A NOTE ON VANISHING PROPERTIES OF CERTAIN THETA CONSTANTS OF GENUS TWO

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Abstract We study the vanishing of Riemann's theta constants $\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)$ for $\varepsilon, \varepsilon' \in \frac{1}{k}\mathbb{Z}^2$, $k = 4, 6$ and derive identities like

$$\lambda \prod_{\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\in A^-} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)^2 = \prod_{\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\in E} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)^{96}$$

for some $\lambda \in \mathbb{Q}(\zeta_{36})^\times$, where

$$\begin{aligned} A^- = \Bigg\{ \begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} :\ & -\frac{1}{2} \le \varepsilon_i,\ \varepsilon_i' < \frac{1}{2},\ i = 1, 2, \\ & \begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \text{ is of exact order 6 in } \mathbb{R}^4/\mathbb{Z}^4,\ 36\ {}^t\varepsilon\cdot\varepsilon' \equiv 1 \pmod 2 \Bigg\} \end{aligned}$$

and

$$E = \left\{ \begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} :\ \varepsilon_i, \varepsilon_i' \in \left\{0, \frac{1}{2}\right\},\ i = 1, 2,\ 4\ {}^t\varepsilon\cdot\varepsilon' \equiv 0 \pmod 2 \right\}.$$

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Definition. Define *Siegel's upper half space* of genus g, $\mathfrak{H}_g$, by

$$\mathfrak{H}_g = \{\tau \in \mathrm{Mat}_{g\times g}(\mathbb{C})|\ {}^t\tau = \tau, \mathrm{Im}(\tau) \text{ is positive definite}\}.$$

For $\varepsilon, \varepsilon' \in \mathbb{R}^g$, $z \in \mathbb{C}^g$, $\tau \in \mathfrak{H}_g$ define Riemann's theta function with characteristic $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$

$$(1)\quad \Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau) = \sum_{N\in\mathbb{Z}^g} \exp\left\{2\pi i\left(\frac{1}{2}\ {}^t(N+\varepsilon)\tau(N+\varepsilon) +\ {}^t(N+\varepsilon)(z+\varepsilon')\right)\right\}.$$

We call $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ a rational characteristic if $\varepsilon, \varepsilon' \in \mathbb{Q}^g$ and an *integral characteristic* if $2\varepsilon, 2\varepsilon' \in \mathbb{Z}^g$.

It is well known that (1) defines a holomorphic function on $\mathbb{C}^g \times \mathfrak{H}_g$ (see [FK] §VI.1. Note that our definition of $\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau)$ is slightly different from the definition in [FK] which is also customary). Let $\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau) = \Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(0,\tau)$. The following transformation formulas are classical ([Mum1] p. 123):

$$(2)\quad \Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau) = \exp\left\{2\pi i\left(\frac{1}{2}\ {}^t\varepsilon\tau\varepsilon +\ {}^t\varepsilon z +\ {}^t\varepsilon\varepsilon'\right)\right\}\cdot\Theta\left(z+(\tau I)\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix},\tau\right)$$

$$(3)\qquad \Theta\begin{bmatrix} \varepsilon+\mu \\ \varepsilon'+\mu' \end{bmatrix}(z,\tau) = \exp\{2\pi i(\ {}^t\varepsilon\mu')\}\cdot\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau) \qquad \begin{bmatrix} \mu \\ \mu' \end{bmatrix} \in \mathbb{Z}^{2g}$$

(4)
$$\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}\left(z+(\tau I)\begin{bmatrix} \mu \\ \mu' \end{bmatrix},\tau\right) = \exp\left\{2\pi i\left(-\frac{1}{2}\ {}^t\mu(\tau\mu+2z+2\varepsilon') +\ {}^t\varepsilon\mu'\right)\right\}\cdot\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau).$$

If $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ is an integral characteristic then

$$(5)\qquad \Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(-z,\tau) = \exp\{2\pi i(2{}^t\varepsilon\varepsilon')\}\cdot\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau).$$

Remark. One often thinks about the functions $\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau)$ and $\Theta\begin{bmatrix} \varepsilon+\mu \\ \varepsilon'+\mu' \end{bmatrix}(z,\tau)$ as the same if $\mu, \mu' \in \mathbb{Z}^g$. Indeed, by (3), $\Theta\begin{bmatrix} \varepsilon+\mu \\ \varepsilon'+\mu' \end{bmatrix}(z,\tau) = \exp(2\pi i\ {}^t\varepsilon\cdot\mu')\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z,\tau)$ and therefore, at least for questions concerning their divisors, these two theta functions could be identified. In this context we remark that if we associate to $\begin{bmatrix}\delta\\ \delta'\end{bmatrix}$ the point $(\tau I)\begin{bmatrix}\delta\\ \delta'\end{bmatrix}$ on $\mathbb{C}^g/(\tau I)\mathbb{Z}^{2g}$, then the points associated to $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ and $\begin{bmatrix}\varepsilon+\mu\\ \varepsilon'+\mu'\end{bmatrix}$ are the same.

