SUPERSINGULAR CURVES ON PICARD MODULAR SURFACES MODULO AN INERT PRIME

EHUD DE SHALIT AND EYAL Z. GOREN

Abstract. We study the supersingular curves on Picard modular surfaces modulo a prime p which is inert in the underlying quadratic imaginary field. We analyze the automorphic vector bundles in characteristic p, and as an application derive a formula relating the number of irreducible components in the supersingular locus to the second Chern class of the surface.

The characteristic p fibers of Shimura varieties of PEL type admit special subvarieties over which the abelian varieties which they parametrize become supersingular. These special subvarieties and the automorphic vector bundles over them have become the subject of extensive research in recent years. The purpose of this note is to analyze one of the simplest examples beyond Shimura curves, namely Picard modular surfaces, at a prime p which is inert in the underlying quadratic imaginary field. The supersingular locus on these surfaces has been studied in depth by Bültel, Vollaard and Wedhorn ([6],[31],[32]) and our debt to these authors should be obvious. It consists of a collection of Fermat curves of degree $p+1$, intersecting transversally at the *superspecial* points. We complement their work by analyzing the automorphic vector bundles in characteristic p, in relation to the supersingular strata. Although some of our results (e.g. the construction of a Hasse invariant, see [12]) have been recently generalized to much larger classes of Shimura varieties, focusing on this simple example allows us to give more details. As an example, we derive the following theorem (3.3).

Theorem 0.1. *Let $\mathcal{K}$ be a quadratic imaginary field and let $\bar{S}$ be the Picard modular surface of level $N \geq 3$ associated with $\mathcal{K}$ (for a precise definition see the text below). Let p be a prime which is inert in $\mathcal{K}$ and relatively prime to $2N$. Then the number of irreducible components in the supersingular locus of $\bar{S} \bmod p$ is $c_2(\bar{S})/3$, where $c_2(\bar{S})$ is the second Chern class of $\bar{S}$.*

Thanks to results of Holzapfel, this number is expressed in terms of the value of the L-function $L(s, (D_{\mathcal{K}}/\cdot))$ at $s = 3$, see Theorem 1.2.

Although the uniformization of the supersingular locus by Rapoport-Zink spaces yields a group-theoretic parametrization of the irreducible components by a certain double coset space, this parametrization in itself is not sufficient to yield the theorem. Our proof goes along different lines, invoking intersection theory on $\bar{S}$.

Note that the number of irreducible components comes out to be independent of p. A similar result was obtained in [2] for Hilbert modular surfaces. This is

Date: February 22, 2015.
1991 *Mathematics Subject Classification.* 40.040, 60.060.
Key words and phrases. Picard surface, supersingular locus, Hasse invariant.

different from the situation with the Siegel modular variety A_g. We are still lacking a conceptual understanding of this independence of p.

The first section introduces notation and background. The second contains most of the analysis modulo p. The Picard surface modulo p has 3 strata: the μ-ordinary (generic) stratum $\bar{S}_\mu$, the general supersingular locus S_{gss} and the superspecial points S_{ssp}. There are two basic automorphic vector bundles to consider, a rank 2 bundle $\mathcal{P}$ and a line bundle $\mathcal{L}$. Of particular importance are the Verschiebung homomorphisms $V_\mathcal{P} : \mathcal{P} \to \mathcal{L}^{(p)}$ and $V_\mathcal{L} : \mathcal{L} \to \mathcal{P}^{(p)}$. It turns out that outside the superspecial points $V_\mathcal{P}$ and $V_\mathcal{L}$ are both of rank 1, but that the supersingular locus is *defined* by the equation $V_\mathcal{P}^{(p)} \circ V_\mathcal{L} = 0$. The Hasse invariant $h_{\bar{\Sigma}} = V_\mathcal{P}^{(p)} \circ V_\mathcal{L}$ is a canonical global section of $\mathcal{L}^{p^2-1}$ whose divisor is $S_{ss} = S_{gss} \cup S_{ssp}$. On S_{ss}, in turn, we construct a canonical section h_{ssp} of $\mathcal{L}^{p^3+1}$, which vanishes precisely at the superspecial points S_{ssp} (to a high order). This secondary Hasse invariant is related to more general work of Boxer [5].

In the last section we use the information carried by h_{ssp}, together with some intersection theoretic computations, to prove the above theorem.

We would like to thank the referee for a thorough reading of the manuscript.

1. Background

1.1. Notation. We recall some classical facts about unitary groups in three variables and set up some notation.

1.1.1. *The quadratic imaginary field.* Let $\mathcal{K}$ be an imaginary quadratic field, contained in $\mathbb{C}$. We denote by $\Sigma : \mathcal{K} \hookrightarrow \mathbb{C}$ the inclusion and by $\bar{\Sigma} : \mathcal{K} \hookrightarrow \mathbb{C}$ its complex conjugate. We use the following notation:

- $d_\mathcal{K}$ - the square free integer such that $\mathcal{K} = \mathbb{Q}(\sqrt{d_\mathcal{K}})$.
- $D_\mathcal{K}$ - the discriminant of $\mathcal{K}$, equal to $d_\mathcal{K}$ if $d_\mathcal{K} \equiv 1 \bmod 4$ and $4d_\mathcal{K}$ if $d_\mathcal{K} \equiv 2, 3 \bmod 4$.
- $\delta_\mathcal{K} = \sqrt{\mathcal{D}_\mathcal{K}}$ - the square root with positive imaginary part, a generator of the different of $\mathcal{K}$, sometimes simply denoted δ.
- $\omega_\mathcal{K} = (1 + \sqrt{d_\mathcal{K}})/2$ if $d_\mathcal{K} \equiv 1 \bmod 4$, otherwise $\omega_\mathcal{K} = \sqrt{d_\mathcal{K}}$, so that $\mathcal{O}_\mathcal{K} = \mathbb{Z} + \mathbb{Z}\omega_\mathcal{K}$.
- $\bar{a}$ - the complex conjugate of $a \in \mathcal{K}$.
- $\mathrm{Im}_\delta(a) = (a - \bar{a})/\delta$, for $a \in \mathcal{K}$.

We fix an integer $N \geq 3$ (the "tame level") and let $R_0 = \mathcal{O}_\mathcal{K}[1/2d_\mathcal{K}N]$. This is our *base ring*. If R is any R_0-algebra and M is any R-module with $\mathcal{O}_\mathcal{K}$-action, then M becomes an $\mathcal{O}_\mathcal{K} \otimes R$-module and we have a canonical *type decomposition*

$$M = M(\Sigma) \oplus M(\bar{\Sigma}) \tag{1.1}$$

where $M(\Sigma) = e_\Sigma M$ and $M(\bar{\Sigma}) = e_{\bar{\Sigma}} M$, and where the idempotents e_Σ and $e_{\bar{\Sigma}}$ are defined by

$$e_\Sigma = \frac{1 \otimes 1}{2} + \frac{\delta \otimes \delta^{-1}}{2}, \quad e_{\bar{\Sigma}} = \frac{1 \otimes 1}{2} - \frac{\delta \otimes \delta^{-1}}{2}. \tag{1.2}$$

Then $M(\Sigma)$ (resp. $M(\bar{\Sigma})$) is the part of M on which $\mathcal{O}_\mathcal{K}$ acts via Σ (resp. $\bar{\Sigma}$). The same notation will be used for sheaves of modules on R-schemes, endowed with an $\mathcal{O}_\mathcal{K}$ action. If M is locally free, we say that it has *type* (p, q) if $M(\Sigma)$ is of rank p and $M(\bar{\Sigma})$ is of rank q.

1.1.2. *The unitary group.* Let $V = \mathcal{K}^3$ and endow it with the hermitian pairing

$$(u,v) = {}^t\bar{u}\begin{pmatrix} & & \delta^{-1} \\ & 1 & \\ -\delta^{-1} & & \end{pmatrix} v. \tag{1.3}$$

We identify $V_{\mathbb{R}}$ with $\mathbb{C}^3$ ($\mathcal{K}$ acting via the natural inclusion Σ). It then becomes a hermitian space of signature $(2,1)$. Conversely, any 3-dimensional hermitian space over $\mathcal{K}$ whose signature at the infinite place is $(2,1)$ is isomorphic to V after rescaling the hermitian form by a positive rational number.

Let

$$\mathbf{G} = \mathbf{U}(V, (,)) \tag{1.4}$$

be the unitary group of V, regarded as an algebraic group over $\mathbb{Q}$. For any $\mathbb{Q}$-algebra A we have

$$\mathbf{G}(A) = \{g \in GL_3(A \otimes \mathcal{K}) |\ (gu, gv) = (u,v)\ \forall u, v \in V_A\}. \tag{1.5}$$

We write $G = \mathbf{G}(\mathbb{Q})$, $G_\infty = \mathbf{G}(\mathbb{R})$ and $G_p = \mathbf{G}(\mathbb{Q}_p)$.

We also introduce an alternating $\mathbb{Q}$-linear pairing $\langle,\rangle : V \times V \to \mathbb{Q}$ defined by $\langle u, v\rangle = \mathrm{Im}_\delta(u,v)$. We then have the formulae

$$\langle au, v\rangle = \langle u, \bar{a}v\rangle, \quad 2(u,v) = \langle u, \delta v\rangle + \delta\langle u, v\rangle. \tag{1.6}$$

We call $\langle,\rangle$ the *polarization form.*

1.1.3. *The hermitian symmetric domain.* The group G_∞ acts on $\mathbb{P}^2_{\mathbb{C}} = \mathbb{P}(V_{\mathbb{R}})$ by projective linear transformations, and preserves the open subdomain $\mathfrak{X}$ of negative definite lines (in the metric $(,)$). This $\mathfrak{X}$ is biholomorphic to the open unit ball in $\mathbb{C}^2$, and G_∞ acts on it transitively. Every negative definite line is represented by a unique vector ${}^t(z, u, 1)$, and such a vector represents a negative definite line if and only if

$$\lambda(z,u) \stackrel{\mathrm{def}}{=} \mathrm{Im}_\delta(z) - u\bar{u} > 0. \tag{1.7}$$

One refers to the realization of $\mathfrak{X}$ as the set of points $(z,u) \in \mathbb{C}^2$ satisfying this inequality as a *Siegel domain of the second kind.* It is convenient to think of the point $x_0 = (\delta/2, 0)$ as the "center" of $\mathfrak{X}$.

If we let K_∞ be the stabilizer of x_0 in G_∞, then K_∞ is compact and isomorphic to $U(2) \times U(1)$. Since G_∞ acts transitively on $\mathfrak{X}$, we may identify $\mathfrak{X}$ with G_∞/K_∞.

1.2. **Picard modular surfaces over $\mathbb{C}$.**

We next recall some classical facts about Picard modular surfaces.

1.2.1. *Lattices and their arithmetic groups.* Let $L \subset V$ be the lattice

$$L = Span_{\mathcal{O}_\mathcal{K}}\{\delta e_1, e_2, e_3\}. \tag{1.8}$$

This L is *self-dual* in the sense that

$$L = \{u \in V |\ \langle u, v\rangle \in \mathbb{Z}\ \forall v \in L\}. \tag{1.9}$$

Equivalently, L is its own $\mathcal{O}_\mathcal{K}$-dual with respect to the hermitian pairing $(,)$.

With $N \geq 3$ as before let

$$\Gamma = \Gamma(N) = \{\gamma \in G |\ \gamma L = L \text{ and } \gamma(u) \equiv u \bmod NL\ \forall u \in L\}. \tag{1.10}$$

Then Γ is a discrete torsion-free subgroup of G which acts freely and faithfully on $\mathfrak{X}$. As $N \geq 3$, it can be seen that $\Gamma \subset SU(V, (,))$, i.e. $\det \gamma = 1$ for all $\gamma \in \Gamma$.

1.2.2. *The cusps.* Let $\mathcal{C}$ be the set of isotropic lines in V. Equivalently, $\mathcal{C}$ may be described as the set of vectors ${}^t(z, u, 1)$ with $z, u \in \mathcal{K}$ and $\lambda(z, u) = 0$, together with the unique cusp at infinity $c_\infty = {}^t(1, 0, 0)$, or simply as the $\mathcal{K}$-rational points on $\partial\mathfrak{X}$. The group Γ acts on $\mathcal{C}$ and its orbits $\mathcal{C}_\Gamma = \Gamma\backslash\mathcal{C}$ are called the *cusps* of Γ. The set $\mathcal{C}_\Gamma$ is finite.

1.2.3. *Picard modular surfaces and their smooth compactifications.* We denote by X_Γ the complex surface $\Gamma\backslash\mathfrak{X}$. Since the action of Γ on $\mathfrak{X}$ is free, X_Γ is smooth. For the following theorem see, for example, [7].

