Some Consequences of the Riemann Hypothesis for Varieties over Finite Fields

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Abstract. We deduce from Deligne’s form of the Riemann hypothesis and the hard Lefschetz theorem in ℓ-adic cohomology the corresponding facts for any “reasonable” cohomology theory, in particular for crystalline cohomology, and give some applications to algebraic cycles.

I.

Let $X$ be a projective smooth absolutely irreducible variety of dimension $n$ over $\mathbb{F}_q$. Fix a prime number $\ell \neq p = \text{char}(\mathbb{F}_q)$, and denote by $H^i(X)$ the étale cohomology groups $H^i(\overline{X}, \mathbb{Q}_\ell)$, and by $F$ the Frobenius relative to $\mathbb{F}_q$. For any polynomial $g(T) = \Pi(1-\alpha_i T)$, and any integer $r \geq 1$, we denote by $g(T)^{(r)}$ the polynomial $\Pi(1-(\alpha_i)^r T)$.

Deligne has proven that:

D1. For every integer $i \geq 0$, the polynomial
$$P^i(X/\mathbb{F}_q, T) = \det(1 - TF|H^i(X))$$
lies in $\mathbb{Z}[T]$, and its reciprocal zeroes all have complex absolute value $q^{i/2}$.

D2. For every integer $d \geq 2$, and every Lefschetz pencil $\{X_t\}_{t \in \mathbb{P}^1}$ of hypersurface sections of degree $d$ of $X$, the polynomial $P^{n-1}(X/\mathbb{F}_q, T)$ may be reconstructed as the least common multiple of all complex polynomials $f(T)$ such that whenever $t \in \mathbb{F}_q^*$ is a parameter value such that $X_t$ is smooth, the polynomial $f(T)^{(r)}$ divides $P^{n-1}(X_t/\mathbb{F}_q^r, T)$.

D3. Let $L \in H^2(X)$ denote the class of a hyperplane. Then for $i \leq n$, $L^i: H^{n-i}(X) \to H^{n+i}(X)$ is an isomorphism.

We should point out that although D3 is a consequence of D2 in any “reasonable” theory (as we shall see), Deligne deduced D2 from D3 via his monodromy techniques.

II.

Now let $\mathbb{H}$ be any cohomology theory defined for projective smooth absolutely irreducible varieties over finite extensions of $\mathbb{F}_p$ with values in finite-dimensional graded anticommutative algebras over a coefficient field $K$ of characteristic zero, which satisfies

Poincaré Duality. Let $X/\mathbb{F}_q$ be as above, $n = \text{dim } X$. Then $\mathbb{H}^{2n}(X)$ is one-dimensional, $\mathbb{H}^i(X) \otimes \mathbb{H}^{2n-i} \to \mathbb{H}^{2n}(X)$ is a perfect pairing, and
Frobenius $F$ relative to $\mathbb{F}_q$ acts as multiplication by $q^n$ [this implies that $F$ is an automorphism of each $\mathcal{H}^i(X)$].

Weak Lefschetz. Given $X$, there is an integer $d_0 = d_0(X)$ such that if $f: Y \hookrightarrow X$ is any smooth hypersurface section of $X$ of degree $d \geq d_0$, then $f^*: \mathcal{H}^i(X) \rightarrow \mathcal{H}^i(Y)$ is an isomorphism for $i \leq n - 2$, and is injective for $i = n - 1$.

Zeta-Function Formula. For $X$ as above, let

$$\mathcal{P}^i(X/\mathbb{F}_q, T) = \det(1 - TF|\mathcal{H}^i(X)).$$

Then the zeta function $Z(X/\mathbb{F}_q, T)$ is given by the formula

$$Z(X/\mathbb{F}_q, T) = \prod_{i=0}^{2n} (\mathcal{P}^i(X/\mathbb{F}_q, T))(-1)^{i+1}.$$  

We should remark that $\ell$-adic cohomology, $\ell = p$, and crystalline cohomology are such theories!

**Theorem 1.** For any theory $\mathcal{H}$ as above, for every $X/\mathbb{F}_q$ as above, we have

$$\mathcal{P}^i(X/\mathbb{F}_q, T) = P^i(X/\mathbb{F}_q, T) \quad \text{for every } i.$$

**Proof.** It suffices to prove the equality after an arbitrary extension of scalars from $\mathbb{F}_q$ to $\mathbb{F}_q^*$, i.e. to prove that $\mathcal{P}^{i(d)} = P^{i(d)}$ for some $d \geq 1$. For then the reciprocal zeroes of each $\mathcal{P}^i$ will be algebraic integers all of whose conjugates have complex absolute value $q^{1/2}$, and the cohomological expression of the zeta function in the theory $\mathcal{H}$ shows that for $i$ odd (resp. for $i$ even), the reciprocal roots of $\mathcal{P}^i$ are precisely those reciprocal zeroes (resp. poles) of the zeta function of $X/\mathbb{F}_q$ all of whose conjugates have complex absolute value $q^{1/2}$. As the reciprocal roots of $P^i$ admit the same description, we have $\mathcal{P}^i = P^i$.

