need to work with sheaves of Λ-modules, where $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$. But then $H^r_c(U,\mathcal{E})$ need not be a free Λ-module, and so it is no longer clear how to define the trace. Nevertheless, we state the theorem:

THEOREM 29.15. *Let U_0 be a nonsingular affine geometrically connected curve over $\mathbb{F}_q$, and let $\mathcal{E}_0$ be a flat constructible locally constant sheaf of Λ-modules on U_0. Then*

$$\sum_{x \in U^F} \operatorname{Tr}(F_x|\mathcal{E}_x) = \sum_r (-1)^r \operatorname{Tr}(F|H^r_c(U,\mathcal{E})).$$

The conditions on $\mathcal{E}$ mean that it corresponds to a $\pi_1(U_0)$-module E that is free and finitely generated as a Λ-module. Our first task will be to explain what the right hand side means. Recall that a locally constant sheaf $\mathcal{E}$ of $\mathbb{Q}_\ell$-vector spaces on U_0 is a family $(\mathcal{E}_n, f_n)$ in which each $\mathcal{E}_n$ is a flat constructible locally constant sheaf of $\mathbb{Z}/\ell^n\mathbb{Z}$-modules. Theorem 29.4 is proved by applying Theorem 29.15 to each $\mathcal{E}_n$ and then forming the inverse limits (of course, it has to be checked that this is possible).

REMARK 29.16. It will be useful to note that we need to prove Theorem 29.15 only in the case that U^F is empty. Let V_0 be an open subset of U_0 omitting all $\mathbb{F}_q$-rational points of U_0. Then Theorem 29.15 for the pair $(V_0,\mathcal{E})$ and the pair $(U_0 \setminus V_0, \mathcal{E})$ implies it for $(U_0,\mathcal{E})$ (cf. the discussion at the start of this section). Since $U \setminus V$ is finite, that the Lefschetz fixed-point formula for $(U_0 \setminus V_0, \mathcal{E})$ is is essentially (29.2).

Perfect complexes. Let R be a ring (Noetherian as always). A complex

$$\cdots \to M^r \to M^{r+1} \to \cdots$$

of R-modules is said to be *perfect* if it is bounded (only finitely many of the M^r are nonzero) and each M^r is a finitely generated projective R-module. Recall that for modules over commutative Noetherian local rings

projective and finitely generated = free and finitely generated.

Because such a module has a finite basis, it is possible to define the trace of an endomorphism.

Proposition 29.17. *Let R be a commutative local Noetherian ring. Let $M^\bullet$ be a complex of R-modules, and let $\alpha\colon\ M^\bullet \to M^\bullet$ be an endomorphism $M^\bullet$. For any quasi-isomorphism $\gamma\colon\ P^\bullet \to M^\bullet$ with $P^\bullet$ perfect, there exists an endomorphism β of $P^\bullet$, unique up to homotopy, such that $\gamma \circ \beta = \alpha \circ \gamma$:*

$$\begin{array}{ccc} P^\bullet & \xrightarrow{\beta} & P^\bullet \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\gamma} \\ M^\bullet & \xrightarrow{\alpha} & M^\bullet \end{array}$$

Moreover, the element

$$\operatorname{Tr}(\beta|P^\bullet) \stackrel{df}{=} \sum_r (-1)^r \operatorname{Tr}(\beta|P^r)$$

of R is independent of the choice of $P^\bullet$, γ, and β. When the R-modules $H^r(M^\bullet)$ are all free,

$$\operatorname{Tr}(\beta|P^\bullet) = \sum (-1)^r \operatorname{Tr}(\beta|H^r(M)).$$

If R is an integral domain with field of fractions K, then

$$\operatorname{Tr}(\beta|P^\bullet) = \sum (-1)^r \operatorname{Tr}(\beta|H^r(M) \otimes_R K).$$

Proof. The proof is elementary — see, for example, EC VI 13.10. $\square$

We next need a criterion on a complex $M^\bullet$ to ensure the existence of a perfect complex $P^\bullet$ and a quasi-isomorphism $P^\bullet \to M^\bullet$. Since $H^r(P^\bullet)$ is finite for all r and zero for all but finitely many r, the $H^r(M^\bullet)$ must satisfy the same conditions. However, these conditions are not sufficient: for example, if $\Lambda = \mathbb{Z}/\ell^2\mathbb{Z}$ and $M^0 = \mathbb{Z}/\ell\mathbb{Z}$ and is 0 otherwise then no $P^\bullet$ exists, because it is not possible to truncate

$$\begin{array}{ccccccccccccc} \cdots & \xrightarrow{\ell} & \Lambda & \xrightarrow{\ell} & \Lambda & \xrightarrow{\ell} & \cdots & \xrightarrow{\ell} & \Lambda & \to & 0 & \to & \cdots \\ & & & & & & & & \| & & & & \\ & & & & & & & & \mathbb{Z}/\ell\mathbb{Z} & & & & \end{array}$$

at the left.

Proposition 29.18. *Let R be local Noetherian ring, and let $M^\bullet$ be a complex of R-modules such that $H^r(M^\bullet)$ is finitely generated for all r and zero for $r > m$, some m. Then there exists a quasi-isomorphism $Q^\bullet \to M^\bullet$ with $Q^\bullet$ a complex of finitely generated free R-modules such that $Q^r = 0$ for $r > m$. If $H^r(Q^\bullet \otimes_R N) = 0$ for $r < 0$ and all finitely generated R-modules N, then there exists a quasi-isomorphism $\gamma\colon\ Q^\bullet \to P^\bullet$ with $P^\bullet$ a perfect complex of R-modules such that P^r is nonzero only for $0 \le r \le m$.*

Proof. The existence of $Q^\bullet$ is a standard result (see, for example, Mumford, Abelian Varieties, Lemma 1, p47).

Consider the sequence

$$Q^{-2} \to Q^{-1} \to Q^0.$$

Under the hypothesis that $H^r(Q^\bullet \otimes_R N) = 0$ for $r < 0$ and all N, this is exact, and remains so after it has been tensored with N. Let B be the image of Q^{-1} in Q^0. Then

$$Q^{-2} \otimes_R N \to Q^{-1} \otimes_R N \to B \otimes_R N$$

is a complex, and $Q^{-1} \otimes_R N \to B \otimes_R N$ is surjective (tensor products are right exact). It follows that $B \otimes_R N \to Q^0 \otimes_R N$ is injective, and identifies $B \otimes_R N$ with the image of $Q^{-1} \otimes_R N$ in $Q^0 \otimes_R N$. Hence, for any submodule N' of N, $B \otimes_R N'$ is a submodule of $B \otimes_R N$. This implies that B is flat. As Q^0 is also flat, and $B \otimes N \to Q^0 \otimes N$ is injective for every N, it follows that Q^0/B is flat (as $\mathrm{Tor}_r(Q^0/B, N) = 0$ for $r > 0$). Define $P^\bullet$ to be the complex

$$\cdots \to 0 \to Q^0/B \to Q^1 \to Q^2 \to \cdots .$$

$\square$

Note that, if $M^r = 0$ for $r < 0$, then the map $Q^\bullet \to M^\bullet$ induces a map $P^\bullet \to M^\bullet$.

Let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, and let $\mathcal{E}$ be a flat constructible sheaf of Λ-modules on a variety U over an algebraically closed field. Let $j\colon\ U \hookrightarrow X$ be a completion of U, and let $j_!\mathcal{E} \to \mathcal{I}^\bullet$ be an injective resolution of $j_!\mathcal{E}$. Theorems 15.1, 19.1, and the Künneth formula show that $M^\bullet \overset{\mathrm{df}}{=} \Gamma(X, \mathcal{I}^\bullet)$ (and $Q^\bullet$) satisfy the hypotheses of the Proposition. We write $P^\bullet(U, \mathcal{E})$ for any perfect complex of Λ-modules for which there is a quasi-isomorphism $P^\bullet(U, \mathcal{E}) \to \Gamma(X, \mathcal{I}^\bullet)$. Thus, $P^\bullet(U, \mathcal{E})$ is a perfect complex representing the cohomology of $\mathcal{E}$ on U with compact support:

$$H^r(P^\bullet(U, \mathcal{E})) = H^r_c(U, \mathcal{E}).$$

In the situation of Theorem 29.15, the constructions that gave us endomorphisms F of $H^r_c(U, \mathcal{E})$ give us an endomorphism F of $\Gamma(X, \mathcal{I}^\bullet)$. We choose an endomorphism F of $P^\bullet(U, \mathcal{E})$ lying over F, and define

$$\sum (-1)^r \operatorname{Tr}(F | H^r_c(U, \mathcal{E})) = \operatorname{Tr}(F | P^\bullet(U, \mathcal{E})).$$

It is possible to choose the complexes $P^\bullet(U, \mathbb{Z}/\ell^n\mathbb{Z})$ for n varying to form a projective system such that the map $P^r(U, \mathbb{Z}/\ell^{n+1}\mathbb{Z}) \to P^r(U, \mathbb{Z}/\ell^n\mathbb{Z})$ induces an isomorphism $P^r(U, \mathbb{Z}/\ell^{n+1}\mathbb{Z})/\ell^n P^r(U, \mathbb{Z}/\ell^n\mathbb{Z}) \to P^r(U, \mathbb{Z}/\ell^n\mathbb{Z})$ for all r and n. We let $P^\bullet(U, \mathbb{Z}_\ell)$ be the inverse limit. This is a perfect complex of $\mathbb{Z}_\ell$-modules such that $H^r(P^\bullet(U, \mathbb{Z}_\ell)) = H^r_c(U, \mathbb{Z}_\ell)$.

The proof of Theorem 29.15. Fix a finite Galois covering $\pi_0\colon\ V_0 \to U_0$ such that $\mathcal{E}_0 | V_0$ is constant, and let G denote the Galois group of V_0 over U_0. Then $\mathcal{E}_0$ corresponds to a finite free Λ-module E endowed with an action of G. Let $V = V_0 \times_{\operatorname{Specm} \mathbb{F}_q} \mathbb{F}$ — it is a variety (not necessarily connected) over $\mathbb{F}$, and $\pi\colon\ V \to U$ is again a Galois covering with Galois group G. Let $R = \Lambda[G]$. Then R acts on the cohomology groups $H^r(V, \mathcal{E})$ through the action of G on V and on $\mathcal{E}$.

Let G_{-1} be the set of regular maps $\alpha\colon\ V \to V$ such that $F \circ \pi = \pi \circ \alpha$. Clearly, the Frobenius map $F\colon\ V \to V$ lies in G_{-1}, and $G_{-1} = \{F \circ g \mid g \in G\}$. The group G acts on G_{-1} by conjugation, and for any $\alpha \in G_{-1}$ we let $z(\alpha)$ be the order of the centralizer $Z(\alpha)$ of α in G.

Proposition 29.19. *For all $\alpha \in G_{-1}$, the ℓ-adic integer $\mathrm{Tr}_{\mathbb{Z}_\ell}(\alpha|P^\bullet(V,\mathbb{Z}_\ell))$ is divisible by $z(\alpha)$, and*

$$\mathrm{Tr}(F|P^\bullet(U,\mathcal{E})) = \sum_{\alpha\in G_{-1}/G} \frac{\mathrm{Tr}_{\mathbb{Z}_\ell}(\alpha|P^\bullet(V,\mathbb{Z}_\ell))}{z(\alpha)} \cdot \mathrm{Tr}_\Lambda(F^{-1}\circ\alpha|E)$$

(equality of elements of $\mathbb{Z}/\ell^n\mathbb{Z}$).

The sum is over a set of representatives α for the orbits of G acting on G_{-1}. Since $F^{-1}\circ\alpha \in G$ and E is a finitely generated free Λ-module, $\mathrm{Tr}(F^{-1}\circ\alpha|E)$ is a well-defined element of $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$. Neither trace depends on the choice of α in the orbit. Thus the right hand side is a well-defined element of Λ.

We now explain why the proposition completes the proof of the Theorem 29.15. As noted above, we may assume that U^F is empty, and then have to show that $\mathrm{Tr}(F|P^\bullet(U,\mathcal{E})) = 0$. For this it suffices to show that $\mathrm{Tr}(\alpha|P^\bullet(V,\mathbb{Z}_\ell)) = 0$ for all α, but

$$\mathrm{Tr}(\alpha|P^\bullet(V,\Lambda)) \overset{29.17}{=} \sum (-1)^r \,\mathrm{Tr}(\alpha|H^r_c(V,\mathbb{Q}_\ell)).$$

We wish to apply the Lefschetz fixed-point formula with constant coefficients to show that $\sum(-1)^r\,\mathrm{Tr}(\alpha|H^r_c(V,\mathbb{Q}_\ell)) = 0$. For this, we need to know that

(a) α extends to a regular map $\alpha\colon\ Y \to Y$ where Y is a complete nonsingular curve containing V;
(b) the fixed points of α in Y have multiplicity one;
(c) α has no fixed points in V.

Statements (a) and (b) will show that there is a Lefschetz fixed-point formula with constant coefficients for the noncompact curve V (cf. the discussion at the start of this section), and (c) shows that the trace is zero.

Statement (a) is a general fact about curves and their completions — α defines an endomorphism of the field $\mathbb{F}(V)$ of regular functions on V, which extends to a regular map on the complete nonsingular curve Y canonically attached to $\mathbb{F}(V)$ — see §14. In fact, the action of G on V extends uniquely to an action on Y, and if $\alpha = g\circ F$ on V, then the same equation holds on Y. For any closed point Q of Y,

$$(d\alpha)_Q = (dg)_{F(Q)} \circ (dF)_Q$$

which is zero, because $(dF)_Q = 0$. This implies (b) (cf. 27.2).