Theorem 1.1. *There exists a smooth projective complex surface $\bar{X}_\Gamma$ containing X_Γ as a dense open subset. The irreducible components of $\bar{X}_\Gamma - X_\Gamma$ are elliptic curves with complex multiplication by $\mathcal{O}_\mathcal{K}$, and are in a one-to-one correspondence with $\mathcal{C}_\Gamma$. These conditions determine $\bar{X}_\Gamma$ uniquely.*

Holzapfel studied the Chern classes of the surface $\bar{X}_\Gamma$. He obtained the following result [19, (5A.4.3), p.325].

Theorem 1.2. *Let $c_2(\bar{X}_\Gamma)$ be the second Chern class of $\bar{X}_\Gamma$, which is equal also to the Euler characteristic*

$$e(\bar{X}_\Gamma) = \sum_{i=0}^{4} (-1)^i \dim H^i(\bar{X}_\Gamma, \mathbb{C}). \tag{1.11}$$

Then

$$c_2(\bar{X}_\Gamma) = [\Gamma(1) : \Gamma(N)] \cdot \frac{3|D_\mathcal{K}|^{5/2}}{32\pi^3} L\left(3, \left(\frac{D_\mathcal{K}}{\cdot}\right)\right). \tag{1.12}$$

Here $\Gamma(N) = \Gamma$ and

$$\Gamma(1) = \{\gamma \in G |\, \gamma L = L,\, \det \gamma = 1\}. \tag{1.13}$$

From the functional equation of the L-function $L(s, (D_\mathcal{K}/\cdot))$ we also get

$$c_2(\bar{X}_\Gamma) = -[\Gamma(1) : \Gamma(N)] \cdot \frac{3}{16} L\left(-2, \left(\frac{D_\mathcal{K}}{\cdot}\right)\right). \tag{1.14}$$

Note that the index $[\Gamma(1) : \Gamma(N)] = [vol(\Gamma(N)\backslash\mathfrak{X}) : vol(\Gamma(1)\backslash\mathfrak{X})]$, *except* if $D_\mathcal{K} = -3$, when $\Gamma(1)$ does not act faithfully on $\mathfrak{X}$ as it contains the roots of unity of order 3 in its center. In this case $[\Gamma(1) : \Gamma(N)]$ is equal to 3 times the volume ratio. This accounts for the factor ε in [19, (5A.4.3)].

1.2.4. *The Shimura variety.* The surface X_Γ is a connected component of a certain complex Shimura variety. To describe it let $\tilde{\mathbf{G}}$ be the group of unitary similtudes of $(V, (,))$

$$\tilde{\mathbf{G}} = \mathbf{GU}(V, (,)). \tag{1.15}$$

Let $\tilde{K}_f^0 \subset \tilde{\mathbf{G}}(\mathbb{A}_f)$ be the subgroup stabilizing $\widehat{L} = L \otimes \widehat{\mathbb{Z}}$ and $\tilde{K}_f \subset \tilde{K}_f^0$ the subgroup of elements which furthermore induce the identity on L/NL. Let $\tilde{K}_\infty$ be the stabilizer of $x_0 = (\delta/2, 0)$ in $\tilde{G}_\infty = \tilde{\mathbf{G}}(\mathbb{R})$, and $\tilde{K} = \tilde{K}_f\tilde{K}_\infty$. We have the identification

$$\tilde{G}_\infty/\tilde{K}_\infty = G_\infty/K_\infty = \mathfrak{X}. \tag{1.16}$$

The Shimura variety $Sh_{\tilde{K}}$ is a complex quasi-projective variety identified (as a complex manifold) with the double coset space

$$\begin{aligned} Sh_{\tilde{K}}(\mathbb{C}) &\simeq \tilde{\mathbf{G}}(\mathbb{Q})\backslash\tilde{\mathbf{G}}(\mathbb{A})/\tilde{K} \\ &= \tilde{\mathbf{G}}(\mathbb{Q})\backslash(\mathfrak{X}\times\tilde{\mathbf{G}}(\mathbb{A}_f)/\tilde{K}_f). \end{aligned} \tag{1.17}$$

See [8],[28], or the survey paper [13]. The connected components of $Sh_{\tilde{K}}$ are of the form $\Gamma_j\backslash\mathfrak{X}$ where the Γ_j are discrete and torsion-free, and one of the Γ_j can be taken to be the Γ which we have fixed above.

By the general theory of Shimura varieties, $Sh_{\tilde{K}}$ admits a *canonical model* over its reflex field. In our case, the reflex field turns out to be the field $\mathcal{K}$, and we denote the canonical model by $S_{\mathcal{K}}$. The irreducible components of $Sh_{\tilde{K}}$ are not defined over $\mathcal{K}$, but only over the ray class field $\mathcal{K}_N$ of conductor N over $\mathcal{K}$. More precisely, each irreducible component of $S_{\mathcal{K}_N}=S_{\mathcal{K}}\times_{\mathcal{K}}\mathcal{K}_N$ is already geometrically irreducible. This follows, for example, from the description of $\pi_0(Sh_{\tilde{K}})$ and the Galois action on it given in [8].

1.3. **The Picard modular surface over the ring R_0 and its arithmetic compactification.**

1.3.1. *The moduli problem.* The canonical model $S_{\mathcal{K}}$ of $Sh_{\tilde{K}}$ has an interpretation as a fine moduli scheme classifying quadruples $(A,\lambda,\iota,\alpha)_{/R}$ where R is a $\mathcal{K}$-algebra and:

- A is an abelian three-fold over R,
- $\lambda : A\to A^t$ is a principal polarization,
- $\iota:\mathcal{O}_{\mathcal{K}}\hookrightarrow End(A/R)$ is an embedding on which the Rosati involution induced by λ is given by $\iota(a)\mapsto\iota(\bar{a})$, and which makes $Lie(A)$ into a locally free R-module of type $(2,1)$,
- α is a level-N structure.

For a precise definition of what one means by a "level N structure" and a full discussion see [23], and also the earlier references [25], [26], [3] and [13]. We write $\mathcal{M}(R)$ for the set of isomorphism classes of such quadruples. Thus $S_{\mathcal{K}}$ represents the functor $\mathcal{M}(-)$ from the category of $\mathcal{K}$-algebras to the category of sets.

The moduli problem $\mathcal{M}$ makes sense, and is representable, already over R_0.

Definition 1.1. *Let S be the fine moduli scheme representing $\mathcal{M}$ over R_0.*

The scheme S is smooth of relative dimension 2 over R_0, and $S_{\mathcal{K}}$ is its generic fiber. This allows us to consider the reduction of S modulo primes of R_0 (i.e. primes of $\mathcal{K}$ not dividing $2d_{\mathcal{K}}N$).

1.3.2. *The arithmetic compactification.* Larsen's thesis was the first source to work out the smooth compactification $\bar{S}$ of S over R_0. See [25], with complements and corrections in [3],[4] and [23]. Let R_N be the integral closure of R_0 in the ray class field $\mathcal{K}_N$. Then the irreducible components of $\bar{S}_{R_N}-S_{R_N}$ are already geometrically irreducible, and are elliptic curves (over R_N) with complex multiplication by $\mathcal{O}_{\mathcal{K}}$.

1.3.3. *The universal semi-abelian variety and the automorphic vector bundles.* By its very definition as a fine moduli scheme, the scheme S carries a universal abelian three-fold $\mathcal{A}/S$ (with an additional structure given by λ,ι and α). This $\mathcal{A}$ extends to a semi-abelian scheme (still denoted $\mathcal{A}$) over $\bar{S}$. In fact, Larsen and Bellaïche give the boundary $C = \bar{S} - S$ an interpretation as a moduli space of certain semi-abelian schemes with additional structure, and $\mathcal{A}/C$ is the universal object arising from this interpretation. (The construction of $\mathcal{A}$ over the whole of $\bar{S}$ requires, of course, a little more effort than its construction over S and C separately.) The abelian part of $\mathcal{A}/C$ is an elliptic curve B with complex multiplication by $\mathcal{O}_{\mathcal{K}}$ and CM type Σ. Its toric part is a torus of the form $\mathfrak{a} \otimes \mathbb{G}_m$. Both B and the invertible $\mathcal{O}_{\mathcal{K}}$-module $\mathfrak{a}$ are locally constant along C, but the extension class of B by $\mathfrak{a} \otimes \mathbb{G}_m$ varies continuously. For details, see [3].

What is important for us is that the relative cotangent space at the origin

$$\omega_{\mathcal{A}/\bar{S}} = e^* \Omega^1_{\mathcal{A}/\bar{S}} \tag{1.18}$$

($e : \bar{S} \to \mathcal{A}$ being the zero section) is a locally free sheaf on $\bar{S}$ endowed with an $\mathcal{O}_{\mathcal{K}}$ action. It therefore breaks into a direct sum

$$\omega_{\mathcal{A}/\bar{S}} = \mathcal{P} \oplus \mathcal{L} \tag{1.19}$$

where $\mathcal{P} = \omega_{\mathcal{A}/\bar{S}}(\Sigma)$ is a plane bundle and $\mathcal{L} = \omega_{\mathcal{A}/\bar{S}}(\bar{\Sigma})$ is a line bundle. See (1.1) for the notation. Here we use the signature assumption in the moduli problem.

Along the boundary C of S the semi-abelian structure on $\mathcal{A}/C$ provides a filtration

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0 \tag{1.20}$$

where $\mathcal{P}_0 = \omega_{B/C}$ is the cotangent space at the origin of the abelian part of $\mathcal{A}$ and $\mathcal{P}_\mu$ the Σ-component of the cotangent space of the toric part. The line bundles $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{L}$ are trivial along C, but the extension $\mathcal{P}$ is non-trivial there.

The vector bundles $\mathcal{P}$ and $\mathcal{L}$ are the basic *automorphic vector bundles* on $\bar{S}$. (Vector valued) *modular forms* are global sections of vector bundles belonging to the tensor algebra generated by them. For example, for any $k \geq 0$ and any R_0-algebra R, the space of level-N weight-k (scalar valued) modular forms over R is

$$M_k(N, R) = H^0(\bar{S} \times_{R_0} R, \mathcal{L}^k). \tag{1.21}$$

By the Köcher principle, this is the same as $H^0(S \times_{R_0} R, \mathcal{L}^k)$.

1.4. **Preliminary results on $\mathcal{P}$ and $\mathcal{L}$.** We review some results on the automorphic vector bundles $\mathcal{P}$ and $\mathcal{L}$. Relations which are special to characteristic p will be treated in the next section.

1.4.1. *The relation* $\det \mathcal{P} \simeq \mathcal{L}$. Although "well-known" and crucial for every purpose, we have not been able to find an adequate reference for this relation. See also the remarks following the proof of the proposition.

Proposition 1.3. *One has* $\det \mathcal{P} \simeq \mathcal{L}$ *over* $\bar{S}_{\mathcal{K}}$.

Proof. Since $Pic(\bar{S}_{\mathcal{K}}) \subset Pic(\bar{S}_{\mathbb{C}})$ it is enough to prove the claim over $\mathbb{C}$. By GAGA, it is enough to establish the triviality of $\det \mathcal{P} \otimes \mathcal{L}^{-1}$ in the analytic category. We do it over the connected component $\bar{X}_\Gamma$. (The argument for any other connected component is the same.) The formulae in [30, Section 1] show that both $\det \mathcal{P}$ and $\mathcal{L}$ can be trivialized over $\mathfrak{X}$ so that the resulting factor of automorphy is the same, namely

$$j(\gamma; z, u) = a_3 z + b_3 u + c_3, \tag{1.22}$$

where (a_3, b_3, c_3) is the bottom row of the matrix $\gamma \in \Gamma \subset G_\infty$. If σ and τ are the trivializing sections constructed by Shimura, the section $\sigma \otimes \tau^{-1}$ trivializes $\det \mathcal{P} \otimes \mathcal{L}^{-1}$ and descends to X_Γ. But using [7], for example, one can easily verify that $\sigma \otimes \tau^{-1}$ extends to a nowhere vanishing section in an open (classical) neighborhood of any component of $\bar{X}_\Gamma - X_\Gamma$. Thus $\sigma \otimes \tau^{-1}$ trivializes $\det \mathcal{P} \otimes \mathcal{L}^{-1}$ on the whole of $\bar{X}_\Gamma$.