The proof proceeds by induction on $n = \dim X$. At the expense of an extension of scalars, we may choose a Lefschetz pencil $\{X_t\}$ of hypersurface sections of high ($\geq d_0(X)$) degree defined over $\mathbb{F}_q$, such that at least one of the sections $X_{t_0}$, $t_0 \in \mathbb{F}_q$, is smooth. Using the weak Lefschetz theorem in both theories (for $X_{t_0} \hookrightarrow X$), and induction, we have the equality $\mathcal{P}^i = P^i$ for $i \leq n - 2$, from which it follows for $i \geq n + 2$ by Poincaré duality. Again by the weak Lefschetz theorem, for every parameter value $t \in \mathbb{F}_q$ such that $X_t$ is smooth, we have

$$\mathcal{P}^{n-1}(X/\mathbb{F}_q, T)^{(r)} \mid \text{divides} \mid \mathcal{P}^{n-1}(X_t/\mathbb{F}_q^r, T)$$

by induction

$$P^{n-1}(X_t/\mathbb{F}_q^r, T).$$
Hence by D2, it follows that $\mathcal{P}^{n-1}(X/\mathbb{F}_q, T)$ divides $P^{n-1}(X/\mathbb{F}_q, T)$. By Poincaré Duality, this implies that $\mathcal{P}^{n+1}(X/\mathbb{F}_q, T)$ divides $P^{n+1}(X/\mathbb{F}_q, T)$.

If we equate the cohomological expressions of the zeta function of $X/\mathbb{F}_q$:

$$\Pi((P^i(X/\mathbb{F}_q, T))^{(-1)^{i+1}} = \Pi((\mathcal{P}^i(X/\mathbb{F}_q, T))^{(-1)^{i+1}}$$

then we may cancel the terms with $i \leq n-2$ and $i \geq n+2$, cross-multiply and get

$$R^{n-1} \cdot R^{n+1} = R^n$$

where $R^i = \frac{P^i}{\mathcal{P}^i}$.

This shows that $\mathcal{P}^n$ divides $P^n$. By the Riemann hypothesis D1, the absolute values of the reciprocal zeroes of these three polynomials $R^{n-1}, R^{n+1}, R^n$ are respectively $q^{\frac{n-1}{2}}, q^{\frac{n+1}{2}}, q^n$. Thus the equality $R^{n-1} \cdot R^{n+1} = R^n$ is impossible unless $R^{n-1} = R^{n+1} = R^n = 1$, whence $P^i = \mathcal{P}^i$ for every $i$. QED

**Corollary 1.**

1) $\dim_k \mathcal{H}^i(X) = \dim_{\mathbb{F}_q} H^i(X)$.

2) Deligne’s theorems D1, D2, D3 hold with $H^i$ and $P^i$ replaced by $\mathcal{H}^i$ and $\mathcal{P}^i$.

**Proof:** The first statement follows from the theorem by equating the degrees of $P^i$ and $\mathcal{P}^i$. As D1 and D2 are statements about the $P^i$, they are also true for the $\mathcal{P}^i$. To conclude, we must explain how D3 follows from D2 in any theory $\mathcal{H}$. Let $f: Y \to X$ be the inclusion of a smooth hypersurface section of high degree, defined over $\mathbb{F}_{q^r}$. We must show that for $1 \leq i \leq n$, the bilinear form $(a, b) \mapsto a b L_i^i$ on $\mathcal{H}^{n-1}(X)$ is non-degenerate. For $2 \leq i \leq n$ this follows from D3 on $Y$ by weak Lefschetz and the projection formula $a b L_i^i = f_*(f^*(a) f^*(b) L_i^i)$, valid because $L = f_*(1)$. Let $I = \text{image}(f^*): \mathcal{H}^{n-1}(X) \to \mathcal{H}^{n-1}(Y)$, and let $I^\perp \subset \mathcal{H}^{n-1}(Y)$ be its orthogonal. It remains to show that $I \cap I^\perp = 0$, i.e. that cup-product is non-degenerate on $I$. Consider the exact sequence

$$0 \to I \cap I^\perp \to I \oplus I^\perp \to \mathcal{H}^{n-1}(Y) \to \mathcal{H}^{n-1}(Y)/I + I^\perp \to 0.$$ 

Denoting by $Z(\ )$ the characteristic polynomial of Frobenius relative to $\mathbb{F}_{q^r}$, we obtain the polynomial identity

$$Z(\mathcal{H}^{n-1}(Y)) \cdot Z(I \cap I^\perp) = Z(I) \cdot Z(I^\perp) \cdot Z(\mathcal{H}^{n-1}(Y)/I + I^\perp),$$

or more conveniently.

$$(*) \quad \mathcal{P}^{n-1}(Y/\mathbb{F}_{q^r}, T) \overset{\text{def}}{=} Z(\mathcal{H}^{n-1}(Y)) = Z(I) \cdot Z(I^\perp/I \cap I^\perp) \cdot Z(\mathcal{H}^{n-1}(Y)/I + I^\perp).$$