For (c), note that a fixed point of α in V would lie over a fixed point of F in U, and we are assuming that $U^F = \emptyset$.

It remains to prove Proposition 29.19.

Proposition 29.20. (a) *For each N, the complex $P^\bullet(V,\mathbb{Z}/\ell^N\mathbb{Z})$ can be chosen to be a perfect complex of $(\mathbb{Z}/\ell^N\mathbb{Z})[G]$-modules. For varying N, and they can be chosen to form a projective system whose limit $P^\bullet(V,\mathbb{Z}_\ell)$ is a perfect $\mathbb{Z}_\ell[G]$-complex with the property that $P^\bullet(V,\mathbb{Z}_\ell)/\ell^N P^\bullet(V,\mathbb{Z}_\ell) = P^\bullet(V,\mathbb{Z}/\ell^N\mathbb{Z})$ for all N.*
(b) *There is a quasi-isomorphism $P^\bullet(U,\mathcal{E}) \xrightarrow{\sim} P^\bullet(V,\Lambda)\otimes_R E$.*

Proof. (a) The proof of the first statement is a straightforward extension of that following 29.18. (That R is noncommutative causes no problems; that G acts on both Y and $\mathcal{E}$ compatibly requires one to write out the definitions of what this means.)

(b) The Künneth formula (better the projection formula 22.5), shows that

$$P^\bullet(V,\mathcal{E}) \xrightarrow{\sim} P^\bullet(V,\Lambda)\otimes_\Lambda E.$$

For any G-module P, the trace map $\mathrm{Ind}^G(P) \to P$ induces an isomorphism $\mathrm{Ind}^G(P)_G \to P$. Therefore

$$P^\bullet(V,\mathcal{E})_G \xrightarrow{\sim} P^\bullet(U,\mathcal{E}),$$

and

$$P^\bullet(V,\mathcal{E})_G \xrightarrow{\sim} (P^\bullet(V,\Lambda)\otimes_\Lambda E)_G = P^\bullet(V,\Lambda)\otimes_R E.$$

□

Remark 29.21. Let N and M be R-modules. Then $N\otimes_\Lambda M$ is an R-module with $\sigma\in G$ acting according to the rule: $\sigma(n\otimes m)=\sigma n\otimes\sigma m$. Let M_0 denote M as an R-module with G acting trivially. Then $\sigma\otimes m\mapsto\sigma\otimes\sigma m\colon\ R\otimes_\Lambda M_0\to R\otimes_\Lambda M$ is an isomorphism of R-modules. Therefore, if M is free as a Λ-module then $R\otimes_\Lambda M$ is free as an R-module; it follows that if N is a projective R-module, then $N\otimes_\Lambda M$ is a projective R-module.

Interlude on noncommutative traces. When R is a noncommutative ring. The trace of an endomorphism α of a free R-module M of finite rank is not well-defined as an element of R. Suppose, for example, that M has rank 1, and let e and $e' \overset{\text{df}}{=} be$ be basis elements. If $\alpha(e)=ae$, then $\alpha(e')=\alpha(be)=bae$, which need not equal $a(be)$ because $ab\neq ba$ in general. Let $R^\natural$ be the quotient of the additive group of R by the subgroup generated by the elements of the form $ab-ba$. For an endomorphism α of a free R-module of finite rank M, we define $\mathrm{Tr}(\alpha|M)$ to be the image of $\sum a_{ii}$ in $R^\natural$, where (a_{ij}) is the matrix of α relative to some basis for M — it is independent of the choice of the basis.

Lemma 29.22. *Let M_1 and M_2 be free R-modules of finite rank, and let α be an endomorphism of $M_1\oplus M_2$. Write $\alpha=\begin{pmatrix}\alpha_{11}&\alpha_{12}\\ \alpha_{21}&\alpha_{22}\end{pmatrix}$ where α_{ij} is a map $M_j\to M_i$. Then* $\mathrm{Tr}_R(\alpha|M_1\oplus M_2)=\mathrm{Tr}_R(\alpha_{11}|M_1)+\mathrm{Tr}_R(\alpha_{22}|M_2)$.

Proof. Compute $\mathrm{Tr}_R(\alpha|M_1\oplus M_2)$ relative to the union of a basis for M_1 and a basis for M_2. □

For an endomorphism α of a finitely generated projective R-module M, we define $\mathrm{Tr}(\alpha|M)=\mathrm{Tr}(\alpha\oplus 0|M\oplus N)$ where N is chosen so that $M\oplus N$ is free of finite rank — it is independent of the choice of N.

Let R be the group ring $\Lambda[G]$ where Λ is a commutative (Noetherian) ring, and G is a finite group. The map $\sum a_\sigma\sigma\mapsto a_e$, where e is the identity element of G, induces a map $\varepsilon\colon\ R^\natural\to\Lambda$, and for any endomorphism α of a finitely generated projective R-module M, we define

$$\mathrm{Tr}^G_\Lambda(\alpha|M)=\varepsilon(\mathrm{Tr}_R(\alpha|M)).$$

Lemma 29.23. *For any finitely generated projective R-module M,*

$$\mathrm{Tr}_\Lambda(\alpha|M) = [G]\,\mathrm{Tr}^G_\Lambda(\alpha|M),$$

with $[G]$ equal to the order of G.

Proof. We may assume that M is free, and as the traces depend only on the diagonal terms of the matrix of α, that $M = R$. Then α acts as multiplication on the right by some element $\sum a_\sigma\sigma$ of R. For any $\tau \in G$, $\alpha(\tau) = a_e\tau + \cdots$, and so $\mathrm{Tr}_\Lambda(\alpha|M) = [G]a_e$. Since $\mathrm{Tr}^G_\Lambda(\alpha|M) = a_e$, the lemma is true in this case. □

This lemma explains the significance of noncommutative traces for the proof of Theorem 29.15: they allow one to "divide" the usual trace over Λ by the order of the group, even when Λ is finite.

Lemma 29.24. *Let α and β be endomorphisms of P and M respectively, where P is a finitely generated projective R-module and M is a finitely generated R-module that is free as a Λ-module. Then*

$$\mathrm{Tr}^G_\Lambda(\alpha\otimes\beta|P\otimes_\Lambda M) = \mathrm{Tr}^G_\Lambda(\alpha|P)\cdot\mathrm{Tr}_\Lambda(\beta|M).$$

Proof. Note that, according to (29.21), $P \otimes_\Lambda M$ is a projective R-module, and so all the terms are defined. We need only consider the case that $P = R$. Then α is multiplication on the right by some element $\sum a_\sigma\sigma$, and the isomorphism $R\otimes_\Lambda M \to R\otimes_\Lambda M_0$ of (29.21) transforms $\alpha\otimes\beta$ into the endomorphism

$$r\otimes m \mapsto \sum_\sigma a_\sigma r\sigma\otimes\sigma^{-1}\beta(m)$$

of $R\otimes_\Lambda M_0$. The trace of

$$r\otimes m\mapsto a_\sigma r\sigma\otimes\sigma^{-1}\beta(m)\colon R\otimes_\Lambda M\to R\otimes_L M$$

is $a_e\,\mathrm{Tr}_\Lambda(\beta|M)$ if $\sigma = e$ and is 0 otherwise. This completes the proof. □

Suppose now that there is an exact sequence

$$1\to G\to W\xrightarrow{\text{degree}}\mathbb{Z}\to 1.$$

Define G_{-1} to be the inverse image of -1 in W. Then G acts on G_{-1} by conjugation, and for $\alpha\in G_{-1}$ we let $Z(\alpha)$ be the centralizer of α and $z(\alpha)$ the order of α. Let W^- be set of elements of W mapping to nonpositive integers. Let P be a Λ-module on which the monoid W^- acts Λ-linearly and which is projective when regarded as an R-module. Then P_G is a projective Λ-module, and every $w\in W^-$ defines an endomorphism of P_G that depends only on the degree(w).

Lemma 29.25. *With the above notations,*

$$\mathrm{Tr}_\Lambda(\sum_{\alpha\in G_{-1}}\beta|P) = \mathrm{Tr}_\Lambda(\sum_{\alpha\in G_{-1}}\beta|P^G) = \mathrm{Tr}_\Lambda(\sum_{\alpha\in G_{-1}}\beta|P_G).$$

Proof. Fix an element α_0 of G_{-1}, and let $\nu = \sum_{\sigma\in G}\sigma$. Then

$$P^G\hookrightarrow P\xrightarrow{\nu}P^G$$

is multiplication by $[G]$. This gives the middle equality in

$$\mathrm{Tr}_\Lambda(\sum_{\alpha\in G_{-1}}\alpha|P)=\mathrm{Tr}_\Lambda(\alpha_0\nu|P)=\mathrm{Tr}([G]\alpha_0|P^G)=\mathrm{Tr}(\sum_{\alpha\in G_{-1}}\alpha|P^G).$$

Multiplication by ν defines an isomorphism $P_G\to P^G$ (Serre, J.-P., *Corps Locaux*, VIII.1, Prop. 1), and so P^G may be replaced by P_G. □

PROPOSITION 29.26. *For any $\alpha_0\in G_{-1}$,*

$$\mathrm{Tr}_\Lambda(\alpha_0|P_G)=\sum_{\alpha\in G_{-1}/G}\mathrm{Tr}_\Lambda^{Z(\alpha)}(\alpha|P).$$

PROOF. See EC, VI.13.19. □

Completion of the proof. It remains to prove Proposition 29.19.

$$\begin{array}{rcll}
\mathrm{Tr}_\Lambda(F|P^\bullet(U,\mathcal{E})) & = & \mathrm{Tr}_\Lambda(\varphi^{-1}|P^\bullet(V,\Lambda)\otimes_R E) & (29.20)\\
& = & \mathrm{Tr}_\Lambda(\varphi^{-1}|(P^\bullet(V,\Lambda)\otimes_\Lambda E)_G) & \text{(obvious)}\\
& = & \sum_{\alpha\in G_{-1}/G}(\mathrm{Tr}_\Lambda^{Z(\alpha)}(\alpha|P^\bullet(V,\Lambda)\otimes_\Lambda E)) & (29.26)\\
& = & \sum_{\alpha\in G_{-1}/G}\mathrm{Tr}_\Lambda^{Z(\alpha)}(\alpha|P^\bullet(V,\Lambda))\cdot\mathrm{Tr}_\Lambda(F^{-1}\circ\alpha|E) & (29.24)\\
& = & \sum_{\alpha\in G_{-1}/G}\mathrm{Tr}_{\mathbb{Z}_\ell}^{Z(\alpha)}(\alpha|P^\bullet(V,\mathbb{Z}_\ell)\cdot\mathrm{Tr}_\Lambda(F^{-1}\circ\alpha|E) & (29.20)\\
& = & \sum_{\alpha\in G_{-1}/G}\frac{\mathrm{Tr}_{\mathbb{Z}_\ell}(\alpha|P^\bullet(V,\mathbb{Z}_\ell)}{z(\alpha)}\cdot\mathrm{Tr}_\Lambda(F^{-1}\circ\alpha|E). & (29.23).
\end{array}$$

30. The MAIN Lemma

Review of notations. Again, U_0 is a nonsingular affine curve over $\mathbb{F}_q$ and U denotes U_0 regarded as a curve over $\mathbb{F}$. Thus $U_0 = \operatorname{Specm} A_0$ for some affine $\mathbb{F}_q$-algebra A_0, and we let K be the field of fractions of A_0. Thus K is the field of rational functions on U_0. Choose an algebraic closure Ω of K_0, and identify $\pi_1(U_0)$ with $\mathrm{Gal}(K^{\mathrm{un}}/K)$, where K^{un} is the maximal unramified extension of K in Ω (relative to A_0), and identify $\pi_1(U)$ with $\mathrm{Gal}(K^{\mathrm{un}}/K\cdot\mathbb{F})$. Then a locally constant sheaf $\mathcal{E}_0$ of $\mathbb{Q}_\ell$-vector spaces on U_0 corresponds to a finite-dimensional $\mathbb{Q}_\ell$-vector space E endowed with a continuous action of $\pi_1(U_0)$. The inverse image of $\mathcal{E}$ on U corresponds to E regarded as a $\pi_1(U)$-module. Here E is the stalk $\mathcal{E}_{\bar\eta}$ of $\mathcal{E}$ at the geometric point of U_0 defined by $\operatorname{Specm}\Omega \to \operatorname{Specm} K \to U_0$.

We have an exact sequence

$$0 \to \pi_1(U) \to \pi_1(U_0) \to \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \to 0.$$

For each closed point $x \in U_0$, we have a Frobenius element $\varphi_x \in \pi_1(U_0)$ (well-defined up to conjugacy) that fixes some prime ideal $\mathfrak{P}$ of the integral closure A^{un} of A in K^{un} lying over $\mathfrak{p}$ and acts as $a \mapsto a^{q^{\deg x}}$ on $A^{\mathrm{un}}/\mathfrak{P}$. The image of φ_x in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ is $\varphi^{\deg x}$. Following Deligne, I set

$$F_x = \varphi_x^{-1} \in \pi_1(U_0), \quad F = \varphi^{-1} \in \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q).$$

Thus the actions of F_x and F on $\mathcal{E}_x = E$ and $H^r_c(U,\mathcal{E})$ respectively coincide with those of their namesakes defined geometrically (29.10; 29.12).

Preliminaries on linear algebra.