An alternative proof is to use Theorem 4.8 of [17]. In our case this theorem gives a *functor* $\mathcal{V} \mapsto [\mathcal{V}]$ from the category of $\tilde{\mathbf{G}}(\mathbb{C})$-equivariant vector bundles on the compact dual $\mathbb{P}^2_{\mathbb{C}}$ of $Sh_{\tilde{K}}$ to the category of vector bundles with $\tilde{\mathbf{G}}(\mathbb{A}_f)$-action on the inverse system of Shimura varieties $Sh_{\tilde{K}}$. Here $\mathbb{P}^2_{\mathbb{C}} = \tilde{\mathbf{G}}(\mathbb{C})/\tilde{\mathbf{H}}(\mathbb{C})$, where $\tilde{\mathbf{H}}(\mathbb{C})$ is the parabolic group stabilizing the line $\mathbb{C}\cdot {}^t(\delta/2, 0, 1)$ in $\tilde{\mathbf{G}}(\mathbb{C}) = GL_3(\mathbb{C}) \times \mathbb{C}^\times$, and the irreducible $\mathcal{V}$ are associated with highest weight representations of the Levi factor $\tilde{\mathbf{L}}(\mathbb{C})$ of $\tilde{\mathbf{H}}(\mathbb{C})$. It is straight forward to check that $\det \mathcal{P}$ and $\mathcal{L}$ are associated with the same character of $\tilde{\mathbf{L}}(\mathbb{C})$, up to a twist by a character of $\tilde{\mathbf{G}}(\mathbb{C})$, which affects the $\tilde{\mathbf{G}}(\mathbb{A}_f)$-action (hence the normalization of Hecke operators), but not the structure of the line bundles themselves. The functoriality of Harris' construction implies that $\det \mathcal{P}$ and $\mathcal{L}$ are isomorphic also algebraically. ∎

We do not know if $\det \mathcal{P}$ and $\mathcal{L}$ are isomorphic as algebraic line bundles on $\bar{S}$ (over R_0). This would be equivalent to the statement that for every PEL structure $(A, \lambda, \iota, \alpha) \in \mathcal{M}(R)$, for any R_0-algebra R, $\det(H^1_{dR}(A/R)(\Sigma))$ is the trivial line bundle on $Spec(R)$. In fact, the Hodge filtration gives a short exact sequence of locally free R-modules

$$0 \to H^0(A, \Omega^1_{A/R})(\Sigma) \to H^1_{dR}(A/R)(\Sigma) \to H^1(A, \mathcal{O})(\Sigma) \to 0. \tag{1.23}$$

This, together with the canonical isomorphisms

$$\begin{aligned} H^0(A, \Omega^1_{A/R})(\Sigma) &\simeq \omega_{A/R}(\Sigma) = \mathcal{P} \\ H^1(A, \mathcal{O})(\Sigma) &\simeq Lie(A^t/R)(\Sigma) \simeq \omega^\vee_{A^t/R}(\Sigma) \simeq \omega^\vee_{A/R}(\overline{\Sigma}) = \mathcal{L}^\vee \end{aligned} \tag{1.24}$$

(where the last isomorphism in the second formula is induced by the polarization λ), yield an isomorphism of line bundles over $Spec(R)$

$$\det(H^1_{dR}(A/R)(\Sigma)) \simeq \det \mathcal{P} \otimes \mathcal{L}^{-1}. \tag{1.25}$$

To our regret, we have not been able to establish that this line bundle is always trivial, although a similar statement in the "Siegel case", namely that for any principally polarized abelian scheme (A, λ) over R, $\det H^1_{dR}(A/R)$ is trivial, follows at once from the Hodge filtration. A related remark is that if we take $N = 1$, $\bar{S}$ still makes sense as a stack, but $\det \mathcal{P} \otimes \mathcal{L}^{-1}$ is not expected to be trivial anymore, only torsion. The proposition, however, suffices to guarantee the following corollary, which is all that we will be using in the sequel.

Corollary 1.4. *For any closed point $Spec(k) \to Spec(R_0)$, we have $\det \mathcal{P} \simeq \mathcal{L}$ on $\bar{S}_k$.*

Proof. Since $\bar{S}$ is a regular scheme, Proposition 1.3 implies that $\det \mathcal{P} \otimes \mathcal{L}^{-1} \simeq \mathcal{O}(D)$ for a Weil divisor D which is a $\mathbb{Z}$-linear combination of irreducible components of vertical fibers over R_0. If Z is an irreducible component of the special fiber $\bar{S}_k$, we can modify D by a multiple of the principle divisor (p) so that D and Z become disjoint, showing that $\det \mathcal{P} \otimes \mathcal{L}^{-1}|_Z$ is trivial. ∎

1.4.2. *The Gauss-Manin connection and the Kodaira-Spencer isomorphism.* Let $\pi : \mathcal{A} \to S$ be the structure morphism of the universal abelian scheme over S. The Gauss-Manin connection [20]

$$\nabla : H^1_{dR}(\mathcal{A}/S) \to H^1_{dR}(\mathcal{A}/S) \otimes_{\mathcal{O}_S} \Omega^1_S \tag{1.26}$$

(we write Ω^1_S for Ω^1_{S/R_0}) defines the Kodaira-Spencer map

$$KS : \omega_{\mathcal{A}/S} \to \omega^\vee_{\mathcal{A}^t/S} \otimes_{\mathcal{O}_S} \Omega^1_S. \tag{1.27}$$

Recall that KS is defined by first embedding $\omega_{\mathcal{A}/S} \simeq R^0\pi_*\Omega^1_{\mathcal{A}/S}$ in $H^1_{dR}(\mathcal{A}/S)$, then following it by ∇, and finally using the projection of $H^1_{dR}(\mathcal{A}/S)$ to $R^1\pi_*\mathcal{O}_\mathcal{A}$. The latter is the relative Lie algebra of $\mathcal{A}^t/S$, hence may be identified with $\omega^\vee_{\mathcal{A}^t/S}$. Unlike ∇, KS is $\mathcal{O}_S$-linear. It also commutes with the endomorphisms coming from $\iota(\mathcal{O}_\mathcal{K})$, so defines maps

$$\begin{aligned} KS(\Sigma) \quad &: \quad \omega_{\mathcal{A}/S}(\Sigma) \to \omega^\vee_{\mathcal{A}^t/S}(\Sigma) \otimes_{\mathcal{O}_S} \Omega^1_S \\ KS(\bar\Sigma) \quad &: \quad \omega_{\mathcal{A}/S}(\bar\Sigma) \to \omega^\vee_{\mathcal{A}^t/S}(\bar\Sigma) \otimes_{\mathcal{O}_S} \Omega^1_S. \end{aligned} \tag{1.28}$$

Alternatively, these are pairings, denoted by the same symbols,

$$\begin{aligned} KS(\Sigma) \quad &: \quad \omega_{\mathcal{A}/S}(\Sigma) \otimes_{\mathcal{O}_S} \omega_{\mathcal{A}^t/S}(\Sigma) \to \Omega^1_S \\ KS(\bar\Sigma) \quad &: \quad \omega_{\mathcal{A}/S}(\bar\Sigma) \otimes_{\mathcal{O}_S} \omega_{\mathcal{A}^t/S}(\bar\Sigma) \to \Omega^1_S. \end{aligned} \tag{1.29}$$

Observe that (1.29) are maps between vector bundles of rank 2.

Lemma 1.5. *The map*

$$KS(\Sigma) : \omega_{\mathcal{A}/S}(\Sigma) \otimes_{\mathcal{O}_S} \omega_{\mathcal{A}^t/S}(\Sigma) \to \Omega^1_S \tag{1.30}$$

is an isomorphism, and so is $KS(\bar\Sigma)$.

Proof. This follows from deformation theory. For a self-contained proof in the arithmetic case, see [3], Prop. II.2.1.5. ∎

The type-reversing isomorphism $\lambda^* : \omega_{\mathcal{A}^t/S} \to \omega_{\mathcal{A}/S}$ induced by the principal polarization is an isomorphism

$$\omega_{\mathcal{A}^t/S}(\Sigma) \quad \simeq \quad \omega_{\mathcal{A}/S}(\bar\Sigma) = \mathcal{L} \tag{1.31}$$

$$\omega_{\mathcal{A}^t/S}(\bar\Sigma) \quad \simeq \quad \omega_{\mathcal{A}/S}(\Sigma) = \mathcal{P} \tag{1.32}$$

and satisfies the *symmetry relation*

$$KS(\Sigma)(\lambda^* x \otimes y) = KS(\bar\Sigma)(\lambda^* y \otimes x) \tag{1.33}$$

for all $x \in \omega_{\mathcal{A}^t/S}(\bar\Sigma)$ and $y \in \omega_{\mathcal{A}^t/S}(\Sigma)$. See [11], Prop. 9.1 on p.81 (in the Siegel modular case).

Proposition 1.6. *The Kodaira-Spencer map induces a canonical isomorphism of vector bundles over* S

$$\mathcal{P} \otimes \mathcal{L} \simeq \Omega^1_S. \tag{1.34}$$

Proof. Use λ^* to identify $\omega_{\mathcal{A}^t/S}(\Sigma)$ with $\omega_{\mathcal{A}/S}(\bar\Sigma)$. ∎

Corollary 1.7. *Up to a twist by a fractional ideal of* R, *there is an isomorphism of line bundles* $\mathcal{L}^3 \simeq \Omega^2_S$.

Proof. Take determinants and use $\det(\mathcal{P}\otimes\mathcal{L})\simeq(\det\mathcal{P})\otimes\mathcal{L}^2\simeq\mathcal{L}^3$. Note that we know $\det\mathcal{P}\simeq\mathcal{L}$ only up to a twist by a fractional ideal of R_0 (cf. the discussion following Proposition 1.3). ∎

1.4.3. *More identities over $\bar{S}$.* We have seen that $\Omega^2_S\simeq\mathcal{L}^3$. For the following proposition, compare [3], Lemme II.2.1.7.

Proposition 1.8. *Let $C=\bar{S}-S$. There is an isomorphism*

$$\Omega^2_{\bar{S}}\simeq\mathcal{L}^3\otimes\mathcal{O}(C)^\vee. \tag{1.35}$$

Proof. We base change to an algebraically closed field, so that C becomes the disjoint union of elliptic curves E_i $(1\le i\le h)$. By [18] II.6.5, $\Omega^2_{\bar{S}}\simeq\mathcal{L}^3\otimes\bigotimes_{j=1}^h\mathcal{O}(E_j)^{n_j}$ for some integers n_j, and we want to show that $n_j=-1$ for all j. By the adjunction formula on the smooth surface $\bar{S}$, if we denote by $K_{\bar{S}}$ a canonical divisor, $\mathcal{O}(K_{\bar{S}})=\Omega^2_{\bar{S}}$, then

$$0=2g_{E_j}-2=E_j.(E_j+K_{\bar{S}}). \tag{1.36}$$

We conclude that

$$\deg(\Omega^2_{\bar{S}}|_{E_j})=E_j.K_{\bar{S}}=-E_j.E_j>0. \tag{1.37}$$

Here $E_j.E_j<0$ because E_j can be contracted to a point (Grauert's theorem). As $\mathcal{L}|_{E_j}$ and $\mathcal{O}(E_i)|_{E_j}$ $(i\neq j)$ are trivial we get

$$-E_j.E_j=n_jE_j.E_j, \tag{1.38}$$

hence $n_j=-1$ as desired. ∎

2. The Picard modular surface modulo an inert prime

2.1. **The stratification.**

2.1.1. *The three strata.* Let p be a rational prime which is inert in $\mathcal{K}$ and relatively prime to $2N$. Then $\kappa_0=R_0/pR_0$ is isomorphic to $\mathbb{F}_{p^2}$. We fix an algebraic closure κ of κ_0 and consider the characteristic p fiber

$$\bar{S}_\kappa=\bar{S}\times_{R_0}\kappa. \tag{2.1}$$

Unless otherwise specified, in this section we let S and $\bar{S}$ denote the characteristic p fibers S_κ and $\bar{S}_\kappa$. We also use the abbreviation $\omega_{\mathcal{A}}$ for $\omega_{\mathcal{A}/\bar{S}}$ etc.

Recall that an abelian variety over an algebraically closed field of characteristic p is called *supersingular* if the Newton polygon of its p-divisible group has a constant slope $1/2$. It is called *superspecial* if it is isomorphic to a product of supersingular elliptic curves. The following theorem combines various results proved in [6], [31] and [32].

Theorem 2.1. *(i) There exists a closed reduced 1-dimensional subscheme $S_{ss}\subset\bar{S}$ (the supersingular locus), disjoint from the cuspidal divisor (i.e. contained in S), which is uniquely characterized by the fact that for any geometric point x of S, the abelian variety $\mathcal{A}_x$ is supersingular if and only if x lies on S_{ss}. The scheme S_{ss} is defined over κ_0.*

(ii) Let S_{ssp} be the singular locus of S_{ss}. Then x lies in S_{ssp} if and only if $\mathcal{A}_x$ is superspecial. If $x\in S_{ssp}$ then

$$\widehat{\mathcal{O}}_{S_{ss},x}\simeq\kappa[[u,v]]/(u^{p+1}+v^{p+1}). \tag{2.2}$$

(iii) Assume that N is large enough (depending on p). Then the irreducible components of S_{ss} are nonsingular, and in fact are all isomorphic to the Fermat curve $\mathcal{C}_p$ given by the equation

$$x^{p+1}+y^{p+1}+z^{p+1}=0. \tag{2.3}$$

There are p^3+1 points of S_{ssp} on each irreducible component, and through each such point pass $p+1$ irreducible components. Any two irreducible components are either disjoint or intersect transversally at a unique point.