Notice that $\mathcal{H}^{n-1}(Y)/I + I^\perp$ is dual to $I \cap I^\perp$, and $I \cap I^\perp$ is isomorphic by $f^*$ with Ker$(L: \mathcal{H}^{n-1}(X) \to \mathcal{H}^{n+1}(X))$. Thus if we write

$$\det(1 - TF|\text{Ker } L) = \Pi(1 - \alpha_i T),$$
and define \( g(T) = \Pi (1 - (q^{n-1}/x_i) T) \), then (recalling that we are over \( \mathbb{F}_q \)) we obtain the formula

\[
Z(\mathcal{H}^{n-1}(Y)/I + I^1) = g(T)^{(r)}.
\]

Again because we are over \( \mathbb{F}_q \), we have

\[
Z(I) = \mathcal{P}^{n-1}(X/\mathbb{F}_q, T)^{(r)}.
\]

Using these last two formulas to substitute into (\( \ast \)), we see that \( \mathcal{P}^{n-1}(Y/\mathbb{F}_q, T) \) is divisible by \( g(T)^{(r)} \cdot \mathcal{P}^{n-1}(X/\mathbb{F}_q, T)^{(r)} \). Replacing if necessary \( q \) by \( q^d \), and letting \( Y \) vary in a Lefschetz pencil defined over \( \mathbb{F}_q \), this contradicts D2 unless \( g(T)^{(d)} = 1 \), i.e., unless

\[
\text{Ker}(L: \mathcal{H}^{n-1}(X) \to \mathcal{H}^{n+1}(X))
\]

is zero. QED

**Corollary** (Ogus). If \( f: Y \to X \) is the inclusion of a smooth hypersurface section of any degree, then \( f^*: \mathcal{H}^i(X) \to \mathcal{H}^i(Y) \) is an isomorphism if \( i \leq n - 2 \), and injective for \( i = n - 1 \).

**Proof.** By the weak Lefschetz theorem in \( \ell \)-adic cohomology and Corollary I, 1), we know that for \( i \leq n - 2 \), \( \dim \mathcal{H}^i(X) = \dim \mathcal{H}^i(Y) \), so it suffices to show that for \( i \leq n - 1 \), \( f^*: \mathcal{H}^i(X) \to \mathcal{H}^i(Y) \) is injective. This follows from the commutative diagram

\[
\begin{array}{ccc}
\mathcal{H}^i(X) & \xrightarrow{f^*} & \mathcal{H}^i(Y) \\
\downarrow \mathcal{P}^{n-i} & & \downarrow \mathcal{P}^{n-i} \\
\mathcal{H}^{2n-i}(X)
\end{array}
\]

in which the oblique arrow is an isomorphism by D3.

### III. Application to Cycles

**Theorem 2.** 1) Assume further that \( \mathcal{H} \) is either a "Weil cohomology" in the sense of [3], or is crystalline cohomology tensored with the fraction field of the Witt vectors of the algebraic closure of \( \mathbb{F}_q \) (in the crystalline theory, the "class of an algebraic cycle" is presently defined only for smooth subvarieties). Let \( X \) be a projective smooth absolutely irreducible variety over \( \mathbb{F}_q \) of dimension \( n \). Then the Küneth components of the diagonal \( \Delta \) on \( X \times X \) are rationally algebraic cycles independent of the theory \( \mathcal{H} \); in fact they are \( \mathbb{Q} \)-linear combination of the graphs of Frobenius and its iterates.

2) If \( \mathcal{H} \) is a Weil cohomology, then for any integrally algebraic cycle \( Z \) on \( X \times X \) of codimension \( n \), the induced endomorphism of each \( \mathcal{H}^i(X) \) has a characteristic polynomial which lies in \( \mathbb{Z}[T] \) and is independent
of the theory $\mathcal{H}$. For any integrally algebraic cycle $Z$ on $X \times X$, the characteristic polynomial of the induced total endomorphism of $\bigoplus_i \mathcal{H}^i(X)$ lies in $\mathbb{Z}[T]$ and is independent of the theory $\mathcal{H}$.

Proof. 1) By D1 and Theorem 1, it follows that the polynomials $G^i(T) = \det(T1 - F|\mathcal{H}^i(X))$ are pairwise relatively prime in $\mathbb{Q}[T]$. Hence for each $i$ we can find a polynomial $P^i(T) \in \mathbb{Q}[T]$ which is divisible by $G^j(T)$ for $j \neq i$, and which is congruent to 1 modulo $G^i(T)$. Letting $F$ denote the graph of Frobenius, it follows from the Cayley-Hamilton theorem that the rationally algebraic cycle $P^i(F)$ defines the endomorphism "projection onto $\mathcal{H}^i(X)$" of $\bigoplus_j \mathcal{H}^j(X)$. The second assertion 2) follows from 1) for any Weil cohomology, cf. [3, Prop. 2.6]. QED

References


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