Lemma 30.1. *Let α and β be endomorphisms of finite-dimensional K-vector spaces V and W respectively. Then the trace of $\alpha\otimes\beta\colon V\otimes W \to V\otimes W$ is the product of the traces of α and β.*

Proof. Choose bases (e_i) and (f_i) for V and W respectively, and set

$$\alpha e_i = \sum a_{ji} e_j, \quad \beta f_i = \sum b_{ji} f_j.$$

Then $(e_i \otimes f_{i'})$ is basis for $V\otimes W$ and

$$(\alpha\otimes\beta)(e_i\otimes f_{i'}) = \sum_{j,j'} a_{ji} b_{j'i'} e_j \otimes f_{j'}.$$

Therefore, the trace of $\alpha\otimes\beta$ is

$$\mathrm{Tr}(\alpha\otimes\beta) = \sum_{i,i'} a_{ii} b_{i'i'} = (\sum a_{ii})(\sum b_{ii}) = (\mathrm{Tr}(\alpha))(\mathrm{Tr}(\beta)).$$

□

Lemma 30.2. *Let K be a field containing $\mathbb{Q}$, and let $\alpha\colon V\to V$ be an endomorphism of a K-vector space V. The characteristic polynomial of α has coefficients in $\mathbb{Q}$ if and only if there is a basis for V relative to which the matrix of α has its entries in $\mathbb{Q}$.*

PROOF. Certainly, if the matrix A of α with respect to some basis of V has entries in $\mathbb{Q}$, then the characteristic polynomial $\det(1-At)$ lies in $\mathbb{Q}[t]$. The converse follows from the theory of rational canonical forms for matrices. $\square$

DEFINITION 30.3. An endomorphism of a vector space over a field containing $\mathbb{Q}$ is said to be *rational* when it satisfies the equivalent conditions of Lemma 30.2.

For a vector space V, $V^{\otimes m}$ denotes the tensor product of m copies of V. An endomorphism α of V defines an endomorphism $\alpha^{\otimes m}$ of $V^{\otimes m}$. From Lemma 30.1, we see that

$$\mathrm{Tr}(\alpha^{\otimes m}|V^{\otimes m}) = (\mathrm{Tr}(\alpha|V))^m.$$

Clearly $\alpha^{\otimes m}$ is rational if α is. Often, we write α for $\alpha^{\otimes m}$.

Statement of the Theorem.

DEFINITION 30.4. Let $n \in \mathbb{Z}$.

(a) A locally constant sheaf $\mathcal{E}_0$ of $\mathbb{Q}_\ell$-vector spaces on U_0 is said to have *weight* n if for all closed points x of U_0, each eigenvalue of $F_x\colon\ \mathcal{E}_x \to \mathcal{E}_x$ is an algebraic number whose complex conjugates have absolute value $q^{\deg(x)n/2}$.

(b) A finite-dimensional $\mathbb{Q}_\ell$-vector space E endowed with a continuous action of $\pi_1(U_0)$ has *weight* n if, for all closed points x of U_0 (i.e., nonzero prime ideals $\mathfrak{p}$ of A_0), each eigenvalue of $\varphi_x\colon\ E_x \to E_x$ is an algebraic number whose complex conjugates have absolute value $q^{-\deg(x)n/2}$.

When E is the $\pi_1(U_0)$-module corresponding to $\mathcal{E}_0$, these notions coincide: $\mathcal{E}_0$ has weight n if and only if E has weight n.

EXAMPLE 30.5. (a) The sheaf $\mathbb{Q}_\ell(1)$ corresponds to the $\pi_1(U_0)$-module $(\varprojlim_n \mu_{\ell^n}(K^{\mathrm{un}})) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$. Because $\ell \neq p$, $\mu_{\ell^n}(\mathbb{F}) = \mu_{\ell^n}(K^{\mathrm{un}})$ is a free $\mathbb{Z}/\ell^n\mathbb{Z}$-module of rank 1: $m \in \mathbb{Z}/\ell^n\mathbb{Z}$ acts according to the rule $(m,\zeta) \mapsto \zeta^m$. Therefore $\varprojlim_n \mu_{\ell^n}(\mathbb{F})$ is a free $\mathbb{Z}_\ell$-module of rank 1, and $\mathbb{Q}_\ell(1)$ is a $\mathbb{Q}_\ell$-vector space of dimension 1. The group $\pi_1(U_0)$ acts on $\mathbb{Q}_\ell(1)$ through its quotient $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, and the canonical generator $\varphi = (z \mapsto z^q)$ of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ acts on $\mathbb{Q}_\ell(1)$ as multiplication by q. The Frobenius element φ_x has image $\varphi^{\deg x}$ in $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$, and so acts on $\mathbb{Q}_\ell(1)$ as $q^{\deg x}$. Thus, $\mathbb{Q}_\ell(1)$ has weight -2.

(b) If $\mathcal{E}_0$ is of weight β, then $\mathcal{E}_0^{\otimes m}$ is of weight $m\beta$. For example, $\mathbb{Q}_\ell(m)$ has weight $-2m$.

The next theorem is often referred to as the MAIN Lemma (restricted form).

THEOREM 30.6. *Let E be the $\pi_1(U_0)$-module corresponding to a locally constant sheaf $\mathcal{E}_0$ of $\mathbb{Q}_\ell$-vector spaces. Let n be an integer. Assume:*

(a) *(Rationality.) For all closed points $x \in U_0$, the action of F_x on E is rational (30.3);*

(b) *There exists a nondegenerate $\pi_1(U_0)$-invariant skew-symmetric form*

$$\psi\colon E \times E \to \mathbb{Q}_\ell(-n).$$

(c) *(Big geometric monodromy.) The image of $\pi_1(U)$ in $Sp(E,\psi)$ is open (for the ℓ-adic topology).*

Then:

(a) *E is of weight n, i.e., the eigenvalues of F_x acting on V have absolute value $(q^{\deg x})^{n/2}$.*
(b) *The action of F on $H^1_c(U,\mathcal{E})$ is rational, and its eigenvalues all have absolute value $\le q^{n/2+1}$.*
(c) *Let j be the inclusion of U into $\mathbb{P}^1$. The action of F on $H^1(\mathbb{P}^1, j_*\mathcal{E})$ is rational, and its eigenvalues α satisfy*

$$q^{n/2} < |\alpha| < q^{n/2+1}.$$

Here $Sp(E,\psi)$ denotes the symplectic group of ψ, i.e.,

$$Sp(E,\psi) = \{\alpha \in \mathrm{GL}(E) \mid \psi(\alpha e, \alpha e') = \psi(e,e'), \quad e, e' \in E\}.$$

Let $\sigma \in \pi_1(U_0)$ be such that $\sigma|\mathbb{F} = \varphi^m$ for some $m \in \mathbb{Z}$. That ψ is $\pi_1(U_0)$-invariant means that

$$\psi(\sigma e, \sigma e') = q^{-nm}\psi(e,e'), \qquad e, e' \in E.$$

In particular, if $\sigma \in \pi_1(U)$, then it acts on E as an element of the symplectic group, and so (c) makes sense.

EXAMPLE 30.7. Let d be an odd integer, and suppose we have a regular map

$$\pi\colon Y_0 \to U_0$$

such that, for each closed point x of U_0, $\pi^{-1}(x)$ is a nonsingular hypersurface $Y(x)_0$ in $\mathbb{P}^{d+1}$ defined over $\kappa(x) = \mathbb{F}_{q^{\deg x}}$. Let $\mathcal{E} = R^d\pi_*\mathbb{Q}_\ell$. Then $\mathcal{E}$ is locally constant (by the proper-smooth base change theorem 20.2), and for any closed point x of U_0, $\mathcal{E}_x \cong H^d(Y(x),\mathbb{Q}_\ell)$ (proper base change theorem 17.2). Moreover F_x acts on $\mathcal{E}_x$ as the Frobenius map of $Y(x)$ acts on $H^d(Y(x),\mathbb{Q}_\ell)$, and so

$$\det(1 - F_x t|\mathcal{E}_x) = \det(1 - Ft|H^d(Y(x),\mathbb{Q}_\ell)).$$

Now, $H^r(Y(x),\mathbb{Q}_\ell)$ is zero for odd $r \ne d$ and equals $H^r(\mathbb{P}^d,\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-\frac{r}{2})$ for r even (see 16.4) and so

$$Z(Y(x)_0,t) = \frac{\det(1 - Ft|H^d(Y(x),\mathbb{Q}_\ell)}{(1-t)(1-qt) \quad \cdots \quad (1-q^d t)}.$$

As $Z(Y(x)_0,t) \in \mathbb{Q}(t)$ (see 27.8), $\det(1 - Ft|H^d(Y(x),\mathbb{Q}_\ell)$ has rational coefficients, and so $\mathcal{E}$ satisfies condition (a) of the theorem.

There is a canonical pairing of sheaves

$$R^d\pi_*\mathbb{Q}_\ell \times R^d\pi_*\mathbb{Q}_\ell \to R^{2d}\pi_*\mathbb{Q}_\ell \cong \mathbb{Q}_\ell(-d)$$

which on each stalk becomes cup-product

$$H^d(Y(x),\mathbb{Q}_\ell) \times H^d(Y(x),\mathbb{Q}_\ell) \to H^{2d}(Y(x),\mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-d).$$

Therefore, the pairing on sheaves corresponds to a pairing

$$\psi\colon E \times E \to \mathbb{Q}_\ell(-d)$$

which is skew-symmetric (because d is odd) non-degenerate (by Poincaré duality for the geometric generic fibre of π defined by $\operatorname{Spec}\Omega \to U_0$) and $\pi_1(U_0)$-invariant (because it is defined by a morphism of sheaves on U_0). Therefore, $\mathcal{E}$ satisfies condition

(b) of the theorem. If it satisfies condition (c), then it has weight d and the Riemann hypothesis holds for every $Y(x)$!

In fact it *is* possible to realize every nonsingular hypersurface Y_0 of odd dimension over $\mathbb{F}_q$ as a member of a family satisfying (c) (after possibly extending $\mathbb{F}_q$), and so prove the Riemann hypothesis for Y_0 this way. This is a geometric problem: if we can find a family $\pi\colon\ Y \to U \subset \mathbb{A}^1$ of nonsingular hypersurfaces over $\mathbb{F}$ containing $Y_{0/\mathbb{F}}$ as a member and satisfying condition (c), then the family will be defined over a finite extension of $\mathbb{F}_q$.

Let δ be the degree of Y_0. We consider the set of all homogeneous polynomials of degree δ in $d+2$ variables considered up to multiplication by a nonzero scalar. This set can be identified with $\mathbb{P}^N$, $N = \begin{pmatrix} d+\delta+1 \\ \delta \end{pmatrix}$ (see AG 5.19). We obtain a map

$$\begin{array}{ccc} H & \subset & \mathbb{P}^N \times \mathbb{P}^{d+1} \\ \downarrow \pi & \swarrow & \mathbb{P}^N \\ \mathbb{P}^N & & \end{array}$$

such that, for each $P \in \mathbb{P}^N$, $\pi^{-1}(P)$ is the hypersurface (possibly reducible) in $\mathbb{P}^{d+1}$ defined by P regarded as a homogeneous polynomial. Now $Y_{0/\mathbb{F}}$ corresponds to a point P in $\mathbb{P}^N$, and the problem is to show that there is a line through P for which the "geometric monodromy" is big.

Outline of the Proof of (a) of the Theorem. Recall that

$$Z(U_0, E^{\otimes 2k}, t) \overset{\text{df}}{=} \prod_{x \in U_0} \frac{1}{\det(1 - F_x t^{\deg x} | E^{\otimes 2k})} = \prod_x \left(\sum_m \operatorname{Tr}(F_x^m | E^{\otimes 2k}) \frac{t^m}{m} \right).$$

The first equality is the definition, and the second follows from the elementary Lemma 27.5. Condition (a) of the Theorem implies that $Z(U_0, E^{\otimes 2k}, t) \in \mathbb{Q}[[t]]$, and so it makes sense to speak of its radius of convergence for $t \in \mathbb{C}$.

We shall prove (under the hypotheses of the theorem):

(I) For all positive integers k, $(\otimes^{2k} E)_{\pi_1(U)}$ is isomorphic to a direct sum of copies of $\mathbb{Q}_\ell(-kn)$.

(A) If, for all positive integers k, $Z(X_0, E^{\otimes 2k}, t)$ converges for $|t| < \frac{1}{q^{kn+1}}$, then E has weight n.

We explain how the two statements imply the theorem. As have already noted, $Z(U_0, E^{\otimes 2k}, t)$ is a power series with coefficients in $\mathbb{Q}$. The multiplicative form of the Lefschetz fixed-point formula (29.6),

$$Z(U_0, E^{\otimes 2k}, t) = \frac{\det(1 - Ft | H_c^1(U, \mathcal{E}^{\otimes 2k}))}{\det(1 - Ft | H_c^2(U, \mathcal{E}^{\otimes 2k}))},$$

shows that $Z(U_0, E, t) \in \mathbb{Q}_\ell(t) \cap \mathbb{Q}[[t]] = \mathbb{Q}(t)$. Recall (29.9) that $H_c^2(U, \mathcal{E}) \cong E_{\pi_1(U)}(-1)$, and so (I) implies that

$$\det(1 - Ft | H_c^2(U, \mathcal{E}^{\otimes 2k})) = (1 - q^{kn+1} t)^N, \text{some } N.$$

Thus

$$Z(U_0, E^{\otimes 2k}, t) = \frac{\text{polynomial in } \mathbb{Q}[t]}{(1 - q^{kn+1}t)^N}.$$

Obviously, this converges for $|q^{kn+1}t| < 1$, i.e., for $|t| < 1/q^{kn+1}$. Now we can apply (A) to show that E has weight n.

The proof of (A). Throughout, E satisfies the condition (a) and (b) of the theorem.