(iv) Without the assumption of N being large (but under $N\geq 3$ as usual) the irreducible components of S_{ss} may have multiple intersections with each other, including self-intersections. Their normalizations are nevertheless still isomorphic to $\mathcal{C}_p$.

Proof. Points (i) and (ii) follow from [32 (7.3)(b)] ($d=3$) and [31, Theorem 3]. The structure of $\widehat{\mathcal{O}}_{S_{ss},x}$ is also proved there, but we shall recover it in (2.32) below.

Point (iii) is [31, Theorem 4]. Let $\mathcal{T}$ be the Bruhat-Tits building of the group G_p. This is a biregular tree, whose even vertices are of degree p^3+1, and whose odd vertices are of degree $p+1$. Fix a connected component Z of S_{ss}. Vollaard identifies the incidence graph, whose even vertices correspond to the irreducible components of Z, and whose odd vertices correspond to the points of $Z\cap S_{ssp}$ (edges denoting incidence relations) with the quotient of $\mathcal{T}$ by a certain discrete cocompact arithmetic subgroup of G_p. The condition on N being large is necessary to guarantee that the action of that group on $\mathcal{T}$ is "good".

Point (iv) is not explicitly stated in [31], but follows from the discussion there. Since this point is used later on, we explain it here. We refer to the notation of [31, Section 6]. Our "N large" condition is Vollaard's "C^p small". This is used by her in two ways. First, it is needed to guarantee that the groups

$$\Gamma_j=I(\mathbb{Q})\cap g_j^{-1}C^pg_j \tag{2.4}$$

are torsion free. For this, it is enough to have $N\geq 3$, by Serre's lemma. Second, Vollaard needs the assumption "C^p small" in the proof of Theorem 6.1, to guarantee that the above-mentioned action of Γ_j on the tree $\mathcal{T}$ is good. For that, she needs the distances $u(\Lambda,g\Lambda)$ to be at least 6 for every $1\neq g\in\Gamma_j^0$ and $\Lambda\in\mathcal{L}_0$. This measure of smallness translates to $N\geq N_0(p)$ where $N_0(p)$ depends on p.

If we only assume $N\geq 3$, then Theorem 6.1 of [31] does not hold. However, Γ_j still acts freely on the set of irreducible components of $\mathcal{N}^{red}$. Indeed, the stabilizer of any given irreducible component is a finite subgroup of Γ_j, hence trivial. It follows that while the irreducible components of $\mathcal{N}^{red}$, which are all isomorphic to $\mathcal{C}_p$, may acquire self-intersections in $\Gamma_j\backslash\mathcal{N}^{red}$, their normalizations will still be $\mathcal{C}_p$. ∎

We call $\bar{S}_\mu=\bar{S}-S_{ss}$ (or $S_\mu=\bar{S}_\mu\cap S$) the *μ-ordinary* or *generic* locus, $S_{gss}=S_{ss}-S_{ssp}$ the *general supersingular locus*, and S_{ssp} the *superspecial locus.* Then $\bar{S}=\bar{S}_\mu\cup S_{gss}\cup S_{ssp}$ is a stratification: the three strata are of pure dimension 2,1, and 0 respectively, the closure of each stratum contains the lower dimensional ones, and each of the three is open in its closure.

2.1.2. *The p-divisible group.* Let $x:Spec(k)\to S$ (k an algebraically closed field) be a geometric point of S, $\mathcal{A}_x$ the corresponding fiber of $\mathcal{A}$, and $\mathcal{A}_x(p)$ its p-divisible group. Let $\mathfrak{G}$ be the p-divisible group of a supersingular elliptic curve over k

(the group denoted by $G_{1,1}$ in the Manin-Dieudonné classification). The following theorem can be deduced from [6] and [31].

Theorem 2.2. *(i) If $x \in S_\mu$ then*

$$\mathcal{A}_x(p) \simeq (\mathcal{O}_\mathcal{K} \otimes \mu_{p^\infty}) \times \mathfrak{G} \times (\mathcal{O}_\mathcal{K} \otimes \mathbb{Q}_p/\mathbb{Z}_p). \tag{2.5}$$

(ii) If $x \in S_{ss}$ then $\mathcal{A}_x(p)$ is isogenous to $\mathfrak{G}^3$, and $x \in S_{ssp}$ if and only if the two groups are isomorphic.

While the p-divisible group of a μ-ordinary geometric fiber actually splits as a product of its multiplicative, local-local and étale parts, over the whole of S_μ we only get a filtration

$$0 \subset Fil^2\mathcal{A}(p) \subset Fil^1\mathcal{A}(p) \subset Fil^0\mathcal{A}(p) = \mathcal{A}(p) \tag{2.6}$$

by $\mathcal{O}_\mathcal{K}$-stable p-divisible groups. Here $gr^2 = Fil^2$ is of multiplicative type, $gr^1 = Fil^1/Fil^2$ is a local-local group and $gr^0 = Fil^0/Fil^1$ is étale, each of height 2 ($\mathcal{O}_\mathcal{K}$-height 1).

2.2. New relations between $\mathcal{P}$ and $\mathcal{L}$ in characteristic p.

2.2.1. *The line bundles $\mathcal{P}_0$ and $\mathcal{P}_\mu$ over $\bar{S}_\mu$.* Consider the universal semi-abelian variety $\mathcal{A}$ over the Zariski open set $\bar{S}_\mu$. Over the cuspidal divisor $C = \bar{S} - S$, where $\mathcal{A}$ becomes an extension of a supersingular elliptic curve B (with $\mathcal{O}_\mathcal{K}$-signature $(1,0)$) by a 2-dimensional torus (with $\mathcal{O}_\mathcal{K}$-signature $(1,1)$), $\mathcal{P} = \omega_\mathcal{A}(\Sigma)$ admits a canonical filtration

$$0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0. \tag{2.7}$$

Here $\mathcal{P}_0$ is the cotangent space to the abelian part B of $\mathcal{A}$, and $\mathcal{P}_\mu$ is the Σ-component of the cotangent space to the toric part of $\mathcal{A}$. This filtration exists already in characteristic 0, but when we reduce the Picard surface modulo p it extends, as we now show, to the whole of $\bar{S}_\mu$.

Let $\mathcal{A}[p]^0$ be the connected part of the subgroup scheme $\mathcal{A}[p]$ over $\bar{S}_\mu$. Then $\mathcal{A}[p]^0$ is finite flat of rank p^4. (It is clearly flat and quasi-finite, and the fiber rank can be computed separately on C and on S_μ. Since the rank is constant, the morphism to $\bar{S}_\mu$ is actually finite, *cf.* [9], Lemme 1.19.) Let

$$0 \subset \mathcal{A}[p]^\mu \subset \mathcal{A}[p]^0 \tag{2.8}$$

be the maximal subgroup-scheme of multiplicative type. Since at every closed point of $\bar{S}_\mu$, $\mathcal{A}[p]^\mu$ is of rank p^2, this subgroup is also finite flat over $\bar{S}_\mu$. It is also $\mathcal{O}_\mathcal{K}$-invariant. Over the cuspidal divisor C, $\mathcal{A}[p]^\mu$ is the p-torsion in the toric part of $\mathcal{A}$, and over S_μ

$$\mathcal{A}[p]^\mu = \mathcal{A}[p] \cap Fil^2\mathcal{A}(p). \tag{2.9}$$

As $\omega_\mathcal{A}$ is killed by p, we have $\omega_\mathcal{A} = \omega_{\mathcal{A}[p]} = \omega_{\mathcal{A}[p]^0}$. Let $\omega_\mathcal{A}^\mu = \omega_{\mathcal{A}[p]^\mu}$, a rank-2 $\mathcal{O}_\mathcal{K}$-vector bundle of type $(1,1)$. The kernel of $\omega_{\mathcal{A}[p]^0} \to \omega_{\mathcal{A}[p]^\mu}$ is then a line bundle $\mathcal{P}_0$ of type $(1,0)$ and we get the short exact sequence

$$0 \to \mathcal{P}_0 \to \omega_\mathcal{A} \to \omega_\mathcal{A}^\mu \to 0 \tag{2.10}$$

over the whole of $\bar{S}_\mu$. Decomposing according to types and setting $\mathcal{P}_\mu = \omega_\mathcal{A}^\mu(\Sigma)$, we get the desired filtration on $\mathcal{P}$.

2.2.2. *Frobenius and Verschiebung.* Let $\mathcal{A}^{(p)}$ be the base change of $\mathcal{A}$ with respect to the absolute Frobenius morphism of degree p of $\bar{S}$. In other words, if we denote by ϕ the homomorphism $x \mapsto x^p$ (of any $\mathbb{F}_p$-algebra), and by $\Phi : \bar{S} \to \bar{S}$ the corresponding map of schemes (which is *not* κ_0-linear), then

$$\mathcal{A}^{(p)} = \mathcal{A} \times_{\bar{S},\Phi} \bar{S}. \tag{2.11}$$

If $\mathcal{M}$ is a coherent sheaf on $\mathcal{A}$, we denote by $\mathcal{M}^{(p)} = \Phi^*\mathcal{M}$ its base-change to $\mathcal{A}^{(p)}$. If $\mathcal{M}$ is endowed with an $\mathcal{O}_\mathcal{K}$-action, so is, via base-change, $\mathcal{M}^{(p)}$. However, if $\mathcal{M}$ is a vector bundle of type (a, b) then $\mathcal{M}^{(p)}$ is of type (b, a), because $x \mapsto x^p$ interchanges Σ with $\overline{\Sigma}$.

The *relative Frobenius* is an $\mathcal{O}_{\bar{S}}$-linear isogeny $Frob_\mathcal{A} : \mathcal{A} \to \mathcal{A}^{(p)}$, characterized by the fact that $pr_1 \circ Frob_\mathcal{A}$ is the absolute Frobenius morphism of $\mathcal{A}$. Over S (but not over the boundary C) we have the dual abelian scheme $\mathcal{A}^t$, and the *Verschiebung* $Ver_\mathcal{A} : \mathcal{A}^{(p)} \to \mathcal{A}$ is the $\mathcal{O}_S$-linear isogeny which is dual to $Frob_{\mathcal{A}^t} : \mathcal{A}^t \to (\mathcal{A}^t)^{(p)}$.

We clearly have $\omega_{\mathcal{A}^{(p)}} = \omega_\mathcal{A}^{(p)}$, and we let

$$F : \omega_\mathcal{A}^{(p)} \to \omega_\mathcal{A}, \ \ V : \omega_\mathcal{A} \to \omega_\mathcal{A}^{(p)} \tag{2.12}$$

be the $\mathcal{O}_{\bar{S}}$-linear maps of vector bundles induced by the isogenies $Frob_\mathcal{A}$ and $Ver_\mathcal{A}$ on the cotangent spaces. Note, however, that while F is defined everywhere, V is so far defined only over S.

To extend the definition of V over $\bar{S}$ we consider the finite flat group scheme $\mathcal{G} = \mathcal{A}[p]^0$ over $\bar{S}_\mu$ as in the previous subsection. We now have the $\mathcal{O}_{\bar{S}}$-linear homomorphism $Ver_\mathcal{G} : \mathcal{G}^{(p)} \to \mathcal{G}$ also over the boundary. Under Cartier duality $Ver_\mathcal{G}$ is dual to $Frob_{\mathcal{G}^D}$, where we denote by $\mathcal{G}^D$ the Cartier dual of $\mathcal{G}$. Over S_μ it coincides with the homomorphism induced by $Ver_\mathcal{A}$. This follows at once from the identification of $\mathcal{A}[p]^D$ with $\mathcal{A}^t[p]$ (Weil pairing).

Recall that $\omega_\mathcal{G} = \omega_\mathcal{A}$, and similarly $\omega_{\mathcal{G}^{(p)}} = \omega_{\mathcal{A}^{(p)}} = \omega_\mathcal{A}^{(p)}$. The morphism $Ver_\mathcal{G} : \mathcal{G}^{(p)} \to \mathcal{G}$ therefore induces a homomorphism of vector bundles over $\bar{S}_\mu$, $V : \omega_\mathcal{A} \to \omega_\mathcal{A}^{(p)}$, which extends the previously defined map V to $\bar{S}$.

Taking Σ-components we get

$$V : \mathcal{P} = \omega_\mathcal{A}(\Sigma) \to \omega_\mathcal{A}^{(p)}(\Sigma) = \omega_\mathcal{A}(\bar{\Sigma})^{(p)} = \mathcal{L}^{(p)}. \tag{2.13}$$

Over $\bar{S}_\mu$ this map fits in a commutative diagram

$$\begin{array}{ccccccccc} 0 & \leftarrow & \omega_\mathcal{A}^\mu(\Sigma) = \mathcal{P}_\mu & \leftarrow & \mathcal{P} & \leftarrow & \mathcal{P}_0 & \leftarrow & 0 \\ & & \downarrow \simeq & & \downarrow V & & \downarrow & & \\ 0 & \leftarrow & (\omega_\mathcal{A}^\mu(\bar{\Sigma}))^{(p)} & \leftarrow & \mathcal{L}^{(p)} & \leftarrow & 0 & \leftarrow & 0 \end{array}. \tag{2.14}$$

The right vertical arrow is 0 since V kills $\mathcal{P}_0$, as $\mathcal{A}[p]^0/\mathcal{A}[p]^\mu$ is of local-local type, hence $Ver_\mathcal{G}$ acts on it nilpotently. The left vertical map is an isomorphism, since Ver is an isomorphism on p-divisible groups of multiplicative type. We conclude that over $\bar{S}_\mu$

$$\mathcal{P}_0 = \ker(V : \mathcal{P} \to \mathcal{L}^{(p)}). \tag{2.15}$$

2.2.3. *Relations between $\mathcal{P}_0, \mathcal{P}_\mu$ and $\mathcal{L}$ over $\bar{S}_\mu$.* We first recall a general lemma.

Lemma 2.3. *Let $\mathcal{M}$ be a line bundle over a scheme S in characteristic p. Let $\Phi : S \to S$ be the absolute Frobenius and $\mathcal{M}^{(p)} = \Phi^*\mathcal{M}$. Then the map $\mathcal{M}^{(p)} \to \mathcal{M}^p$*

$$a \otimes m \mapsto a \cdot m \otimes \cdots \otimes m \tag{2.16}$$

is an isomorphism of line bundles over S.

Since $\mathcal{L}^{(p)} \simeq \mathcal{L}^p$ by the lemma, we have

$$\mathcal{L}^p \simeq \mathcal{P}/\mathcal{P}_0 = \mathcal{P}_\mu. \tag{2.17}$$

Finally, from $\mathcal{P}_0 \otimes \mathcal{P}_\mu \simeq \det \mathcal{P} \simeq \mathcal{L}$ we get

$$\mathcal{P}_0 \simeq \mathcal{L}^{1-p}. \tag{2.18}$$

We have proved:

Proposition 2.4. *Over $\bar{S}_\mu$, $\mathcal{P}_\mu \simeq \mathcal{L}^p$ and $\mathcal{P}_0 \simeq \mathcal{L}^{1-p}$.*

In the same vein we get a commutative diagram for the $\bar{\Sigma}$ parts

$$\begin{array}{ccccccccc} 0 & \leftarrow & \omega^\mu_{\mathcal{A}}(\bar{\Sigma}) & \leftarrow & \mathcal{L} & \leftarrow & 0 & \leftarrow & 0 \\ & & \downarrow \simeq & & \downarrow V & & \downarrow & & \\ 0 & \leftarrow & (\omega^\mu_{\mathcal{A}}(\Sigma))^{(p)} & \leftarrow & \mathcal{P}^{(p)} & \leftarrow & \mathcal{P}_0^{(p)} & \leftarrow & 0 \end{array} \tag{2.19}$$

and deduce that V is injective on $\mathcal{L}$ and

$$\mathcal{P}^{(p)} = \mathcal{P}_0^{(p)} \oplus V(\mathcal{L}). \tag{2.20}$$

Thus over $\bar{S}_\mu$, $\mathcal{P}$ has a canonical filtration by $\mathcal{P}_0$, but the induced filtration on $\mathcal{P}^{(p)}$ already splits as a direct sum.

Remark 2.1. *Restricting attention to a connected component E of C, $\mathcal{P}|_E$ is a* non-split *extension of $\mathcal{P}_\mu$ by $\mathcal{P}_0$. However, both $\mathcal{P}_\mu$ and $\mathcal{P}_0$ are trivial on E, so the extension is described by a non-zero class $\xi \in H^1(E, \mathcal{O}_E)$. The extension $\mathcal{P}^{(p)}|_E$ is then described by $\xi^{(p)}$. The semilinear map $\xi \mapsto \xi^{(p)}$ is the Cartier-Manin operator, and since E is a supersingular elliptic curve, $\xi^{(p)} = 0$ and $\mathcal{P}^{(p)}|_E$ splits. Thus at least over C, the splitting of $\mathcal{P}^{(p)}$ is consistent with what we already know.*

Since V induces an isomorphism of $\mathcal{L}$ onto $\mathcal{P}^{(p)}/\mathcal{P}_0^{(p)} \simeq (\mathcal{P}/\mathcal{P}_0)^p$ and $\mathcal{P}/\mathcal{P}_0 \simeq \mathcal{L}^p$ we conclude that over $\bar{S}_\mu$, $\mathcal{L} \simeq \mathcal{L}^{p^2}$. In the next section we realize this isomorphism via the Hasse invariant. Combining what was proved so far we get the following.

Proposition 2.5. *Over $\bar{S}_\mu$, $\mathcal{L}^{p^2} \simeq \mathcal{L}$. For $k \geq 1$ odd, $\mathcal{P}^{(p^k)} \simeq \mathcal{L}^{p-1} \oplus \mathcal{L}$. For $k \geq 2$ even, $\mathcal{P}^{(p^k)} \simeq \mathcal{L}^{1-p} \oplus \mathcal{L}^p$, but for $k = 0$ we only have an exact sequence*

$$0 \to \mathcal{L}^{1-p} \to \mathcal{P} \to \mathcal{L}^p \to 0. \tag{2.21}$$

Corollary 2.6. *Over $\bar{S}_\mu$, $\mathcal{L}^{p^2-1}, \mathcal{P}_\mu^{p^2-1}$ and $\mathcal{P}_0^{p+1}$ are trivial line bundles.*

2.2.4. *Extending the filtration on $\mathcal{P}$ over S_{gss}.* In order to determine to what extent the filtration on $\mathcal{P}$ and the relation between $\mathcal{L}$ and the two graded pieces of the filtration extend into the supersingular locus, we have to employ Dieudonné theory.

Proposition 2.7. *Let $\mathcal{P}_0 = \ker(V : \mathcal{P} \to \mathcal{L}^{(p)})$. Then over the whole of $\bar{S} - S_{ssp}$, $V(\mathcal{P}) = \mathcal{L}^{(p)}$ and $\mathcal{P}_0$ is a rank 1 submodule. Let $\mathcal{P}_\mu = \mathcal{P}/\mathcal{P}_0$. Then $\mathcal{P}_\mu \simeq \mathcal{L}^p$, $\mathcal{P}_0 \simeq \mathcal{L}^{1-p}$ and the filtration (2.7) is valid there.*

Proof. Everything is a formal consequence of the fact that V maps $\mathcal{P}$ *onto* $\mathcal{L}^{(p)}$, and the relation $\det \mathcal{P} \simeq \mathcal{L}$. Over $\bar{S}_\mu$ the proposition was verified in the previous subsection, so it is enough to prove that $V(\mathcal{P}) = \mathcal{L}^{(p)}$ in the fiber of any geometric point $x \in S_{gss}(k)$ (k algebraically closed). We use the description of $H^1_{dR}(\mathcal{A}_x/k)$ given in Lemma 2.8 below, due to Bültel and Wedhorn. In the notation of that lemma, $\mathcal{P}_x$ is spanned over k by e_1 and e_2 and $\mathcal{L}_x$ by f_3, while $V(e_1) = 0$, $V(e_2) = f_3^{(p)}$. This concludes the proof. ∎

For the next lemma let $D_0 = H^1_{dR}(\mathcal{A}_x/k)$, where $x \in S_{gss}(k)$ and k is algebraically closed. We identify D_0 with the reduction modulo p of the (contravariant) Dieudonné module of $\mathcal{A}_x$. It is therefore equipped with k-linear maps $F : D_0^{(p)} \to D_0$ and $V : D_0 \to D_0^{(p)}$ where $D_0^{(p)} = k \otimes_{\phi,k} D_0$ as usual.

Lemma 2.8. *There exists a basis $e_1, e_2, f_3, f_1, f_2, e_3$ of D_0 with the following properties. Denote by $e_1^{(p)} = 1 \otimes e_1 \in D_0^{(p)}$ etc.*

(i) $\mathcal{O}_\mathcal{K}$ acts on the e_i via Σ and on the f_i via $\bar{\Sigma}$ (hence it acts on the $e_i^{(p)}$ via $\bar{\Sigma}$ and on the $f_i^{(p)}$ via Σ).

(ii) The symplectic pairing on D_0 induced by the principal polarization λ_x satisfies

$$\langle e_i, f_j\rangle = -\langle f_j, e_i\rangle = \delta_{ij}, \quad \langle e_i, e_j\rangle = \langle f_i, f_j\rangle = 0. \tag{2.22}$$

(iii) The vectors e_1, e_2, f_3 form a basis for the cotangent space $\omega_\mathcal{A}$ at x. Hence e_1 and e_2 span $\mathcal{P}$ and f_3 spans $\mathcal{L}$.

(iv) $\ker(V)$ is spanned by e_1, f_2, e_3. Hence $\mathcal{P}_0 = \mathcal{P} \cap \ker(V)$ is spanned by e_1.

(v) $Ve_2 = f_3^{(p)}$, $Vf_3 = e_1^{(p)}$, $Vf_1 = e_2^{(p)}$.

(vi) $Ff_1^{(p)} = -e_3$, $Ff_2^{(p)} = -e_1, Fe_3^{(p)} = -f_2$.

Proof. Up to a slight change of notation, this is the unitary Dieudonné module which Bültel and Wedhorn call a "braid of length 3" and denote by $\bar{B}(3)$, cf [6] (3.2). The classification in loc. cit. Proposition 3.6 shows that the Dieudonné module of a μ-ordinary abelian variety is isomorphic to $\bar{B}(2) \oplus \bar{S}$, that of a gss abelian variety is isomorphic to $\bar{B}(3)$ and in the superspecial case we get $\bar{B}(1) \oplus \bar{S}^2$. ∎

Proposition 2.9. *Over the whole of $\bar{S} - S_{ssp}$, V maps $\mathcal{L}$ injectively onto a sub-line-bundle of $\mathcal{P}^{(p)}$.*

Proof. Once again, we know it already over $\bar{S}_\mu$, and it remains to check the assertion fiber-wise on S_{gss}. We refer again to Lemma 2.8, and find that $V(f_3) = e_1^{(p)}$, which proves our claim. ∎

The emerging picture is this: Outside the superspecial points, V maps $\mathcal{L}$ injectively onto a sub-line-bundle of $\mathcal{P}^{(p)}$, and $V^{(p)}$ maps $\mathcal{P}^{(p)}$ surjectively onto $\mathcal{L}^{(p^2)}$. However, the line $V(\mathcal{L})$ coincides with the line $\mathcal{P}_0^{(p)} = \ker(V^{(p)})$ only on the general supersingular locus, while on its complement $\bar{S}_\mu$ the two lines make up a frame for $\mathcal{P}^{(p)}$ (2.19). One can be a little more precise. The equation

$$V(\mathcal{L}) = \mathcal{P}_0^{(p)} \tag{2.23}$$

(i.e. $V^{(p)} \circ V(\mathcal{L}) = 0$) is the *defining equation* of S_{ss} in the sense that when expressed in local coordinates it defines S_{ss} with its *reduced* subscheme structure. See Proposition 2.11 below.

2.2.5. *Non-extendibility of the filtration across S_{ssp}.* For a superspecial x, $\mathcal{A}_x$ is isomorphic to a product of three supersingular elliptic curves, so V vanishes on the whole of $\omega_\mathcal{A}$ at x. The analysis of the last paragraph breaks up. To complete the picture, we shall now prove that there does not exist any way to extend the filtration (2.7) across such an x.

Proposition 2.10. *It is impossible to extend the filtration* $0 \to \mathcal{P}_0 \to \mathcal{P} \to \mathcal{P}_\mu \to 0$ *along* S_{ss} *in a neighborhood of a superspecial point* x.

Proof. At any superspecial point there are $p+1$ branches of S_{ss} meeting transversally. We shall prove the proposition by showing that along any one of these branches (labelled by ζ, a $p+1$-st root of -1) $\mathcal{P}_0$ approaches a line $\mathcal{P}_x[\zeta] \subset \mathcal{P}_x$, but these $p+1$ lines are distinct. In other words, on the normalization of S_{ss} we can extend the filtration uniquely, but the extension does not descend to S_{ss}.

Before we go into the proof a word of explanation is needed. In order to study the deformation of the action of V on $\omega_{\mathcal{A}}$ near a *general* supersingular point $x \in S_{gss}$, the first infinitesimal neighborhood of x suffices, and one ends up using Grothendieck's crystalline deformation theory (see 2.3.2 below). At a point $x \in S_{ssp}$, in contrast, we need to work in the full formal neighborhood of x in S, or at least in an Artinian neighborhood which no longer admits a divided power structure. The reason is that the singularity of S_{ss} at x is formally of the type $Spec(\kappa[[u,v]]/(u^{p+1}+v^{p+1}))$. Crystalline deformation theory is inadequate, and we need to use Zink's "displays". As the theory of displays is covariant, we start with the covariant Cartier module of $A = \mathcal{A}_x$ rather than the contravariant Dieudonné module, and look for its universal deformation.