LEMMA 30.8. *For all positive integers k, the coefficients of the power series*

$$\frac{1}{\det(1 - F_x t^{\deg x} | E^{\otimes 2k})}$$

are positive rational numbers.

PROOF. According to the elementary Lemma 27.5, we have

$$\log \frac{1}{\det(1 - F_x t^{\deg x} | E^{\otimes 2k})} = \sum_{m=1}^{\infty} \operatorname{Tr}(F_x^m | E^{\otimes 2k}) \frac{t^{m \deg(x)}}{m}.$$

But $\operatorname{Tr}(F_x^m | E^{\otimes 2k}) = \operatorname{Tr}(F_x^m | E)^{2k}$. Under the hypothesis (a) of the theorem, $\operatorname{Tr}(F_x^m | E) \in \mathbb{Q}$, and so $\operatorname{Tr}(F_x^m | E^{\otimes 2k})$ is a positive rational number. Thus the coefficients of the power series $\log \frac{1}{\det(1 - F_x t^{\deg x} | E^{\otimes 2k})}$ are positive rational numbers, and the same is true of exp of it. □

LEMMA 30.9. *If, for all positive integers k, $Z(X_0, E^{\otimes 2k}, t)$ converges for $|t| < \frac{1}{q^{kn+1}}$, then E has weight n.*

PROOF. Consider

$$Z(U_0, E^{\otimes 2k}, t) \stackrel{\text{df}}{=} \prod_{x \in U_0} \frac{1}{\det(1 - F_x t^{\deg x} | E^{\otimes 2k})}.$$

If a_m is the coefficient of t^m in the power series expansion of $Z(U_0, E^{\otimes 2k}, t)$, and $a_{m,x}$ is the coefficient of t^m in the expansion of $1/\det(1 - F_x t^{\deg x} | E^{\otimes 2k})$, then $a_m \geq a_{m,x}$ (because $a_{0,x} = 1$ and $a_{m,x} \geq 0$ all x). Therefore the radius of convergence of $Z(U_0, E^{\otimes 2k}, t)$ is $\leq$ the radius of convergence of $1/\det(1 - F_x t^{\deg x} | E^{\otimes 2k})$. The hypothesis of the lemma therefore implies that $1/\det(1 - F_x t^{\deg x} | E^{\otimes 2k})$ converges for $|t| < 1/q^{kn+1}$.

If α is an eigenvalue of F_x on E, α^{2k} is an eigenvalue of F_x on $E^{\otimes 2k}$, and so $1/\alpha^{\frac{2k}{\deg x}}$ is a pole of $1/\det(1 - F_x t^{\deg\ x} | E^{\otimes 2k})$. Therefore,

$$\left|\frac{1}{\alpha^{\frac{2k}{\deg x}}}\right| \geq \frac{1}{q^{kn+1}}$$

and so $|\alpha^{2k}| \leq (q^{\deg x})_x^{kn+1}$. On taking the $2k^{th}$ root and letting $k \to \infty$, we find that $|\alpha| \leq (q^{\deg x})^{n/2}$ for all eigenvalues α of F_x on E. The existence of the pairing in (b) of the theorem shows that, for each eigenvalue α there is an eigenvalue α' such that $\alpha\alpha' = (q^{\deg x})^n$, and this completes the proof. □

Proof of (I).

Lemma 30.10. *Condition (c) of the theorem implies that* $\pi_1(U)$ *is Zariski dense in* $Sp(\psi)$*, and therefore that*

$$\mathrm{Hom}(E^{\otimes 2k}, \mathbb{Q}_\ell)^{\pi_1(U)} = \mathrm{Hom}(E^{\otimes 2k}, \mathbb{Q}_\ell)^{Sp(\psi)}.$$

Proof. Let $\bar{\pi}$ be the image of $\pi_1(U)$ in $Sp(\psi, \mathbb{Q}_\ell)$. Because $\pi_1(U)$ is compact (it is a Galois group), $\bar{\pi}$ is compact, and therefore closed in $Sp(\psi, \mathbb{Q}_\ell)$. The ℓ-adic version of Cartan's theorem (Serre, Lie Algebras and Lie Groups, LG 5.42) then says that it a Lie subgroup of $Sp(\psi, \mathbb{Q}_\ell)$. Since it is open, it has the same dimension as $Sp(\psi, \mathbb{Q}_\ell)$ as a Lie group. Let $G \subset Sp$ be the Zariski closure of $\bar{\pi}$ in Sp (so G is the smallest closed subvariety of Sp defined over $\mathbb{Q}_\ell$ such that $G(\mathbb{Q}_\ell) \supset \bar{\pi}$. Then G is an algebraic subgroup of Sp, and

$$\mathrm{Tgt}_e(\bar{\pi}) \subset \mathrm{Tgt}_e(G) \subset \mathrm{Tgt}_e(Sp).$$

But, $\mathrm{Tgt}_e(\bar{\pi}) = \mathrm{Tgt}_e(Sp)$, and so $\dim G = \dim Sp$. As Sp is connected, this implies that $G = Sp$, and so $\bar{\pi}$ is Zariski dense in Sp.

Finally, let $f\colon\ E^{\otimes 2k} \to \mathbb{Q}_\ell$ be a linear map fixed by $\pi_1(U)$. For $g \in \mathrm{SL}(V)$ to fix f is an algebraic condition on g: if f is fixed by $\bar{\pi}$, then it is fixed by the Zariski closure of $\bar{\pi}$. $\square$

Obviously,

$$\mathrm{Hom}(E^{\otimes 2k}, \mathbb{Q}_\ell)^{Sp(\psi)} = \mathrm{Hom}((E^{\otimes 2k})_{Sp(\psi)}, \mathbb{Q}_\ell).$$

If $f_1, f_2, \ldots, f_N$ is a basis for $\mathrm{Hom}(E^{\otimes 2k}, \mathbb{Q}_\ell)^{Sp(\psi)}$, then the map

$$a \mapsto (f_1(a), \ldots, f_N(a))\colon E^{\otimes 2k} \to \mathbb{Q}_\ell^N$$

induces an isomorphism

$$(E^{\otimes 2k})_{Sp(\psi)} \to (\mathbb{Q}_\ell)^N.$$

We shall use invariant theory to choose a basis (f_i) for which we shall be able to see how the action of $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ transfers to an action on $(\mathbb{Q}_\ell)^N$.

Consider the following general question: given a vector space V over a field K of characteristic zero and a nondegenerate skew-symmetric form ψ on V, what are the Sp-invariant linear forms $f\colon\ V^{\otimes 2k} \to K$? That is, what is $\mathrm{Hom}(V^{\otimes 2k}, K)^{Sp}$? Note that a linear form $f\colon\ V^{\otimes 2k} \to K$ can be regarded as a multi-linear form

$$f\colon V \times V \times \cdots \times V \to K.$$

For example,

$$(v_1, \ldots, v_{2k}) \mapsto \psi(v_1, v_2) \cdots \psi(v_{2k-1}, v_{2k})$$

is such a multi-linear form, and it is obviously Sp-invariant. More generally, for any partition of $\{1, ..., 2k\}$ into k-disjoint sets $P\colon\ \{\{a_1, b_1\}, ..., \{a_k, b_k\}\}$, $a_i < b_i$, we get an invariant form:

$$f_P\colon V^{2k} \to K,\ (v_1, ..., v_{2k}) \to \prod \psi(v_{a_i}, v_{b_i}).$$

Proposition 30.11. *The invariant forms* f_P *span* $\mathrm{Hom}(V^{\otimes 2k}, K)^{Sp}$.

PROOF. See H. Weyl, *The Classical Groups*, PUP, VI.1, 1939 (Rev. edn 1946). Alternatively, see Fulton, W., and Harris, J., *Representation Theory*, Springer Verlag, 1991, Appendix F. □

A basis of forms of the type f_P for $\mathrm{Hom}(V^{\otimes 2k}, K)^{Sp}$, gives a $\pi_1(U_0)$-equivariant map

$$E^{\otimes 2k} \to \mathbb{Q}_\ell(-kn)^N$$

inducing an isomorphism $(E^{\otimes 2k})_{\pi_1(U)} \to \mathbb{Q}_\ell(-kn)^N$. This proves (I).

Proof of (b) of the theorem. As U is affine, $H_c^0(U, \mathcal{E}) = 0$. On the other hand,

$$H_c^2(U, \mathcal{E}) = E_{\pi_1(U)},$$

which is zero because hypothesis (c) implies that $\pi_1(U)$ is Zariski dense in $Sp(E, \psi)$ and there are no $Sp(E, \psi)$-invariant linear forms $E \to \mathbb{Q}_\ell$ (or on $E^{\otimes n}$ for any odd n, by invariant theory). Therefore, the Lefschetz Fixed Point Formula 29.6 shows that

$$Z(U_0, \mathcal{E}_0, t) = \det(1 - Ft|H_c^1(U, \mathcal{E})).$$

By definition

$$Z(U_0, \mathcal{E}_0, t) = \prod_{x \in U_0} \frac{1}{(1 - F_x t^{\deg x}|E)},$$

and so

$$\frac{1}{\det(1 - Ft|H_c^1(U, \mathcal{E}))} = \prod_x \det(1 - F_x t^{\deg x}|E).$$

Hypothesis (a) implies that, when expanded out, the right hand side is a power series in t with coefficients in $\mathbb{Q}$. To complete the proof, we show that it converges for $|t| < 1/q^{\frac{n}{2}+1}$.

Recall from complex analysis, that if $p_1, p_2, \ldots$ is an infinite sequence of complex numbers, none of which is zero, then $\prod p_n$ is said to converge if the partial products converge to a nonzero complex number. An infinite product $\prod(1 + a_n)$ converges absolutely if and only if the series $\sum |a_n|$ converges.

Let $d =$ dimension of E, and let $a_{x,i}$, $1 \leq i \leq d$, be the eigenvalues of F_x acting on E, so that

$$\frac{1}{Z(U_0, \mathcal{E}_0, t)} = \prod_{x,i}(1 - a_{x,i} t^{\deg x}).$$

We shall show that

$$\sum_{x,i} |a_{x,i} t^{\deg x}|$$

converges for $|t| < 1/q^{\frac{n}{2}+1}$. The two facts we shall need are:

- $|a_{x,i}| = (q^{\deg x})^{\frac{n}{2}}$;
- the number of closed points x on U_0 of degree m is $\leq q^m$.

The first statement was proved in (a). The second follows from the fact that each closed point x of U_0 of degree m contributes at least 1 (in fact exactly m) elements to $U_0(\mathbb{F}_{q^m})$, which has $\leq q^m$ elements since $U_0 \subset \mathbb{A}^1$. Put $|t| = \frac{1}{q^{\frac{n}{2}+1+\varepsilon}}$, $\varepsilon > 0$. The first fact implies that

$$\sum_i |\alpha_{i,x} t^{\deg x}| \leq \frac{d}{(q^{\deg x})^{1+\varepsilon}},$$

and the second that

$$\sum_x \frac{1}{(q^{\deg x})^{1+\varepsilon}} \leq \sum_m \frac{q^m}{q^{m(1+\varepsilon)}} = \sum_m \frac{1}{q^{m\varepsilon}} < \infty.$$

Note that the proof used only that E_0 has weight n and that $E_{\pi_1(U)} = 0$.

Proof of (c) of the theorem. From the cohomology sequence of

$$0 \to j_!\mathcal{E} \to j_*\mathcal{E} \to i_*i^*j_*\mathcal{E} \to 0,$$

(see 8.15), we obtain a surjection

$$H^1_c(U, \mathcal{E}) \to H^1(\mathbb{P}^1, j_*\mathcal{E}).$$

Hence (b) implies that F acts rationally on $H^1(\mathbb{P}^1, j_*\mathcal{E})$ and its eigenvalues satisfy

$$|\alpha| < q^{\frac{n}{2}+1}.$$

The sheaf $\check{\mathcal{V}}(1)$ satisfies the same hypotheses as $\mathcal{E}$ with n replaced by $-2-n$. Therefore, F acts rationallly on $H^1(\mathbb{P}^1, j_*\check{\mathcal{V}}(1))$ and its eigenvalues β satisfy

$$|\beta| < q^{\frac{-2-n}{2}+1} = q^{-\frac{n}{2}}.$$

Now the duality theorem (32.3) gives a canonical nondegenerate pairing

$$H^r(\mathbb{P}^1, j_*\mathcal{E}) \times H^{2-r}(\mathbb{P}^1, j_*\check{\mathcal{V}}(1)) \to H^2(\mathbb{P}^1, \mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell.$$

Hence, each α is the inverse of a β, and so

$$|\alpha| > q^{\frac{n}{2}}.$$

31. The Geometry of Lefschetz Pencils

In this section, we see how to fibre a variety over $\mathbb{P}^1$ in such a way that the fibres have only very simple singularities, and in the next section we use the fibring to study the cohomology of the variety. This approach to the study of the cohomology of varieties goes back to Lefschetz in the complex case. Throughout this section, we work over an algebraically closed field k.

Definition. A linear form $H = \sum_{i=0}^{m} a_i T_i$ defines hyperplane in $\mathbb{P}^m$, and two linear forms define the same hyperplane if and only if one is a nonzero multiple of the other. Thus the hyperplanes in $\mathbb{P}^m$ form a projective space, called the *dual projective space* $\check{\mathbb{P}}^m$.