Let us review the (confusing) functoriality of these two modules. For the moment, let A be any abelian variety over κ. Here κ can be any perfect field of characteristic p. If D is the (contravariant) Dieudonné module of A and M is its (covariant) Cartier module, then $D/pD = H^1_{dR}(A)$ and $M/pM = H^1_{dR}(A^t)$ are set in duality. The dual of

$$V : D/pD \to (D/pD)^{(p)} \tag{2.24}$$

($V = Ver_A^*$, Ver_A being the Verschiebung isogeny from $A^{(p)}$ to A) is the map

$$F : (M/pM)^{(p)} \to M/pM \tag{2.25}$$

($F = Frob_{A^t}^*$, $Frob_{A^t}$ being the Frobenius isogeny from A^t to $A^{t(p)}$). As usual, since κ is perfect, we may view V as a ϕ^{-1}-linear endomorphism of D/pD, and F as a ϕ-linear endomorphism of M/pM. Replacing A by A^t we then also have semi-linear endomorphisms F of D/pD and V of M/pM. The Hodge filtration $\omega_A \subset H^1_{dR}(A)$ is $(D/pD)[F]$. Its dual is the quotient $Lie(A) = H^1(A^t, \mathcal{O})$ of $H^1_{dR}(A^t)$, identified with M/VM. Compare [29], Corollary 5.11.

This reminder tells us that when we pass from the contravariant theory to the covariant one, instead of looking for the deformation of V on ω_A we should look for the deformation of F on $Lie(A) = M/VM$. At a superspecial point F annihilates $Lie(A)$, but at nearby points in S it need not annihilate it anymore.

Now let $x \in S_{ssp}$ and $A = \mathcal{A}_x$. The Cartier module (modulo p) M/pM of A admits a symplectic basis $f_3, e_1, e_2, e_3, f_1, f_2$ where $\mathcal{O}_\mathcal{K}$ acts on the e_i via Σ and on the f_i via $\bar{\Sigma}$, where the polarization pairing is $\langle e_i, f_j \rangle = -\langle f_j, e_i \rangle = \delta_{ij}$ and $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$, and where f_3, e_1, e_2 project to a basis of $Lie(A) = M/VM$. With an appropriate choice of the basis, the Frobenius F on M/pM is the ϕ-linear map whose matrix with respect to the basis $f_3, e_1, e_2, e_3, f_1, f_2$ is

$$\left(\begin{array}{cccccc} 0 & & & 0 & & \\ & 0 & & & 0 & \\ & & 0 & & & 0 \\ 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{array}\right). \tag{2.26}$$

All this can be deduced from [6], 3.2.

To construct the universal display we follow the method of [15]. See also [1]. With local coordinates u and v we write $\widehat{S} = Spf\kappa[[u,v]]$ for the formal completion of S at x. We study the deformation of F to

$$F : H^1_{dR}(\mathcal{A}^t/\widehat{S})^{(p)} \to H^1_{dR}(\mathcal{A}^t/\widehat{S}). \tag{2.27}$$

If A/R is any abelian scheme over an $\mathbb{F}_p$-algebra R and a any endomorphism of A/R, then $Frob : A \to A^{(p)}$ satisfies

$$Frob \circ a = a^{(p)} \circ Frob \tag{2.28}$$

where $a^{(p)} = 1 \otimes a$ is the base-change of a to $\mathcal{A}^{(p)}$. Thus F commutes with the $\mathcal{O}_{\mathcal{K}}$-structure on de Rham cohomology. Note however that $H^1_{dR}(\mathcal{A}^t/\widehat{S})^{(p)}(\Sigma) = H^1_{dR}(\mathcal{A}^t/\widehat{S})(\overline{\Sigma})^{(p)}$ and vice versa.

We use a basis $f_3, e_1, \dots, f_2$ of $H^1_{dR}(\mathcal{A}^t/\widehat{S})$ satisfying the same assumptions as above with respect to the $\mathcal{O}_{\mathcal{K}}$-type and the polarization pairing. A bit of elementary algebra, which we skip, shows that one can modify the local coordinates u and v, *and* the basis of $H^1_{dR}(\mathcal{A}^t/\widehat{S})$ (keeping our assumptions on the $\mathcal{O}_{\mathcal{K}}$-type and the polarization), so that the universal Frobenius is given by the (Hasse-Witt) matrix

$$F = \left(\begin{array}{cccccc} 0 & u & v & 0 & & \\ u & 0 & & & 0 & \\ v & & 0 & & & 0 \\ 1 & & & 0 & & \\ & 1 & & & 0 & \\ & & 1 & & & 0 \end{array}\right). \tag{2.29}$$

This means, as usual, that

$$F(f_3^{(p)}) = ue_1 + ve_2 + e_3, \tag{2.30}$$

etc. Since the first three vectors (f_3, e_1, e_2) project onto a basis of $Lie(\mathcal{A})$, the matrix of $F : Lie(\mathcal{A})^{(p)} \to Lie(\mathcal{A})$ is the 3×3 upper left block, and the matrix of $F^2 (= F \circ F^{(p)})$ is (note the semilinearity)

$$\left(\begin{array}{ccc} u^{p+1} + v^{p+1} & & \\ & u^{p+1} & uv^p \\ & vu^p & v^{p+1} \end{array}\right). \tag{2.31}$$

Thus on $Lie(\mathcal{A})(\bar{\Sigma})^{(p^2)} = \mathcal{L}^{(p^2)\vee}$ the action of F^2 is given by multiplication by $u^{p+1} + v^{p+1}$. As the supersingular locus is the locus where the action of F on the Lie algebra is nilpotent, we recover the fact that the local (formal) equation of S_{ss} at x is

$$u^{p+1} + v^{p+1} = 0. \tag{2.32}$$

Note that this equation guarantees also that the lower 2×2 block, representing the action of F^2 on the Σ-part of the Lie algebras is (semi-linearly) nilpotent, i.e.

$$\left(\begin{array}{cc} u^{p+1} & uv^p \\ vu^p & v^{p+1} \end{array}\right)\left(\begin{array}{cc} u^{p^2(p+1)} & u^{p^2}v^{p^3} \\ v^{p^2}u^{p^3} & v^{p^2(p+1)} \end{array}\right) = 0. \tag{2.33}$$

We write $\widehat{S}_{ss} = Spf(\kappa[[u,v]]/(u^{p+1}+v^{p+1}))$ for the formal completion of S_{ss} at x. Letting ζ run over the $p+1$ roots of -1 we recover the $p+1$ formal branches through x as the "lines"

$$u = \zeta v. \tag{2.34}$$

We write $\widehat{S}_{ss}[\zeta] = Spf(\kappa[[u,v]]/(u-\zeta v))$ for this branch. When we restrict (pull back) the vector bundle $Lie(\mathcal{A})$ to $\widehat{S}_{ss}[\zeta]$, $\ker(F : Lie(\mathcal{A})^{(p)} \to Lie(\mathcal{A}))$ (the dual of $\omega_{\mathcal{A}}^{(p)}/V(\omega_{\mathcal{A}})$, which outside x is just $\mathcal{P}_{\mu}^{(p)}$) becomes

$$\ker\left(\begin{array}{ccc} 0 & \zeta v & v \\ \zeta v & 0 & 0 \\ v & 0 & 0 \end{array}\right). \tag{2.35}$$

When $v \neq 0$ (i.e. outside the point x) this is the line (note again the semi-linearity, and the relation $\zeta^{-p} = -\zeta$)

$$\kappa\left(\begin{array}{c} 0 \\ 1 \\ \zeta^{-1} \end{array}\right). \tag{2.36}$$

As these lines are distinct, the filtration of $\mathcal{P}$ can not be extended across x. ∎

2.3. The Hasse invariant $h_{\bar{\Sigma}}$.

2.3.1. *Definition of* $h_{\bar{\Sigma}}$. The construction and main properties of the Hasse invariant that we are about to describe have been given (for any unitary Shimura variety) by Goldring and Nicole in [12] (see also [22]), but in our case they can be also obtained easily from the discussion of the previous subsection. Let R be any κ-algebra, and $(A, \lambda, \iota, \alpha) \in \mathcal{M}(R)$.

Definition 2.1. *The Hasse invariant of A*

$$h_{\bar{\Sigma}}(A) \in Hom(\omega_{A/R}(\bar{\Sigma}), \omega_{A/R}(\bar{\Sigma})^{(p^2)}) \tag{2.37}$$

is the map $h_{\bar{\Sigma}} = V^{(p)} \circ V$.

Applying the same definition to the universal semi-abelian scheme $\mathcal{A}/\bar{S}$ we get (note $\mathcal{L}^{(p^2)} \simeq \mathcal{L}^{p^2}$)

$$h_{\bar{\Sigma}} \in Hom_{\bar{S}}(\mathcal{L}, \mathcal{L}^{p^2}) = H^0(\bar{S}, \mathcal{L}^{p^2-1}) = M_{p^2-1}(N, \kappa). \tag{2.38}$$

Thus the Hasse invariant is a modular form of weight $p^2 - 1$ defined over κ.

Theorem 2.11. *The Hasse invariant is invertible on $\bar{S}_{\mu}$ and vanishes on S_{ss} to order one. More precisely, when we endow S_{ss} with its induced reduced subscheme structure, it becomes the Cartier divisor $div(h_{\bar{\Sigma}})$.*

Proof. The Hasse invariant vanishes precisely where $V(\mathcal{L})$ is contained in $\ker(V^{(p)} : \mathcal{P}^{(p)} \to \mathcal{L}^{(p^2)})$. We have already seen that over $\bar{S}_\mu$ the latter is the line bundle $\mathcal{P}_0^{(p)}$ and that V sends $\mathcal{L}$ isomorphically onto a direct complement of $\mathcal{P}_0^{(p)}$, *cf* (2.19). Thus the Hasse invariant does not vanish on $\bar{S}_\mu$. To prove that $h_{\bar{\Sigma}}$ vanishes on S_{ss} to order 1 we must study the Dieudonné module at an infinitesimal neighborhood of a point $x \in S_{gss}$ and compute $V^{(p)} \circ V$ using local coordinates there. In Lemma 2.8 we described the (contravariant) Dieudonné module at x. In subsection 2.3.2 below we describe its infinitesimal deformation. Using the local coordinates u and v introduced there, $f_3 - uf_1 - vf_2$ becomes a basis for $\mathcal{L}$ over the first infinitesimal neighborhood of x. We then compute

$$\begin{aligned} V(f_3 - uf_1 - vf_2) &= e_1^{(p)} - ue_2^{(p)} \\ V^{(p)}(e_1^{(p)} - ue_2^{(p)}) &= -uf_3^{(p^2)} = -u \cdot (f_3 - uf_1 - vf_2)^{(p^2)}. \end{aligned} \tag{2.39}$$

It follows that after $\mathcal{L}$ has been locally trivialized, the equation $h_{\bar{\Sigma}} = 0$ becomes $u = 0$, which is the local equation for S_{gss}. ∎

2.3.2. *Infinitesimal deformations.* Let $x \in S_{gss}$. We shall study the infinitesimal deformation of the Dieudonné module of $\mathcal{A}$ at x. Let $\mathcal{O}_{S,x}$ be the local ring of S at x, $\mathfrak{m}$ its maximal ideal, and $R = \mathcal{O}_{S,x}/\mathfrak{m}^2$. Thus $Spec(R)$ is the first infinitesimal neighborhood of x in S, and $R \simeq \kappa[u,v]/(u^2, uv, v^2)$, although the choice of the local parameters u and v lies still at our disposal. The module of Kähler differentials of R is the 3-dimensional vector space over κ

$$\Omega_R^1 = \Omega_{R/\kappa}^1 = Rdu + Rdv / \langle udu, vdv, udv + vdu \rangle . \tag{2.40}$$

Let A be the restriction of $\mathcal{A}$ to $Spec(R)$ and $A_0 = \mathcal{A}_x$ its special fiber. Let $D = H^1_{dR}(A/R)$ (a free R-module of rank 6) and

$$D_0 = H^1_{dR}(A_0/\kappa) = \kappa \otimes_R D,$$

identified with the Dieudonné module of A_0 modulo p (see Lemma 2.8).

The Gauss-Manin connection [20] is in general defined only for abelian schemes over a base which is smooth over a field. In our case, despite the fact that R is not smooth over κ, the Gauss-Manin connection of $\mathcal{A}/S$ yields, by base change, also a connection

$$\nabla : D \to \Omega_R^1 \otimes_R D. \tag{2.41}$$

Caution must be exercised, though, because $\Omega_R^1 \neq \Omega_S^1 \otimes_{\mathcal{O}_S} R$. The Kodaira-Spencer map over R, for example, is not an isomorphism.