A line D in $\check{\mathbb{P}}^m$ is called a *pencil* of hyperplanes in $\mathbb{P}^m$. If H_0 and H_∞ are any two distinct hyperplanes in D, then the pencil consists of all hyperplanes of the form $\alpha H_0 + \beta H_\infty$ with $(\alpha\colon\ \beta) \in \mathbb{P}^1(k)$. If $P \in H_0 \cap H_\infty$, then it lies in every hyperplane in the pencil — the *axis* A of the pencil is defined to be the set of such P. Thus

$$A = H_0 \cap H_\infty = \cap_{t \in D} H_t.$$

The axis of the pencil is a linear subvariety of codimension 2 in $\mathbb{P}^m$, and the hyperplanes of the pencil are exactly those containing the axis. Through any point in $\mathbb{P}^m$ not on A, there passes exactly one hyperplane in the pencil. Thus, one should imagine the hyperplanes in the pencil as sweeping out $\mathbb{P}^m$ as they rotate about the axis.

Let X be a nonsingular projective variety of dimension $d \geq 2$, and embed X in some projective space $\mathbb{P}^m$. By the square of an embedding, we mean the composite of $X \hookrightarrow \mathbb{P}^m$ with the Veronese mapping (AG 5.18)

$$(x_0\colon\ \dots : x_m) \mapsto (x_0^2\colon\ \dots : x_i x_j\colon\ \dots : x_m^2)\colon \mathbb{P}^m \to \mathbb{P}^{\frac{(m+2)(m+1)}{2}}.$$

Definition 31.1. A line D in $\check{\mathbb{P}}^m$ is said to be a *Lefschetz pencil* for $X \subset \mathbb{P}^m$ if

(a) the axis A of the pencil $(H_t)_{t \in D}$ cuts X transversally;
(b) the hyperplane sections $X_t \overset{\text{df}}{=} X \cap H_t$ of X are nonsingular for all t in some open dense subset U of D;
(c) for $t \notin U$, X_t has only a single singularity, and the singularity is an ordinary double point.

Condition (a) means that, for any closed point $P \in A \cap X$, $\mathrm{Tgt}_P(A) \cap \mathrm{Tgt}_P(X)$ has codimension 2 in $\mathrm{Tgt}_P(X)$.

The intersection $X \cap H_t$ in (b) should be taken scheme-theoretically, i.e., if X is defined by the homogeneous ideal $\mathfrak{a}$, then $X \cap H_t$ is defined by $\mathfrak{a} + (H_t)$. Condition (b) means that X_t is reduced and nonsingular as an algebraic variety.

A point P on a variety X of dimension d is an *ordinary double point* if the tangent cone at P is isomorphic to the subvariety of $\mathbb{A}^{d+1}$ defined by a nondegenerate quadratic form $Q(T_1, \dots, T_{d+1})$, or, equivalently, if

$$\hat{\mathcal{O}}_{X,P} \approx k[[T_1, \dots, T_{d+1}]]/(Q(T_1, \dots, T_{d+1})).$$

THEOREM 31.2. *There exists a Lefschetz pencil for X (after possibly replacing the projective embedding of X by its square).*

PROOF. (Sketch). Let $Y \subset X \times \check{\mathbb{P}}^m$ be the closed variety whose points are the pairs (x, H) such that H contains the tangent space to X at x. For example, if X has codimension 1 in $\mathbb{P}^m$, then $(x, H) \in Y$ if and only if H is the tangent space at x. In general,

$$(x, H) \in Y \iff x \in H \text{ and } H \text{ does not cut } X \text{ transversally at } x.$$

The image of Y in $\check{\mathbb{P}}^m$ under the projection $X \times \check{\mathbb{P}}^m \to \check{\mathbb{P}}^m$ is called the *dual variety* $\check{X}$ of X. The fibre of $Y \to X$ over x consists of the hyperplanes containing the tangent space at x, and these hyperplanes form an irreducible subvariety of $\check{\mathbb{P}}^m$ of dimension $m-(\dim X+1)$; it follows that Y is irreducible, complete, and of dimension $m-1$ (see AG 8.8) and that $\check{X}$ is irreducible, complete, and of codimension ≥ 1 in $\check{\mathbb{P}}^m$ (unless $X = \mathbb{P}^m$, in which case it is empty). The map $\varphi\colon\ Y \to \check{X}$ is unramified at (x, H) if and only if x is an ordinary double point on $X \cap H$ (see SGA 7, XVII 3.7). Either φ is generically unramified, or it becomes so when the embedding is replaced by its square (so, instead of hyperplanes, we are working with quadric hypersurfaces) (ibid. 3.7). We may assume this, and then (ibid. 3.5), one can show that for $H \in \check{X} \setminus \check{X}_{\text{sing}}$, $X \cap H$ has only a single singularity and the singularity is an ordinary double point. Here $\check{X}_{\text{sing}}$ is the singular locus of $\check{X}$.

By Bertini's theorem (Hartshorne II.8.18) there exists a hyperplane H_0 such that $H_0 \cap X$ is irreducible and nonsingular. Since there is an $(m-1)$-dimensional space of lines through H_0, and at most an $(m-2)$-dimensional family will meet $\check{X}_{\text{sing}}$, we can choose H_∞ so that the line D joining H_0 and H_∞ does not meet $\check{X}_{\text{sing}}$. Then D is a Lefschetz pencil for X. $\square$

THEOREM 31.3. *Let $D = (H_t)$ be a Lefschetz pencil for X with axis $A = \cap H_t$. Then there exists a variety X^* and maps*

$$X \leftarrow X^* \xrightarrow{\pi} D.$$

such that:

(a) *the map $X^* \to X$ is the blowing up of X along $A \cap X$;*
(b) *the fibre of $X^* \to D$ over t is $X_t = X \cap H_t$.*

Moreover, π is proper, flat, and has a section.

PROOF. (Sketch) Through each point x of $X \setminus A \cap X$, there will be exactly one H_x in D. The map

$$\varphi\colon X \setminus A \cap X \to D,\ x \mapsto H_x,$$

is regular. Take the closure of its graph Γ_φ in $X \times D$; this will be the graph of π. $\square$

REMARK 31.4. The singular X_t may be reducible. For example, if X is a quadric surface in $\mathbb{P}^3$, then X_t is curve of degree 2 in $\mathbb{P}^2$ for all t, and such a curve is singular if and only if it is reducible (look at the formula for the genus). However, if the embedding $X \hookrightarrow \mathbb{P}^m$ is replaced by its cube, this problem will never occur.

References. The only modern reference I know of is SGA 7, Exposé XVII. (Perhaps one day I'll include it in AG.)

32. The Cohomology of Lefschetz Pencils

Throughout this section, k is an algebraically closed field of characteristic p (possibly 0).

Preliminaries on locally constant sheaves. Let U be an open subset of a complete curve X, and let Λ be a commutative ring (for example, $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, $\Lambda = \mathbb{Z}_\ell$, $\Lambda = \mathbb{Q}_\ell$). We have seen that to give a constructible locally constant sheaf of Λ-modules on U is to give a finitely generated Λ-module on which $\pi_1(U)$ acts continuously.

When we consider constructible sheaves of Λ-modules on X whose restriction to U is locally constant, the picture is more complicated: it is possible to change the stalks of the sheaf over the points of $X \setminus U$ almost at will (see 8.17). However, there is a special class of such sheaves for which $\mathcal{F}$ is determined by $\mathcal{F}|U$, namely, those for which the canonical map $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism. Here, as usual, j is the inclusion $U \hookrightarrow X$.

Proposition 32.1. *Let U be an open subset of a complete nonsingular curve X over an algebraically closed field k, and let $\Lambda = \mathbb{Z}/\ell^n\mathbb{Z}$, $\mathbb{Z}_\ell$, or $\mathbb{Q}_\ell$ with ℓ distinct from the characteristic of k.*

(a) *Let $\mathcal{F}$ be a sheaf of Λ-modules on X; the canonical map $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism if and only if, for all $s \in X \setminus U$, the cospecialization map $\mathcal{F}_s \to \mathcal{F}_{\bar\eta}$ is injective and has image $\mathcal{F}_{\bar\eta}^{I_s}$ (here $\bar\eta$ is a generic geometric point, and $I_s \subset \pi_1(U, \bar\eta)$ is an inertia group at s).*

(b) *For any locally constant sheaf of $\mathbb{Q}_\ell$-vector spaces $\mathcal{F}$ on U, the cup-product pairing*

$$H^r(X, j_*\mathcal{F}) \times H^{2-r}(X, j_*\check{\mathcal{F}}(1)) \to H^2(X, \mathbb{Q}_\ell(1)) \cong \mathbb{Q}_\ell$$

is nondegenerate.

Proof. (a) The map $\mathcal{F} \to j_*j^*\mathcal{F}$ induces an isomorphism on the stalks for all $x \in U$. For $s \in X \smallsetminus U$, the stalks of the two sheaves are $\mathcal{F}_s$ and $\mathcal{F}_{\bar\eta}^{I_s}$ respectively, and so the map induces an isomorphism on the stalks at s if and only if the cospecialization map is an isomorphism $\mathcal{F}_s \to \mathcal{F}_{\bar\eta}^{I_s}$.

(b) This can be deduced from the usual Poincaré duality theorem 14.7 by a local calculation. See EC V 2.2b. □

The tame fundamental group.

The local case. Let $K = k((T))$, the field of Laurent series over k. Then K is the field of fractions of the complete valuation ring $k[[T]]$, and so we can study its extension fields as we do in number theory. In one respect this field is simpler: since the residue field is algebraically closed, there are no unramified extensions. Thus, for a finite extension L of K, the ramification degree is equal to the degree. In particular, the tamely ramified extensions of K are precisely those of degree prime to p. The tamely ramified extensions are exactly the Kummer extensions of the form $K[T^{\frac{1}{d}}]$ for $\gcd(p, d) = 1$.

Fix an algebraic closure K^{al} of K, and let K^{tame} be the composite of all the tame extensions of K contained in K^{al}. For each $\sigma \in \mathrm{Gal}(K^{\mathrm{tame}}/K)$ and d not divisible by p,

$\sigma T^{\frac{1}{d}} = \zeta T^{\frac{1}{d}}$ for some $\zeta \in \mu_d(k)$. Thus, we have a well-defined map $\mathrm{Gal}(K^{\mathrm{tame}}/K) \to \mu_d$. On passing to the limit over d, we obtain an isomorphism

$$t\colon\ \mathrm{Gal}(K^{\mathrm{tame}}/K) \to \prod_{\ell \neq p} \mathbb{Z}_\ell(1).$$

The global case. Now let $K = k(T)$, the field of rational functions in the symbol T over k. Then K is the field of fractions of the Dedekind domain $k[T]$, and we can study its extensions as in the number field case. The prime ideals of $k[T]$ are of the form $(T-a)$ for some $a \in k$, and the completion of $k(T)$ with respect to the valuation defined by $(T-a)$ is $k((T-a))$. There is one additional prime, namely, the "prime at ∞" corresponding to the prime ideal (T^{-1}) in $k[T^{-1}]$. Let S be a finite set of prime ideals of $k[T]$.

Fix an algebraic closure K^{al} of K, and let K^{tame} be the composite of all the finite extensions L/K contained in K^{al} that are unramified at all primes of $k[T]$ not in S and tamely ramified at all primes $s \in S$. Let K_s be the completion of K at the prime $s \in S$, and let K_s^{tame} be a maximal tamely ramified extension of K_s. The choice of an extension of the embedding of K into K_s to K^{tame}, i.e., to a commutative diagram

$$\begin{array}{ccc} K^{\mathrm{tame}} & \hookrightarrow & K_s^{\mathrm{tame}} \\ \uparrow & & \uparrow \\ K & \hookrightarrow & K_s \end{array}$$

determines an injective homomorphism $\mathrm{Gal}(K_s^{\mathrm{tame}}/K_s) \to \mathrm{Gal}(K^{\mathrm{tame}}/K)$. Its image, I_s, is uniquely determined by s up to conjugation. For each $s \in S$, we have a subgroup I_s of $\mathrm{Gal}(K^{\mathrm{tame}}/K)$ and a surjective homomorphism $t\colon\ I_s \to \mathbb{Z}_\ell(1)$. The subgroups I_s needn't generate $\mathrm{Gal}(K^{\mathrm{tame}}/K)$, because there may be a proper extension of K unramified at all primes in $k[T]$ but wildly ramified at the infinite prime (see 3.2).

The geometric case. Let U be an open subset of the projective line $\mathbb{P}^1$ over k, and let S be the complement of U in $\mathbb{P}^1$. We wish to study finite maps $V \to U$ that are unramified over the points of U and tamely ramified over the points in S. Essentially, this is the same as the last case, except that we have one extra prime corresponding to the point at infinity. Fix an algebraic closure K^{al} of $K \overset{\mathrm{df}}{=} k(\mathbb{P}^1)$, and let $\bar{\eta} \to X$ be the corresponding geometric point. Let $\pi_1^{\mathrm{tame}}(U,\bar{\eta}) = \mathrm{Gal}(K^{\mathrm{tame}}/K)$ where K^{tame} is the composite of the subfields of K^{al} that are unramified at all primes corresponding to points of U and tamely ramified at those corresponding to points in S. Then $\pi_1^{\mathrm{tame}}(U,\bar{\eta})$ contains a subgroup I_s for each $s \in S$, and it is now generated by these subgroups (see 3.2). For each s, there is a canonical epimorphism $t\colon\ I_s \to \mathbb{Z}_\ell(1)$.

The Cohomology. We wish to the study how the cohomology varies in a Lefschetz pencil, but first we should look at the case of curves. Throughout this subsection, the base field k will be algebraically closed.