We claim that D has a basis of horizontal sections for ∇. This follows from the crystalline nature of H^1, but for completeness we give the easy argument. If $x \in D$ and

$$\nabla x = du \otimes x_1 + dv \otimes x_2 \tag{2.42}$$

($x_i \in D$) then $\tilde{x} = x - ux_1 - vx_2$ satisfies

$$\nabla \tilde{x} = -u \nabla x_1 - v \nabla x_2. \tag{2.43}$$

But if $\nabla x_1 = du \otimes x_{11} + dv \otimes x_{12}$ and $\nabla x_2 = du \otimes x_{21} + dv \otimes x_{22}$ then

$$0 = \nabla^2 x = du \wedge dv \otimes (x_{21} - x_{12}) \tag{2.44}$$

hence $x_{21} - x_{12} \in \mathfrak{m}D$. As $\mathfrak{m}^2 = 0$, $udu = vdv = 0$ and $udv = -vdu$, it follows that

$$\begin{aligned} \nabla \tilde{x} &= -udv \otimes x_{12} - vdu \otimes x_{21} \\ &= du \otimes v(x_{12} - x_{21}) = 0. \end{aligned} \tag{2.45}$$

This means that $\tilde{x}$ is a horizontal section having the same specialization as x in the special fiber, so the horizontal sections span D over R by Nakayama's lemma.

Let $e_1, e_2, f_3, f_1, f_2, e_3$ be any six horizontal sections over R, specializing to a basis of D_0. Identify D_0 with their κ-span in D. As we have just seen,

$$R \otimes_\kappa D_0 \to D \tag{2.46}$$

is surjective, hence, by a dimension count, an isomorphism, and $\nabla = d \otimes 1$ on the left hand side. Since $R^{d=0} = \kappa$, it follows that $D_0 = D^\nabla$, i.e. there are no more horizontal sections besides D_0. Thus every $x_0 \in H^1_{dR}(A_0/\kappa)$ has a *unique* extension to a horizontal section $x \in H^1_{dR}(A/R)$.

The pairing $\langle , \rangle$ on D induced from the polarization is horizontal for ∇, i.e.

$$d \langle x, y \rangle = \langle \nabla x, y \rangle + \langle x, \nabla y \rangle . \tag{2.47}$$

We conclude that we may regard the basis of D_0 given in Lemma 2.8 also as a basis of horizontal sections spanning D over R, and that the action of $\iota(\mathcal{O}_\mathcal{K})$ and the pairing $\langle , \rangle$ are given by the formulae of the lemma also over R.

As $p \neq 2$, R has a canonical divided power structure, and Grothendieck's crystalline deformation theory [16] tells us that A/R is completely determined by A_0 and by the Hodge filtration $\omega_{A/R} \subset D = R \otimes_\kappa D_0$. Since A *is* the universal infinitesimal deformation of A_0, we may choose the coordinates u and v so that

$$\mathcal{P} = Span_R\{e_1 + ue_3, e_2 + ve_3\}. \tag{2.48}$$

The fact that $\omega_{A/R}$ is isotropic for $\langle , \rangle$ implies then that

$$\mathcal{L} = Span_R\{f_3 - uf_1 - vf_2\}. \tag{2.49}$$

Consider the abelian scheme $A^{(p)}$. This is *not* the universal deformation of $A_0^{(p)}$ over R. In fact, the p-power map $\phi : R \to R$ factors as

$$R \xrightarrow{\pi} \kappa \xrightarrow{\phi} \kappa \xrightarrow{i} R, \tag{2.50}$$

and therefore $A^{(p)}$, *unlike* A, is constant: $A^{(p)} = A_0^{(p)} \times_\kappa R$ (intuitively, Φ is contracting on the base, so the pull-back of $\mathcal{A}$ becomes constant on Artinian neighborhoods which are contracted to a point). As with D, $D^{(p)} = R \otimes_\kappa D_0^{(p)}$, $\nabla = d \otimes 1$, but this time the basis of horizontal sections can be obtained also from the trivalization of $A^{(p)}$, and $\omega_{A^{(p)}/R} = Span_R\{e_1^{(p)}, e_2^{(p)}, f_3^{(p)}\}$.

The isogenies $Frob$ and Ver, like any isogeny, take horizontal sections with respect to the Gauss-Manin connection to horizontal sections, e.g. if $x \in D$ and $\nabla x = 0$ then $Vx \in D^{(p)}$ satisfies $\nabla(Vx) = 0$. Since V and F preserve horizontality, e_1, f_2, e_3 span $\ker(V)$ over R in D, and the relations in *(v)* and *(vi)* of Lemma 2.8 continue to hold. Indeed, the matrix of V in the basis at x prescribed by that lemma continues to represent V over $Spec(R)$ by "horizontal continuation". The matrix of F is then derived from the relation

$$\langle Fx, y \rangle = \langle x, Vy \rangle^{(p)} \tag{2.51}$$

$(x \in D^{(p)}, y \in D)$.

The Hodge filtration nevertheless varies, so we conclude that

$$\mathcal{P}_0 = \mathcal{P} \cap \ker(V) = Span_R\{e_1 + ue_3\}. \tag{2.52}$$

The condition $V(\mathcal{L}) = \mathcal{P}_0^{(p)}$, which is the "equation" of the closed subscheme $S_{gss} \cap Spec(R)$ means

$$V(f_3 - uf_1 - vf_2) = e_1^{(p)} - ue_2^{(p)} \in R \cdot e_1^{(p)} \tag{2.53}$$

and this holds if and only if $u = 0$. We have proved the following lemma, which completes the proof of Theorem 2.11.

Lemma 2.12. *Let $x \in S_{gss}$ and the coordinates u, v be as above. Then the closed subscheme $S_{gss} \cap Spec(R)$ is given by the equation $u = 0$.*

2.4. **A secondary Hasse invariant on the supersingular locus.** In his forthcoming Ph.D. thesis [5], Boxer develops a general theory of secondary Hasse invariants defined on lower strata of Shimura varieties of Hodge type. See also [21]. In this section we provide an independent approach, in the case of Picard modular surfaces, affording a detailed study of its properties.

2.4.1. *Definition of h_{ssp}.* As we have seen, along the general supersingular locus S_{gss}, Verschiebung induces isomorphisms

$$V_{\mathcal{L}} : \mathcal{L} \simeq \mathcal{P}_0^{(p)}, \;\; V_{\mathcal{P}} : \mathcal{P}_\mu \simeq \mathcal{L}^{(p)}. \tag{2.54}$$

(The first is unique to S_{gss}, the second holds also on the μ-ordinary stratum.) Consider the isomorphism

$$V_{\mathcal{P}}^{(p)} \otimes V_{\mathcal{L}}^{-1} : \mathcal{P}_\mu^{(p)} \otimes \mathcal{P}_0^{(p)} \simeq \mathcal{L}^{(p^2)} \otimes \mathcal{L} \simeq \mathcal{L}^{p^2+1}. \tag{2.55}$$

Its source is the line bundle $\det \mathcal{P}^{(p)}$ which is identified with $\mathcal{L}^{(p)} \simeq \mathcal{L}^p$. We therefore get a nowhere vanishing section

$$\tilde{h}_{ssp} \in H^0(S_{gss}, \mathcal{L}^{p^2-p+1}). \tag{2.56}$$

Our "secondary" Hasse invariant is the nowhere vanishing section

$$h_{ssp} = \tilde{h}_{ssp}^{p+1} \in H^0(S_{gss}, \mathcal{L}^{p^3+1}). \tag{2.57}$$

We shall show that h_{ssp} extends to a holomorphic section on S_{ss}, and vanishes at the superspecial points (to a high order).

2.4.2. *Computations at the superspecial points.* The goal of this subsection is to show that h_{ssp} (unlike $\tilde{h}_{ssp}$) extends over S_{ss}, and to compute its order of vanishing at the superspecial points. We refer to the computations of Proposition 2.10. Dualizing (to put us back in the contravariant world), and using the letters e_i, f_j to denote *the dual basis* to the basis used there we get the following.

Lemma 2.13. *Let $x \in S_{ssp}$ be a superspecial point. There exist formal coordinates u and v so that the formal completion of S at x is $\widehat{S} = Spf(\kappa[[u, v]])$, and $D = H^1_{dR}(\mathcal{A}/\widehat{S})$ has a basis $f_3, e_1, e_2, e_3, f_1, f_2$ over $\kappa[[u, v]]$ with the following properties:*

(i) f_3, e_1, e_2 is a basis for $\omega_{\mathcal{A}}$

(ii) The basis is symplectic, i.e. the polarization form is $\langle e_i, f_j \rangle = -\langle f_j, e_i \rangle = \delta_{ij}$, $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$.

(iii) $\mathcal{O}_{\mathcal{K}}$ acts on the e_i via Σ and on the f_j via $\bar{\Sigma}$.

(iv) $V : D \to D^{(p)}$ is given by $Vf_3 = ue_1^{(p)} + ve_2^{(p)}$, $Ve_1 = uf_3^{(p)}$, $Ve_2 = vf_3^{(p)}$, $Ve_3 = f_3^{(p)}$, $Vf_1 = e_1^{(p)}$, $Vf_2 = e_2^{(p)}$.

Using the lemma, we compute along $\widehat{S}_{ss}[\zeta]$, where $u = \zeta v$ ($\zeta^{p+1} = -1$). See the discussion following (2.32) for the definition of the formal branch $\widehat{S}_{ss}[\zeta]$. Denote by $\mathcal{P}[\zeta]$, $\mathcal{P}_0[\zeta]$ and $\mathcal{L}[\zeta]$ the pull-backs of the corresponding vector bundles to $\widehat{S}_{ss}[\zeta]$. The map $V_{\mathcal{L}}$ is given by

$$f_3 \mapsto ue_1^{(p)} + ve_2^{(p)} = v \cdot (\zeta e_1^{(p)} + e_2^{(p)}) = v \cdot (\zeta^p e_1 + e_2)^{(p)} \in \mathcal{P}_0^{(p)}[\zeta]. \tag{2.58}$$

Use $e_1 \wedge e_2 = e_1 \wedge (\zeta^p e_1 + e_2)$ as a basis for $\det \mathcal{P}[\zeta]$. Since $V_{\mathcal{P}}^{(p)}$ maps $e_1^{(p)}$ to $(\zeta v)^p f_3^{(p^2)}$, $\tilde{h}_{ssp}$ maps $e_1^{(p)} \wedge e_2^{(p)} = e_1^{(p)} \wedge (\zeta^p e_1 + e_2)^{(p)}$ to

$$\begin{aligned} \tilde{h}_{ssp}(e_1^{(p)} \wedge e_2^{(p)}) &= \zeta^p v^{p-1} f_3^{(p^2)} \otimes f_3 \\ &= \zeta^p v^{p-1} f_3^{p^2+1} = \zeta u^{p-1} f_3^{p^2+1}. \end{aligned} \tag{2.59}$$

Lemma 2.14. *There does not exist a function $g \in \kappa[[u,v]]/(u^{p+1} + v^{p+1})$ on $\widehat{S}_{ss}$ whose restriction to the branch $\widehat{S}_{ss}[\zeta]$ is ζu^{p-1}.*

Proof. Had there been such a function g, represented by a power series $G \in \kappa[[u,v]]$, then we would get $vg = u^p$ on $\widehat{S}_{ss}[\zeta]$ for every ζ, hence

$$vG - u^p \in (u^{p+1} + v^{p+1}) \subset \kappa[[u,v]]. \tag{2.60}$$

But any power series in the ideal $(u^{p+1}+v^{p+1})$ contains only terms of degree $\geq p+1$, while in $vG - u^p$ we can not cancel the term u^p. ∎

The lemma means that $\tilde{h}_{ssp}$ can not be extended over S_{ss} to a section of $Hom(\det \mathcal{P}^{(p)}, \mathcal{L}^{p^2+1}) \simeq \mathcal{L}^{p^2-p+1}$. However, when we raise it to a $p+1$ power the dependence on ζ disappears. It then extends to a section h_{ssp} of $\mathcal{L}^{p^3+1}$ over S_{ss}, given over $\widehat{S}_{ss}$ (the formal completion of S_{ss} at x) by the equation

$$h_{ssp} = \varepsilon u^{p^2-1} f_3^{p^3+1}, \tag{2.61}$$

where $\varepsilon \in \kappa[[u,v]]^\times$ depends on the isomorphism between $\det \mathcal{P}$ and $\mathcal{L}$.

Theorem 2.15. *The secondary Hasse invariant h_{ssp} belongs to $H^0(S_{ss}, \mathcal{L}^{p^3+1})$. It vanishes precisely at the points of S_{ssp}. The subscheme "$h_{ssp} = 0$" of S_{ss} is not reduced. At $x \in S_{ssp}$, with u and v as above, it is the spectrum of*

$$\kappa[[u,v]]/(u^{p+1} + v^{p+1}, u^{p^2-1}, v^{p^2-1}). \tag{2.62}$$

3. On the number of supersingular curves on S

We continue to work over κ, and to ease the notation drop the subscript κ.