Pencils of curves. Consider a proper flat map $\pi\colon\ X \to \mathbb{P}^1$ whose fibres are irreducible curves, nonsingular except for s in a finite set S, and such that X_s for $s \in S$ has a single node as its singularity. Then

$$\chi(X_t, \mathcal{O}_{X_t}) \overset{\mathrm{df}}{=} \dim_k H^0(X_t, \mathcal{O}_{X_t}) - \dim_k H^1(X_t, \mathcal{O}_{X_t})$$

is constant in the family (Mumford, Abelian Varieties, p50), and $H^0(X_t, \mathcal{O}_{X_t}) = k$ for all t (because X_t is irreducible). Therefore, $\dim_k H^1(X_t, \mathcal{O}_{X_t})$ will be constant, equal to g say. By definition, $\dim_k H^1(X_t, \mathcal{O}_{X_t})$ is the arithmetic genus of X_t. When X_t is nonsingular, it is the usual genus, and when X_t has a single node, the genus of the normalization of X_t is $g-1$ (Serre, J.-P., *Groupes Algébriques et Corps de Classes*, Hermann, 1959, IV.7, Proposition 3).

When X_t is nonsingular, $H^1(X_t, \mathbb{Q}_\ell)$ has dimension $2g$ (see 14.2). As in the nonsingular case, in order to compute $H^1(X_s, \mathbb{Q}_\ell)$, $s \in S$, we must first compute the Picard variety of X_s.

Let $\varphi\colon \tilde{X}_s \to X_s$ be the normalization of X_s. It is an isomorphism, except that two points P_1 and P_2 of $\tilde{X}_s$ map to the singular point P on X_s. The map $f \mapsto f \circ \varphi$ identifies the functions on an open neighbourhood U of P to the functions on $\varphi^{-1}(U) \subset \tilde{X}_s$ that take the same value at P_1 and P_2. Therefore, we have an exact sequence of sheaves on $(X_s)_{et}$,

$$0 \to \mathbb{G}_{m,X_s} \underset{f \mapsto f \circ \varphi}{\longrightarrow} \varphi_*(\mathbb{G}_{m,\tilde{X}_s}) \underset{f \mapsto \frac{f(P_1)}{f(P_2)}}{\longrightarrow} \mathbb{G}_{m,P} \to 0.$$

The cohomology sequence of this is

$$0 \to \mathbb{G}_m \to Pic(X_s) \to Pic(\tilde{X}_s) \to 0,$$

from which we can extract an exact sequence

$$0 \to \mathbb{G}_m \to Pic^0(X_s) \to Pic^0(\tilde{X}_s) \to 0.$$

One can show (Serre, ib.) that $Pic^0(X_s)$ is equal to the group of divisors of degree zero on $X_s - \{P\}$ modulo principal divisors of the form (f) with $f(P) = 1$. The first map in the sequence can be described as follows: let $a \in \mathbb{G}_m(k) = k^\times$; because the regular functions on $\tilde{X}_s$ separate points, there exists an $f \in k(\tilde{X}_s) = k(X_s)$ such that $f(P_1) = a$ and $f(P_2) = 1$; the image of a is (f).

Note that descriptions of the maps in the sequence involves choosing an ordering of the points P_1, P_2 mapping to the singular point P. The opposite choice gives the negative of the maps. This sign indeterminacy persists throughout the theory.

From the above exact sequence and the cohomology sequence of the Kummer sequence, we obtain an exact sequence

$$0 \to \mathbb{Q}_\ell(1) \to H^1(X_s, \mathbb{Q}_\ell(1)) \to H^1(\tilde{X}_s, \mathbb{Q}_\ell(1)) \to 0,$$

and hence (twisting by -1, i.e., tensoring with $\mathbb{Q}(-1)$) an exact sequence

$$0 \to \mathbb{Q}_\ell \to H^1(X_s, \mathbb{Q}_\ell) \to H^1(\tilde{X}_s, \mathbb{Q}_\ell) \to 0.$$

In particular, $H^1(X_s, \mathbb{Q}_\ell)$ is of dimension $2g-1$. Write E_s for the kernel of $H^1(X_s, \mathbb{Q}_\ell) \to H^1(\tilde{X}_s, \mathbb{Q}_\ell)$. It is the group of *vanishing cycles*.[34]. Note that $E_s \cong \mathbb{Q}_\ell$ (the isomorphism is well-defined up to sign) — we denote the element of E_s corresponding to 1 by δ_s.

[34]The topologists have a way of visualizing things in which the vanishing cycle (in homology) moves in a family and does vanish at the point s. I have never been able to understand the picture, but look forward to the movie.

Let $\mathcal{V} = R^1\pi_*\mathbb{Q}_\ell$. Thus $\mathcal{V}_{\bar{\eta}} \approx H^1(X_t, \mathbb{Q}_\ell)$ for all $t \notin S$, and $\mathcal{V}_s \cong H^1(X_s, \mathbb{Q}_\ell)$. Let $V = \mathcal{V}_{\bar{\eta}}$. One can show that the cospecialization map $\mathcal{V}_s \to \mathcal{V}_{\bar{\eta}}$ is injective, with image V^{I_s} where $I_s \subset \pi_1(U)$ is the inertia group at s. Moreover, in the cup-product pairing

$$\begin{array}{ccccc} H^1(X_t,\mathbb{Q}_\ell) & \times & H^1(X_t,\mathbb{Q}_\ell) & \to & H^2(X_t,\mathbb{Q}_\ell), \\ \| & & \| & & \| \\ V & & V & & \mathbb{Q}_\ell(-1) \end{array}$$

E_s is the exact annihilator of $\mathcal{V}_s$. In other words, the sequence

$$0 \to H^1(X_s,\mathbb{Q}_\ell) \to H^1(X_{\bar{\eta}},\mathbb{Q}_\ell) \xrightarrow{x \mapsto x\cup\delta} \mathbb{Q}_\ell(-1) \to 0$$

is exact. The theory of Lefschetz pencils shows that there is a similar sequence for any Lefschetz pencil with odd fibre dimension.

Cohomology in a Lefschetz Pencil. Let $\pi\colon\ X^* \to \mathbb{P}^1$ be the map arising from a Lefschetz pencil, and let $S \subset \mathbb{P}^1$ be the subset of $\mathbb{P}^1$ such that X_s is singular. Let $n = 2m+1$ be the dimension of the fibres of π — thus we are assuming that the fibre dimension is odd. We set

$$\begin{aligned} U &= \mathbb{P}^1 \smallsetminus S, \\ \pi_1(U) &= \pi_1^{\text{tame}}(U,\bar{\eta}), \\ I_s &= \text{the tame fundamental group at } s \end{aligned}$$

(subgroup of $\pi_1(U)$ well-defined up to conjugacy).

Let $V = (R^n\pi_*\mathbb{Q}_\ell)_{\bar{\eta}}$, and $V(r) = V \otimes \mathbb{Q}_\ell(r)$. Then the following are true.

- For $r \neq n, n+1$, the sheaves $R^r\pi_*\mathbb{Q}_\ell$ are locally constant (hence constant) on $\mathbb{P}^1$, i.e., they don't "see" the singularities.
- The proper-smooth base change theorem implies that $R^n\pi_*\mathbb{Q}_\ell|U$ is locally constant, and so V is a $\pi_1(U,\bar{\eta})$-module. In fact, the action factors through the tame fundamental group $\pi_1(U)$.
- For each $s \in S$ there is a "vanishing cycle" $\delta_s \in V(m)$ (well-defined up to sign). Let $E(m)$ be the subspace of $V(m)$ generated by the δ_s — then $E \subset V$ is the *space of vanishing cycles.*

If one vanishing cycle is zero, they all are. In the following, I assume that no vanishing cycle is zero. This is the typical case, and the proof of the Riemann hypothesis is much easier in the other case. Under this hypothesis, $R^{n+1}\pi_*\mathbb{Q}_\ell$ also is constant. The only sheaf left to understand is $R^n\pi_*\mathbb{Q}_\ell$.

- For each $s \in S$, the sequence

  $$0 \to H^n(X_s,\mathbb{Q}_\ell) \to H^n(X_{\bar{\eta}},\mathbb{Q}_\ell) \xrightarrow{x\mapsto(x\cup\delta)} \mathbb{Q}_\ell(m-n) \to 0$$

  is exact. Here $(x \cup \delta)$ is the image of (x, δ) under the pairing

  $$H^n(X_{\bar{\eta}},\mathbb{Q}_\ell) \times H^n(X_{\bar{\eta}},\mathbb{Q}_\ell(m)) \to H^{2n}(X_{\bar{\eta}},\mathbb{Q}_\ell(m)) \cong \mathbb{Q}_\ell(m-n).$$

- An element $\sigma_s \in I_s$ acts on V according the following rule:

  $$\sigma_s(x) = x \pm t(\sigma_s)(x \cup \delta_s)\delta_s$$

> **(Picard-Lefschetz formula)**. Here t is the map $I_s \to \mathbb{Z}_\ell(1)$ defined earlier, and so $t(\sigma_s)(x \cup \delta_s)\delta_s \in V(1 + (m-n) + m) = V$. The sign depends only on n modulo 4.

It follows from these two statements, that for all $s \in S$ the map $H^n(X_s, \mathbb{Q}_\ell) \to H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell)$ is injective with image $H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell)^{I_s}$, and hence that $R^n\pi_*\mathbb{Q}_\ell \xrightarrow{\approx} j_*j^*R^n\pi_*\mathbb{Q}_\ell$ (see 32.1).

Let ψ be the form $V \times V \to \mathbb{Q}_\ell(-n)$ defined by cup-product. It is nondegenerate (Poincaré duality) and skew-symmetric (because n is odd).

PROPOSITION 32.2. *The space E of vanishing cycles is stable under the action of π_1. Let $E^\perp$ be the orthogonal complement of E in $H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell)$ under the pairing ψ. Then $E^\perp = H^n(X_{\bar{\eta}}, \mathbb{Q}_\ell)^{\pi_1}$.*

PROOF. Both statements follow from the Picard-Lefschetz formula. For example, the Picard-Lefschetz formula with $\delta_{s'}$ for x

$$\sigma_s(\delta_{s'}) = \delta_{s'} \pm t(\sigma_s)(\delta_{s'} \cup \delta_s)\delta_s, \quad \sigma_s \in I_s,$$

implies the first statement, and

$$\sigma_s(x) - x = \pm t(\sigma_s)(x \cup \delta_s)\delta_s$$

implies the second (because the I_s generate π_1 and the δ_s are nonzero). □

THEOREM 32.3 (Lefschetz in the classical case). *The vanishing cycles are conjugate under the action of π_1 up to sign, i.e., given $s, s' \in S$, there exists a $\sigma \in \pi^1$ such that $\sigma\delta_{s'} = \delta_s$ or $-\delta_s$.*

PROOF. (Sketch). Let $D(= \mathbb{P}^1)$ be the line in the construction of the Lefschetz pencil. Recall that $D \subset \check{\mathbb{P}}^N$ and that $S = D \cap \check{X}$.

Fact 1: The map $\mathbb{P}^1 \backslash S \to \check{\mathbb{P}}^m \backslash \check{X}$ induces a surjective map $\pi_1(\mathbb{P}^1 \backslash S) \to \pi_1(\check{\mathbb{P}}^N \backslash \check{X})$.

This follows from a theorem of Bertini, viz, that if Y is irreducible and $Y \to \check{\mathbb{P}}$ is dominating, then the pull-back of Y to a “generic” line in $\check{\mathbb{P}}$ is also irreducible.

Fact 2: The action of $\pi_1(\mathbb{P}^1 \setminus S)$ on V factors through $\pi_1(\check{\mathbb{P}}^N \setminus \check{X})$.

Fact 3: Choose a generator δ of $\mathbb{Z}_\ell(1)$, and for each $s \in S$ choose a $\sigma_s \in I_s$ such that $t(\sigma_s) = \delta$. Then the σ_s become conjugate in $\pi_1(\check{\mathbb{P}}^N \setminus \check{X})$.

In the classical case, there is a simple geometric proof of Fact 3. The abstract case is more difficult.

We now use the Picard-Lefschetz formula to complete the proof. First note that the formula

$$\sigma_s x - x = \pm t(\sigma_s)(x \cup \delta_s)\delta_s, \quad x \in V,$$

determines δ_s up to sign. Next note that for $\gamma \in \pi_1(\mathbb{P}^1 \setminus S)$,

$$(\gamma\sigma_s\gamma^{-1})x = \gamma(\gamma^{-1}x \pm t(\sigma_s)(\gamma^{-1}x \cup \delta_s)\delta_s) = x \pm t(\sigma_s)(x \cup \gamma\delta_s)\gamma\delta_s.$$

In fact, this formula holds for $\gamma \in \pi_1(\check{\mathbb{P}}^N \setminus \check{X})$, and so the “facts” complete the proof. □

Corollary 32.4. *The space $E/E\cap E^{\perp}$ is an absolutely simple $\pi_1(\mathbb{P}^1\setminus S)$-module, i.e., it contains no nonzero proper submodule stable under $\pi_1(\mathbb{P}^1\setminus S)$ even after it has been tensored with an extension field of $\mathbb{Q}_\ell$.*

Proof. Let $x\in E$. If $x\notin E^{\perp}$, then there exists an s such that $x\cup\delta_s\neq 0$, and so the formula $\sigma_s x = x\pm t(\sigma_s)(x\cup\delta_s)\delta_s$ shows that the space spanned by x and all its transforms by elements of $\pi_1(U)$ contains δ_s, and hence all the vanishing cycles. It therefore equals E. This argument works over any extension of $\mathbb{Q}_\ell$. □

Remark 32.5. Let ψ be the form $V\times V\to\mathbb{Q}_\ell(-n)$ defined by cup-product; it is skew-symmetric because n is odd, and it is nondegenerate on $E/E\cap E^{\perp}$. It is respected by the monodromy group $\pi_1(U_0)$, and so $\pi_1(U)$ maps into $Sp(E/E\cap E^{\perp},\psi)$.

Theorem 32.6 (Kazhdan and Margulis). *(n odd). The image of $\pi_1(\mathbb{P}^1\setminus S,\bar\eta)$ in $Sp(E/E\cap E^{\perp},\psi)$ is open.*

This follows from the results reviewed above and the next lemma. Note that, because $\pi_1(\mathbb{P}^1\setminus S,\bar\eta)$ is compact, its image in $Sp(E/E\cap E^{\perp},\psi)$ is closed.

Lemma 32.7. *Let ψ be a nondegenerate form on a vector space W over $\mathbb{Q}_\ell$. Let $G\subset Sp(W,\psi)$ be a closed subgroup such that:*

(a) *W is a simple G-module;*
(b) *G is generated topologically by automorphisms of the form*

$$x\mapsto x\pm\psi(x,\delta)\delta$$

for certain $\delta\in G$.

Then G contains an open subgroup of $Sp(W,\psi)$.

Proof. We shall need to use a little of the theory of Lie groups over $\mathbb{Q}_\ell$, for which I refer to Serre, J.-P., *Lie Algebras and Lie Groups*, Benjamin, 1965. As G is closed in Sp, it is a Lie group over $\mathbb{Q}_\ell$ (by the ℓ-adic analogue of Cartan's Theorem). Let $L = Lie(G)$ (equal to the tangent space to G at 1). To prove the lemma, it suffices to show that L equals $Lie(Sp)$, because the exponential map sends any sufficiently small neighbourhood of 0 in the Lie algebra of a Lie group onto a neighbourhood of 1 in the Lie group.

There is also a map $\log\colon G\to LieG$ (defined on a neighbourhood of 1). Let $\delta\in W$ and let α be the endomorphism $x\mapsto x\pm\psi(x,\delta)\delta$ of W. Then

$$\log(\alpha)=\log(1-(1-\alpha))=-\sum\frac{(1-\alpha)^n}{n}.$$

But $(1-\alpha)(x)=\pm\psi(x,\delta)\delta$, and so $(1-\alpha)^2=0$ because $\psi(\delta,\delta)=0$. Hence $Lie(G)$ contains the endomorphisms

$$N(\delta)\colon x\mapsto\pm\psi(x,\delta)\delta$$

and it is generated by them. Thus, the following statement about Lie algebras will complete the proof. □

Lemma 32.8. *Let W be a vector space over a field k of characteristic zero, and let ψ be a nondegenerate form on W. Let L be a sub-Lie-algebra of the Lie algebra of $Sp(W,\psi)$ such that*

(a) *W is a simple L-module;*
(b) *L is generated by certain endomorphisms of the form $N(\delta)$, for certain $\delta \in W$.*

Then L equals the Lie algebra of $Sp(W, \psi)$.

The proof is omitted (for the present) — it is about 2 pages and is elementary.

33. Completion of the Proof of the Weil Conjectures.

Let X_0 be a smooth projective variety of even dimension $2m+2 = n+1$ over $\mathbb{F}_q$. After (28.3), it remains to show that:

(*) F acts rationally on $H^{n+1}(X, \mathbb{Q}_\ell)$ and its eigenvalues α satisfy

$$q^{n/2} < |\alpha| < q^{n/2+1}.$$

We prove $(*)$ by induction on m.

We say that an endomorphism $\varphi\colon\ V \to V$ of a finite-dimensional vector space V over $\mathbb{Q}_\ell$ satisfies $*(n)$ if it acts rationally and its eigenvalues α satisfy

$$q^{\frac{n-1}{2}} < |\alpha| < q^{\frac{n+1}{2}}.$$

Lemma 33.1. (a) *If V satisfies $*(n)$ and W is a subspace of V stable under φ, then both W and V/W satisfy $*(n)$.*

(b) *If there exists a filtration*

$$V \supset V_1 \supset \cdots \supset V_r \supset 0$$

stable under the action of φ such that, for all i, the endomorphism of V_i/V_{i+1} defined by φ satisfies $*(n)$, then φ satisfies $*(n)$.

Proof. Easy exercise for the reader. □

The same argument as in the proof of (28.1) shows that we can extend the ground field $\mathbb{F}_q$. This allows us to assume that there is a Lefschetz pencil for X_0 rational over $\mathbb{F}_q$; write D_0 $(= \mathbb{P}^1)$ for the pencil, $S \subset D$ for the set of singular hyperplane sections, and $U_0 = D_0 - S$. We may assume (after extending $\mathbb{F}_q$):

(a) each $s \in S$ is rational over $\mathbb{F}_q$, and the quadratic form defining the tangent cone at s can be expressed (over $\mathbb{F}_q$) as

$$Q(T_1, \dots, T_n) = \sum_{i=1}^{m} T_i T_{i+1} + T_{2m+1}^2;$$

(b) there is a $u_0 \in \mathbb{P}^1(\mathbb{F}_q)$ such that the fibre $X_{u_0} \overset{\text{df}}{=} \pi^{-1}(u_0)$ is nonsingular, and X_{u_0} admits a nonsingular hyperplane section Y_0 defined over $\mathbb{F}_q$.

Note that X has dimension $n+1 = 2m+2$, the fibres have dimension $n = 2m+1$, and Y_0 has dimension $n-1 = 2m$.

Then the variety X^* obtained from X by blowing up along the axis $A \cap X$ is also defined over $\mathbb{F}_q$, and we have a map $\pi_0\colon\ X_0^* \to \mathbb{P}^1$ defined over $\mathbb{F}_q$. We write $\mathbb{P}_0^1$ for the projective line over $\mathbb{F}_q$, and $\mathbb{P}^1$ for the projective line over $\mathbb{F}$.

Let u denote the point of U mapping to u_0 (u_0 as in (b)). Then u can be regarded as a geometric point of U_0. We write $\pi_1(U_0)$ for $\pi_1^{\text{tame}}(U_0, u)$ and $\pi_1(U)$ for $\pi_1^{\text{tame}}(U, u)$. Recall that there is an isomorphism $\pi_1(U_0, \bar{\eta}) \to \pi_1(U_0, u)$ (well-defined up to conjugation).

Lemma 33.2. *It suffices to prove (*) for X^*.*

Proof. The variety X^* is obtained by blowing up X along $A \cap X$. Because A crosses X transversally, $A \cap X$ is a nonsingular subvariety of codimension 2 in X, and so the inverse image $(A \cap X)^*$ of $A \cap X$ in X^* is isomorphic to $\mathbb{P}(N_{X/A\cap X})$ where $N_{X/A\cap X}$ is the normal bundle (see Shafarevich, Vol 2, p73). Now (23.2) shows that

$$H^*((A \cap X)^*, \mathbb{Q}_\ell) \cong H^*(A \cap X, \mathbb{Q}_\ell) \oplus H^{*-2}(A \cap X, \mathbb{Q}_\ell)(-1).$$

The map $\varphi\colon\ X^* \to X$ is proper, and so we can apply the proper base change theorem (§17) to it. We find that the canonical map $\mathbb{Q}_\ell \to \varphi_*\mathbb{Q}_\ell$ is an isomorphism, that $R^2\varphi_*\mathbb{Q}_\ell$ has support on $A \cap X$, and that $R^r\varphi_*\mathbb{Q}_\ell = 0$ for $r \neq 0, 2$. Moreover, $R^2\varphi_*\mathbb{Q}_\ell = i_*(R^2\varphi'_*\mathbb{Q}_\ell)$ where $\varphi' = \varphi|(A \cap X)^*$, and $R^2\varphi'_*\mathbb{Q}_\ell = \mathbb{Q}_\ell(-1)$. The Leray spectral sequence for φ_* degenerates at E_2 (because that for φ' does), and so

$$H^*(X^*, \mathbb{Q}_\ell) \cong H^*(X, \mathbb{Q}_\ell) \oplus H^{*-2}(A \cap X, \mathbb{Q}_\ell)(-1).$$

□

From the Leray spectral sequence of π, we see that it suffices to prove $(*)(n+1)$ for F acting on each of the three groups:

$$H^2(\mathbb{P}^1, R^{n-1}\pi_*\mathbb{Q}_\ell),\ H^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell),\ H^0(\mathbb{P}^1, R^{n+1}\pi_*\mathbb{Q}_\ell).$$

For a constant sheaf $\mathcal{V}$ on $\mathbb{P}^1$,

$$\begin{array}{rcl} H^0(\mathbb{P}^1, \mathcal{V}) & = & \mathcal{V}_u, \\ H^1(\mathbb{P}^1, \mathcal{V}) & = & \mathrm{Hom}(\pi_1(\mathbb{P}^1), \mathcal{V}_u) = 0, \\ H^2(\mathbb{P}^1, \mathcal{V}) & = & H^0(\mathbb{P}^1, \check{\mathcal{V}}(1))\check{} = \mathcal{V}_u(-1). \end{array}$$

The group $H^2(\mathbb{P}^1, R^{n-1}\pi_*\mathbb{Q}_\ell)$. From the theory of Lefschetz pencils $R^{n-1}\pi_*\mathbb{Q}_\ell$ is a constant sheaf on $\mathbb{P}^1$, and $(R^{n-1}\pi_*\mathbb{Q}_\ell)_u = H^{n-1}(X_u, \mathbb{Q}_\ell)$. Therefore,

$$H^2(\mathbb{P}^1, R^{n-1}\pi_*\mathbb{Q}_\ell) = H^{n-1}(X_u, \mathbb{Q}_\ell)(-1).$$

Consider the cohomology sequence

$$\cdots \to H_c^{n-1}(X_u \setminus Y, \mathbb{Q}_\ell) \to H^{n-1}(X_u, \mathbb{Q}_\ell) \to H^{n-1}(Y, \mathbb{Q}_\ell) \to \cdots$$

of

$$0 \to j_!j^*\mathbb{Q}_\ell \to \mathbb{Q}_\ell \to i_*i^*\mathbb{Q}_\ell \to 0.$$

Here Y_0 is the nonsingular hyperplane section of X_0, and so $X_0 \setminus Y_0$ is affine. The Poincaré duality theorem shows that $H_c^{n-1}(X_u \setminus Y, \mathbb{Q}_\ell) \approx H^{n+1}(X_u \setminus Y, \mathbb{Q}_\ell)^\vee$, which the weak Lefschetz theorem shows to be zero. Thus $H^{n-1}(X_u, \mathbb{Q}_\ell) \to H^{n-1}(Y, \mathbb{Q}_\ell)$ is injective. By induction $(*)$ holds for Y, and this implies that the eigenvalues of F on $H^2(\mathbb{P}^1, R^{n-1}\pi_*\mathbb{Q}_\ell)$ are algebraic numbers α with

$$q^{\frac{n}{2}} < |\alpha| < q^{\frac{n}{2}+1}.$$

The group $H^0(\mathbb{P}^1, R^{n+1}\pi_*\mathbb{Q}_\ell)$. Under our assumption that the vanishing cycles are nonzero, $R^{n+1}\pi_*\mathbb{Q}_\ell$ is constant, and $H^0(\mathbb{P}^1, R^{n+1}\pi_*\mathbb{Q}_\ell) = H^{n+1}(X_u, \mathbb{Q}_\ell)$. The weak Lefschetz theorem shows that the Gysin map

$$H^{n-1}(Y, \mathbb{Q}_\ell)(-1) \to H^{n+1}(X_u, \mathbb{Q}_\ell)$$

is surjective, and we can apply the induction hypothesis to Y again.

The group $H^2(\mathbb{P}^1, R^{n-1}\pi_*\mathbb{Q}_\ell)$. Finally, we have to treat $H^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell)$. Let $V = (R^n\pi_*\mathbb{Q}_\ell)_u$. We have a filtration

$$V \supset E \supset E \cap E^\perp \supset 0.$$

It follows from the Picard-Lefschetz formula that $\pi_1(U)$ acts trivially on V/E and on $E/E\cap E^\perp$. In particular, this is a filtration of $\pi_1(U)$-modules, and hence corresponds to a filtration of sheaves on U:

$$\mathcal{V} \supset \mathcal{E} \supset \mathcal{E} \cap \mathcal{E}^\perp \supset 0$$

On applying j_*, we get a filtration

$$\begin{array}{ccccccc} R^n\pi_*\mathbb{Q}_\ell & \supset & j_*\mathcal{E} & \supset & j_*(\mathcal{E}\cap\mathcal{E}^\perp) & \supset & 0. \\ & \text{constant} & & \text{interesting} & & \text{constant} & \end{array}$$

We are using that $R^n\pi_*\mathbb{Q}_\ell \xrightarrow{\approx} j_*j^*R^n\pi_*\mathbb{Q}_\ell$.

The quotient $E/E \cap E^\perp$. We wish to apply the Main Lemma (Theorem 30.6) to $E/E \cap E^\perp$.

LEMMA 33.3 (Rationality Lemma). *For all closed points $x \in U_0$, the action of F_x on $E/E\cap E^\perp$ is rational.*

PROOF. We defer the proof to the next subsection. (The proof is quite intricate; see Deligne 1974, §6). □

The cup-product pairing

$$H^n(X_u, \mathbb{Q}_\ell) \times H^n(X_u, \mathbb{Q}_\ell) \to H^{2n}(X_u, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell(-n)$$

is skew-symmetric (because n is odd) and nondegenerate. Because it is canonical (in fact, it arises from a canonical pairing on $R^n\pi_*\mathbb{Q}_\ell$), it is invariant under $\pi_1(U_0)$. Recall that $V = H^n(X_u, \mathbb{Q}_\ell)$, and so we can rewrite the pairing as

$$V \times V \to \mathbb{Q}_\ell(-n).$$

Now E is a $\pi_1(U_0)$-invariant subspace of V, and $E^\perp$ is the orthogonal complement of E for this pairing. Hence, the pairing induces a nondegenerate pairing

$$\psi\colon E/E\cap E^\perp \times E/E\cap E^\perp \to \mathbb{Q}_\ell(-n).$$

Finally, the theorem of Kazhdan and Margulis (32.6) shows that the image of $\pi_1(U)$ is open in $Sp(E/E\cap E^\perp, \psi)$. Thus, the Main Lemma (30.6) shows that the action of F on $j_*(\mathcal{E}/\mathcal{E}\cap\mathcal{E}^\perp)$ satisfies $*(n+1)$.