3.1. **The connected components of S_{ss}.** The Picard surface $\bar{S}$ is not connected. The supersingular locus S_{ss} is, however, as connected as it could be.

For the next proposition we need, besides the smooth compactification, also the Baily-Borel (singular) compactification S^* of S, see [25] and [3]. Every geometric component of $C = \bar{S} - S$ is contracted in S^* to a point. The surface S^* is known to be normal.

Proposition 3.1. *The scheme $S_\mu^* = S^* - S_{ss}$ is affine and the intersection of S_{ss} with every connected component of S is connected.*

Proof. The line bundle $\mathcal{L}$ is ample on S, even over R_0.[1] Hence for large enough m, which we take to be a multiple of p^2-1, $\mathcal{L}^m$ is very ample, and by [25] the Baily-Borel compactification S^* is the closure of S in the projective embedding supplied by the linear system $H^0(S,\mathcal{L}^m)$. It follows that $\mathcal{L}^m$ has an extension to a line bundle on S^* which we denote $\mathcal{O}_{S^*}(1)$, since it comes from the restriction of the $\mathcal{O}(1)$ of the projective space to S^*. Moreover, Larsen proves that on the smooth compactification $\bar{S}$, $\mathcal{L}^m = \pi^*\mathcal{O}_{S^*}(1)$ where $\pi : \bar{S} \to S^*$.[2]

Replacing $h_{\bar{\Sigma}}$ by its power $h_{\bar{\Sigma}}^{m/(p^2-1)}$, this power becomes a global section of $\mathcal{L}^m$, hence its zero locus S_{ss} a hyperplane section of S^* in the projective embeding supplied by $H^0(S,\mathcal{L}^m)$. Its complement is therefore affine. The second claim follows from the fact [18], III, 7.9, that a positive dimensional hyperplane section of a normal projective variety is connected. ∎

3.2. **The number of irreducible components.**

3.2.1. *The degree of $\mathcal{L}$ along an irreducible component of S_{ss}.* Assume that N is large enough (depending on p) so that Theorem 2.1(iii) holds. Each irreducible component Z of S_{ss} is non-singular, and the secondary Hasse invariant h_{ssp} has a zero of order p^2-1 at every superspecial point of Z, as follows from (2.61). Each component contains p^3+1 superspecial points. It follows that if Z is such a component,

$$\deg(\mathcal{L}^{p^3+1}|_Z) = \deg(div_Z(h_{ssp})) = (p^3+1)(p^2-1). \tag{3.1}$$

We have proved the following lemma.

Lemma 3.2. *Let Z be an irreducible component of S_{ss}, and assume that N is large enough. Then* $\deg(\mathcal{L}|_Z) = p^2-1$.

3.2.2. *A computation of intersection numbers.* Let

$$Z = \bigcup_{i=1}^{n} Z_i \tag{3.2}$$

be the decomposition into irreducible components of a single connected component Z of S_{ss} (recall that Z is the intersection of S_{ss} with a connected component of $\bar{S}$). If N is large, then the Z_i are smooth, and as they are Fermat curves of degree $p+1$, their genus is $g(Z_i) = p(p-1)/2$.

Theorem 3.3. *Let c_2 be the Euler characteristic of the connected component of $\bar{S}$ containing Z, i.e. if over $\mathbb{C}$ this connected component is $\bar{X}_\Gamma$ then*

$$c_2 = \sum_{i=0}^{4} (-1)^i \dim_{\mathbb{C}} H^i(\bar{X}_\Gamma, \mathbb{C}). \tag{3.3}$$

Then the number n of irreducible components of Z is given by

$$3n = c_2. \tag{3.4}$$

---

[1] One way to see it is to use the ampleness of the Hodge bundle $\det \omega_{\mathcal{A}} \simeq \mathcal{L}^2$ (pull back from Siegel space, where it is known to be ample by [Fa-Ch]).

[2] It is not clear that $\mathcal{L}$ itself has an extension to a line bundle on S^*, or that $\pi_*\mathcal{L}$, which is a coherent sheaf extending $\mathcal{L}|_S$, is a line bundle (the problem lying of course only at the cusps). In other words, it is not clear that we can extract an mth root of $\mathcal{O}_{S^*}(1)$ as a line bundle.

This $c_2 = c_2(\bar{X}_\Gamma)$ is given by Holzapfel's formula. To get the theorem quoted in the introduction, one would have to sum over all the connected components.

Proof. We first prove the theorem under the assumption that N is large enough. Computing intersection numbers, and using the fact that every Z_i meets transversally $(p^3+1)p$ other $Z_j's$ (there are p^3+1 intersection points on Z_i, through each of which pass $p+1$ components, including Z_i itself), we get

$$(Z.Z) = n(p^3+1)p + \sum_{i=1}^{n}(Z_i.Z_i). \tag{3.5}$$

Denote by $K_{\bar{S}}$ a canonical divisor on the given connected component of $\bar{S}$. From the adjunction formula,

$$p(p-1)-2 = 2g(Z_i)-2 = Z_i.(Z_i+K_{\bar{S}}). \tag{3.6}$$

As we have seen in Proposition 1.8, $\mathcal{O}(K_{\bar{S}}+C) \simeq \mathcal{L}^3$ where C is the cuspidal divisor (on the given connected component of $\bar{S}$). Hence

$$(Z_i.K_{\bar{S}}) = Z_i.(K_{\bar{S}}+C) = \deg(\mathcal{L}^3|_{Z_i}) = 3(p^2-1) \tag{3.7}$$

by the previous lemma. We get

$$(Z_i.Z_i) = -2p^2 - p + 1. \tag{3.8}$$

Plugging this into the expression for $(Z.Z)$ we get

$$(Z.Z) = n(p^2-1)^2. \tag{3.9}$$

On the other hand, as Z is the divisor of the Hasse invariant on the given connected component of $\bar{S}$, $div(h_{\bar{\Sigma}}) = Z$, and $h_{\bar{\Sigma}}$ is a global section of $\mathcal{L}^{p^2-1}$, we get $\mathcal{O}(Z) = \mathcal{L}^{p^2-1}$. From the relation $(Z.Z) = c_1(\mathcal{O}(Z))^2$ between the self-intersection number and the first Chern class,

$$n = c_1(\mathcal{L})^2. \tag{3.10}$$

From this and $\mathcal{O}(K_{\bar{S}}+C) \simeq \mathcal{L}^3$ we get $9n = (K_{\bar{S}}+C).(K_{\bar{S}}+C)$. Holzapfel [19] (4.3.11′) on p.184, implies

$$9n = 3c_2(X_\Gamma) = 3c_2(\bar{X}_\Gamma) \tag{3.11}$$

proving the theorem when N is large.

We now note that while the assumption of N being large was crucial for the intersection-theoretic computations, the end result $3n = c_2$ holds for a given $N \geq 3$ if and only if it holds for any multiple N' of N. Indeed, the covering $S(N') \to S(N)$ is étale, say of degree $d(N,N')$. The second Chern class, being equal to the Euler characteristic, gets multiplied by $d(N,N')$. But thanks to Theorem 2.1(iv), the same holds true for the number n. We may therefore deduce the validity of our formula for N from its validity for N'. This completes the proof of the theorem. ∎

One can probably get a "mass formula" for the number of irreducible components weighted by the reciprocals of the orders of certain automorphism groups even if $N = 1$. We do not pursue it here.

We easily deduce the following two corollaries.

Corollary 3.4. *(a) Let p be inert in $\mathcal{K}$. Then for N sufficiently large the number of superspecial points on S is*

$$\frac{c_2(\overline{S})}{3} \cdot (p^2 - p + 1). \tag{3.12}$$

(b) The aritmetic genus g_a of a connected component Z of S_{ss} is given by

$$g_a(Z) = \frac{c_2(\overline{S}_Z)}{3} \cdot \frac{(p^4 + p^2 - 2)}{2} + 1, \tag{3.13}$$

where $\overline{S}_Z$ is the connected component of $\overline{S}$ containing Z.

Proof. (a) Each irreducible component of S_{ss} contains $p^3 + 1$ points of S_{ssp} and each point of S_{ssp} is shared by $p + 1$ irreducible components.

(b) This follows from the adjunction formula, which for the singular curve Z takes the form

$$2g_a(Z) - 2 = Z.(Z + K_{\bar{S}}) \tag{3.14}$$

(see [27], Chapter 9, Theorem 1.37), using the computations of $Z.Z$ and $Z.K_{\bar{S}}$ carried out in the proof of the theorem. ∎

Bibliography

[1] F. Andreatta, E. Z. Goren: Hilbert modular forms: mod p and p-adic aspects, Memoirs A.M.S. **819,** 2005.

[2] E. Bachmat, E. Z. Goren: On the non ordinary locus in Hilbert-Blumenthal surfaces, Math. Annalen, **313** (1999) 475-506.

[3] J. Bellaïche: Congruences endoscopiques et représentations Galoisiennes, *Thèse*, Paris XI (Orsay), 2002.

[4] J. Bellaïche: Relévement des formes modulaires de Picard, J. London Math. Soc.(2), **74** (2006), 13-25.

[5] G. Boxer, *Ph.D. Thesis,* Harvard, to appear.

[6] O.Bültel, T.Wedhorn: Congruence relations for Shimura varieties associated to some unitary groups, J. Instit. Math. Jussieu **5**, (2006) 229-261.

[7] J. Cogdell: Arithmetic cycles on Picard modular surfaces and modular forms of nebentypus, J. Reine Angew. Math. **357** (1985), 115-137.

[8] P. Deligne: Travaux de Shimura, Sém. Bourbaki **389** (1971).

[9] P. Deligne, M. Rapoport: Les schémas de modules de courbes elliptiques, *in: Modular Functions of One Variable* II, LNM **349** (1973), 143-316.

[10] M. Demazure: *Lectures on p-divisible groups*, LNM **302**, Springer, 1972.

[11] G. Faltings, C.-L. Chai: *Degeneration of abelian varieties,* Springer, 1990.

[12] W. Goldring, M.-H. Nicole: A μ-ordinary Hasse invariant, *preprint*, arXiv:1302.1614.

[13] B. Gordon: Canonical models of Picard modular surfaces, *in:* [24], 1-29.

[14] E. Z. Goren: Hasse invariants for Hilbert modular varieties, Israel J. Math. **122** (2001), 157-174.

[15] E. Z. Goren, F. Oort: Stratification of Hilbert modular varieties, J. Alg. Geom. **9** (2000), 111-154.

[16] A. Grothendieck: *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Les Presses de l'Université de Montréal, 1974.

[17] M. Harris: Arithmetic vector bundles and automorphic forms on Shimura varieties. I, Invent. Math. **82** (1985), 151-189.

[18] R. Hartshorne: *Algebraic Geometry,* Graduate Texts in Mathematics, No. **52**, Springer, 1977.

[19] R.-P. Holzapfel: *Ball and surface arithmetics*, Aspects of Mathematics, **E29**, Vieweg & Sohn, 1998.

[20] N. Katz, T. Oda: On the differentiation of de Rham cohomology classes with respect to parameters, J. Math. Kyoto Univ. **8** (1968), 199-213.

[21] J.-S. Koskivirta: Sections of the Hodge bundle over Ekedahl Oort strata of Shimura varieties of Hodge type, *preprint*, arXiv:1410.1317.

[22] J.-S. Koskivirta, T. Wedhorn: Hasse invariants for Shimura varieties of Hodge type, *preprint,* arXiv:1406.2178.

[23] K.-W. Lan: Arithmetic compactifications of PEL-type Shimura varieties, London Mathematical Society Monographs **36**, Princeton, 2013.

[24] R. Langlands, D. Ramakrishnan: *The zeta functions of Picard modular surfaces*, Univ. Montréal, Montréal, 1992.

[25] M. Larsen: Unitary groups and l-adic representations, *Ph.D. Thesis*, Princeton University, 1988.

[26] M. Larsen: Arithmetic compactification of some Shimura surfaces *in*: [24], 31-45.

[27] Q. Liu: *Algebraic geometry and arithmetic curves*, Oxford University Press, 2002.

[28] J. Milne: *Introduction to Shimura varieties,* available on the web at: www.jmilne.org/math/xnotes/svi.pdf.

[29] T. Oda: On the first de Rham cohomology group and Dieudonné modules, Ann. Sci. ENS **2** (1969), 63-135.

[30] G. Shimura: The arithmetic of automorphic forms with respect to a unitary group, Ann. of Math. **107** (1978), 569-605.

[31] I. Vollaard: The supersingular locus of the Shimura variety for $GU(1,s)$, Canad. J. Math. **62** (2010), 668-720.

[32] T. Wedhorn: The dimension of Oort strata of Shimura varieties of PEL-type, *in: Moduli of abelian varieties,* Progr. Math. **195**, 441-471, Birkhäuser, 2001.

Hebrew University, Jerusalem, Israel
E-mail address: deshalit@math.huji.ac.il.

McGill University, Montreal, Canada
E-mail address: eyal.goren@mcgill.ca