The constant quotients. Recall that $E/E \cap E^\perp$ is a simple π_1-module. Therefore, either it is nonzero (and no vanishing cycle is in $E \cap E^\perp$), or it is zero and $E \subset E^\perp$.

First case: No vanishing cycle is in $E \cap E^\perp$. This means that for any $s \in S$, E^{I_s} has codimension 1 in E. Since V^{I_s} has codimension 1 in V, and I_s acts trivially on V/E, the sequence

$$0 \to E \to V \to V/E \to 0$$

remains exact when we take I_s-invariants. This implies that when we apply j_* to the corresponding sequence of locally constant sheaves on U,

$$0 \to \mathcal{E} \to \mathcal{V} \to \mathcal{V}/\mathcal{E} \to 0,$$

the sequence we obtain on $\mathbb{P}^1$, namely,

$$0 \to j_*\mathcal{E} \to R^n\pi_*\mathbb{Q}_\ell \to j_*(\mathcal{V}/\mathcal{E}) \to 0,$$

is exact. Because $j_*(\mathcal{V}/\mathcal{E})$ is constant, the map

$$H^1(\mathbb{P}^1, j_*\mathcal{E}) \to H^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell)$$

is surjective, and so it suffices to prove $*(n+1)$ for $H^1(\mathbb{P}^1, j_*\mathcal{E})$.

Again, because E^{I_s} has codimension 1 in E, the sequence

$$0 \to E \cap E^\perp \to E \to E/E \cap E^\perp \to 0$$

remains exact when we take I_s-invariants. Therefore,

$$0 \to j_*(\mathcal{E} \cap \mathcal{E}^\perp) \to j_*\mathcal{E} \to j_*(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp) \to 0$$

is exact, and, because $j_*(\mathcal{E} \cap \mathcal{E}^\perp)$ is constant

$$H^1(\mathbb{P}^1, j_*\mathcal{E}) \to H^1(\mathbb{P}^1, j_*(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp))$$

is injective, and so $*(n+1)$ for $H^1(\mathbb{P}^1, j_*\mathcal{E})$ follows from $*(n+1)$ for $H^1(\mathbb{P}^1, j_*(\mathcal{E}/\mathcal{E} \cap \mathcal{E}^\perp))$.

Second case: The vanishing cycles are in $E \cap E^\perp$, i.e., $E \subset E^\perp$. In this case, we define $\mathcal{F}$ to be the quotient $R^n\pi_*\mathbb{Q}_\ell/j_*\mathcal{E}^\perp$. We then have exact sequences

$$0 \to j_*\mathcal{E}^\perp \to R^n\pi_*\mathbb{Q}_\ell \to \mathcal{F} \to 0$$

$$0 \to \mathcal{F} \to j_*j^*\mathcal{F} \to \oplus_s\mathbb{Q}_\ell(m-n)_s \to 0.$$

The sheaves $j_*\mathcal{E}^\perp$ and $j_*j^*\mathcal{F}$ are constant, and so the corresponding cohomology sequences are

$$0 \to H^1(\mathbb{P}^1, R^n\pi_*\mathbb{Q}_\ell) \to H^1(\mathbb{P}^1, \mathcal{F})$$

$$\oplus_{s\in S}\mathbb{Q}_\ell(m-n) \to H^1(\mathbb{P}^1, \mathcal{F}) \to 0.$$

As F acts on $\mathbb{Q}_\ell(m-n)$ as $q^{n-m} = q^{\frac{n+1}{2}}$.

This completes the proof of the Weil conjectures.

The proof of the rationality lemma. Omitted for the present.

The proof when the vanishing cycles vanish. Since the vanishing cycles are conjugate (up to sign) — see Theorem 32.3 — they are either all zero or all nonzero. In the above, we assumed that they are nonzero (see §32). In the case that the vanishing cycles are all zero, the proof is easier (but is omitted for the present).

34. The Geometry of Estimates

The Weil conjectures show that the number of solutions of a system of equations over a finite field is controlled by the topological properties of the complex variety defined by any lifting of the equations to characteristic zero.

In this section, I explain how estimates of the sizes of exponential (and similar) sums reflect properties of monodromy actions on certain sheaves.

For example, let X be a smooth projective surface over $\mathbb{F}_q$ and let $\pi\colon X \to \mathbb{P}^1$ be a regular map whose fibres X_t are elliptic curves, except for a for a finite number of $t \in \mathbb{F}$. For $t \in \mathbb{P}^1(\mathbb{F}_q)$, let

$$\#X_t(\mathbb{F}_q) = q - e(t) + 1.$$

If we ignore the singular fibres, then the Weil conjectures tell us

$$|2e(t)| < 2\sqrt{q}.$$

Are there similar estimates for

$$\sum_t e(t), \quad \sum_t e(t)^2, \quad \sum_t e(t)e(t+u), \quad \sum_{t_1+t_2=t_3+t_4} e(t_1)e(t_2)e(t_3)e(t_4)?$$

The sums are over the $t \in \mathbb{F}_q$. Analytic number theorists have a heuristic method for guessing estimates for such sums. For example, in $\sum e(t)$ there are q terms, each with size about $2\sqrt{q}$, and so trivially the sum is $\leq Cq^{\frac{3}{2}}$ for some constant C (independent of the power q of p). However, unless the family is constant, one expects the $e(t)$ to vary randomly, and this suggests

$$|\sum_t e(t)| \leq Cq.$$

This particular inequality has an elementary proof, but in general the results one wants are not obtainable by the methods of analytic number theory. The theorems of Deligne give a very powerful approach to obtaining such estimates. One interprets the sum as the trace of a Frobenius operator on the cohomology groups of a sheaf on a curve, and obtains the estimate as a consequence of an understanding of the geometry (monodromy) of the sheaf and Deligne's theorems. For example, the following results can be obtained in this fashion.

Theorem 34.1. *Let $e(t)$ be as above.*

(a) *If the j invariant of the family is not constant, then*

$$|\sum_t e(t)| \leq (\beta_2(X) - 2)q$$

where $\beta_2(X)$ is the second Betti number of X.

(b) *If j is not constant, then*

$$\sum_t e(t)^2 = q^2 + O(q^{\frac{3}{2}}).$$

(c) *Let S be the set of $t \in \mathbb{F}$ for which X_t is singular; if the sets S and $\{s-u \mid s \in S\}$ are disjoint, then*

$$\sum_t e(t)e(t+u) = O(q^{\frac{3}{2}}).$$

The fourth sum is much more difficult, but the following is known. Suppose that π is such that X_t is the intersection of the projective surface

$$X^3 + Y^3 + Z^3 = nW^3$$

with the plane

$$\alpha X + \beta Y + \gamma Z = tW.$$

Then, under some hypotheses on q,

$$\sum_{t_1+t_2=t_3+t_4} e(t_1)e(t_2)e(t_3)e(t_4) \geq 2q^4 - Bq^{7/2}$$

for[35] some constant B (Milne, J., Estimates from étale cohomology, Crelle 328, 1981, 208–220).

For comprehensive accounts of the applications of étale cohomology to the estimation of various sums, see:

Katz, N., Sommes exponentielles. Course taught at the University of Paris, Orsay, Fall 1979. With a preface by Luc Illusie. Notes written by Gérard Laumon. Astérisque, 79. Société Mathématique de France, Paris, 1980. 209 pp.

Katz, N., Gauss sums, Kloosterman sums, and monodromy groups. Annals of Mathematics Studies, 116. Princeton University Press, Princeton, NJ, 1988. x+246 pp.

The correspondence between number-theoretic estimates and the monodromy of sheaves can be used in both directions: N. Katz has used some estimates of Davenport and Lewis concerning the solutions of polynomials over finite fields to prove the following theorem (Monodromy of families of curves: applications of some results of Davenport-Lewis. Seminar on Number Theory, Paris 1979–80, pp. 171–195, Progr. Math., 12, Birkhäuser, Boston, Mass., 1981):

THEOREM 34.2. *Let $f(X,Y) \in \mathbb{C}[X,Y]$ be a polynomial in two variables. Suppose that for indeterminates a, b, c the complete nonsingular model of the affine curve*

$$f(X,Y) + aX + bY + c = 0$$

over the field $\mathbb{C}(a,b,c)$ has genus $g \geq 1$. Then for any nonempty Zariski open set $S \subset \mathbb{A}^3_{\mathbb{C}}$ over which the complete nonsingular model extends "nicely" to a morphism $f\colon C \to S$, the fundamental group of S acts absolutely irreducibly on a general stalk of $R^1 f_* \mathbb{Q}$ (higher direct image for the complex topology).

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[35] The inequality was proved at the request of C. Hooley, and allowed him to obtain an asymptotic estimate for the number of ways an integer can be written as the sum of 3 cubes and 2 squares, a problem he had worked on unsuccessfully for over 20 years. See his plenary talk at International Congress of Mathematicians, Warsaw 1983, which fails to acknowledge the crucial role played by étale cohomology in his final success.

The interplay between the number-theoretic estimates and the geometry of the étale sheaves is fascinating, but requires an understanding of both analytic number theory and étale cohomology for its full appreciation.

Review of Katz 1980 (MR 82m:10059). These notes use etale cohomology to prove two theorems of a very general nature concerning the sizes of exponential sums.

For any $q = p^n$, let $\psi\colon \mathbb{F}_q \to \mathbb{C}^\times$ be the additive character $a \mapsto \exp(\frac{2\pi i}{p}\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} a)$. Let V be a variety (or scheme of finite-type) over $\mathbb{F}_p$, and let $f\colon V \to \mathbb{A}^1_{\mathbb{F}_p}$ be a regular function on V. The exponential sum assocated with V, f, and $q = p^n$ is $S_q(V,f) = \sum \psi(f(x))$ where the sum is over all $x \in V(\mathbb{F}_q)$. For example, if V is defined by $XY = a$ and f is the function $(x,y) \mapsto x+y$, then

$$S_p(V,f) = \sum_{xy=a} \exp(\frac{2\pi i}{p}(x+y)) = \sum_{x\neq 0} \exp(\frac{2\pi i}{p}(x+\frac{a}{x}))$$

is a Kloosterman sum.

The first theorem treats the following question: suppose V is defined by equations with coefficients in $\mathbb{Z}$ and f is a mapping $V \to \mathbb{A}^1_{\mathbb{Z}}$, so that, for all p, there is a pair (V_p, f_p) over $\mathbb{F}_p$ obtained by reducing (V,f) modulo p; then is it possible to uniformly bound the sums $S_q(V_p, f_p)$? The answer given is that, for a fixed (V,f), there is a constant A such that for all sufficiently large p and all q divisible by such a p, $|S_q(V_p,f_p)| \leq Aq^N$, where N is the largest dimension of a geometric fibre of f and the generic fibre is assumed to be geometrically irreducible or have dimension $< N$. The proof uses standard arguments to interpret S_q as an alternating sum of traces of the Frobenius endomorphism on certain étale cohomology spaces, and then uses general results from étale cohomology and from P. Deligne [Inst. Hautes Etudes Sci. Publ. Math. No. 52 (1980), 137 - 252] to bound the dimensions and the weights, respectively, of the spaces.

The second theorem provides, for a fixed p, an explicit constant A in a situation where the geometry is particularly easy to handle. Consider for example a nonsingular projective surface V' defined by a homogeneous equation $P(X_0, X_1, X_2, X_3) = 0$ of degree D with coefficients in $\mathbb{F}_p$. Let H be the hyperplane defined by $X_0 = 0$ and let F be a linear form $a_1X_1 + a_2X_2 + a_3X_3$. Assume that $(D,p) = 1$, that $C = V' \cap H$ is a smooth curve of degree D, and that the plane $F = 0$ cuts C transversally. Let V be the affine surface $V' \smallsetminus C$ (defined by $P(1, X_1, X_2, X_3) = 0$) and let f be the function $(x_1, x_2, x_3) \mapsto \sum a_i x_i \colon V \to \mathbb{A}^1_{\mathbb{F}_p}$. Then it follows fairly directly from Deligne (loc. cit.) and the Grothendieck - Ogg - Shafarevich formula that $|S_q(V,f)| \leq D(D-1)^2 q$. In the notes, this result is generalized to the case that V' is any smooth projective variety, H is the hyperplane defined by an equation $s = 0$, and f is the function on $V = V' \smallsetminus V' \cap H$ defined by F/s^d where F is a homogeneous polynomial of degree d. There are similar primality and transversality assumptions, and the constant A is expressed in terms of various Euler - Poincaré characteristics.

The proofs of these two theorems occupy the last two chapters of the notes. The first three chapters, which will be accessible to those with only a limited knowledge of étale cohomology, contain the following: a brief general discussion of questions, both answered and unanswered, concerning exponential sums; a review of the L-series associated with exponential sums and examples where the Weil conjectures can be applied; an explanation of how to express exponential sums in terms of traces of endomorphisms on étale cohomology groups, and a statement of the main theorems.

James Milne (1-MI).

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