Definition. Let $J=\begin{pmatrix}0 & I_g\\ -I_g & 0\end{pmatrix}$ where I_g is the $g\times g$ identity matrix. We define the *symplectic group*

$$\mathrm{Sp}(2g,\mathbb{Q})=\{M\in \mathrm{Mat}_{2g\times 2g}(\mathbb{Q})|\ {}^tMJM=J\}.$$

We will denote $\mathrm{Sp}(2g,\mathbb{Z})$ also by $\Gamma_g(1)$. For any matrix $M=(m_{ij})_{1\le i,j\le g}$ we let $\Delta(M)={}^t(m_{11},\dots,m_{gg})$ and for a positive integer n we put $\Gamma_g(n)=\{M\in \Gamma_g(1)|\ M\equiv I_{2g} \mod n\}$.

The group $\mathrm{Sp}(2g,\mathbb{R})$ acts as a group of holomorphic automorphisms of $\mathfrak{H}_g$: $\begin{pmatrix}A & B\\ C & D\end{pmatrix}\tau=(A\tau+B)(C\tau+D)^{-1}$. This action extends to $\mathbb{C}^g\times\mathfrak{H}_g$ by $\begin{pmatrix}A & B\\ C & D\end{pmatrix}(z,\tau)=({}^t(C\tau+D)^{-1}z,(A\tau+B)(C\tau+D)^{-1}))$.

Definition. Let $\gamma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in \mathrm{Sp}(2g,\mathbb{Z})$. Define for $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\in\mathbb{R}^{2g}$:

$$(6)\qquad \gamma^*\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}=\left[{}^t\gamma^{-1}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}+\frac{1}{2}\begin{bmatrix}\Delta(C{}^tD)\\ \Delta(A{}^tB)\end{bmatrix}\right]=\begin{bmatrix}D\varepsilon-C\varepsilon'+\frac{1}{2}\Delta(C{}^tD)\\ -B\varepsilon+A\varepsilon'+\frac{1}{2}\Delta(A{}^tB)\end{bmatrix}.$$

It is easy to check that this is a group action on $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$. Let $Ch(k)$ ($k\in\mathbb{N}$) be the set of characteristics $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ such that $k\varepsilon, k\varepsilon'\in\mathbb{Z}^g$. If k is even, then for $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\in Ch(k)$ put $\mathrm{sgn}_k\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}=\mathrm{sgn}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}=(k^2\ {}^t\varepsilon\cdot\varepsilon') \mod 2$. We call $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ *even* if $\mathrm{sgn}\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}=0$ and *odd* otherwise (if needed we will make the dependence on k explicit by saying *k-even, k-odd*). Note that since k is even, $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ is k-even (k-odd) if and only if $\begin{bmatrix}\varepsilon+\mu\\ \varepsilon'+\mu'\end{bmatrix}$ is k-even (k-odd) for any $\begin{bmatrix}\mu\\ \mu'\end{bmatrix}\in\mathbb{Z}^{2g}$. The action of $\mathrm{Sp}(2g,\mathbb{Z})$ preserves both $Ch(2)$ and the sgn function on it. Moreover it acts transitively on the even (odd) integral characteristics $\mod \mathbb{Z}^{2g}$. We have the important transformation formula ([FC] p. 161):

(7)

$$\Theta\gamma^*\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\gamma(z,\tau))=K(\gamma)\cdot\zeta\left(\gamma,\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\right)\det(C\tau+D)^{1/2}\exp(\pi i\,{}^tz(C\tau+D)^{-1}Cz)\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(z,\tau)$$

where $K(\gamma) \in \mu_8$ depends only on γ and the choice of the square root and

(8)
$$\zeta\left(\gamma, \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}\right) = \exp(-\pi i\{\,{}^t\varepsilon{}^tBD\varepsilon' + {}^t\varepsilon'\,{}^tAC\varepsilon' - 2{}^t\varepsilon{}^tBC\varepsilon' - {}^t\Delta(A{}^tB)(D\varepsilon - C\varepsilon')\}).$$

Definition. 1) Let C be a Riemann surface of genus g. An *Abel-Jacobi structure* on C is a triple (B, Ω, c) where $B = \{\gamma_1, \dots, \gamma_g, \delta_1, \dots, \delta_g\}$ is an ordered symplectic basis for $H_1(C, \mathbb{Z})$ (w.r.t. the intersection pairing)[1], $\Omega = \{\theta_1, \dots, \theta_g\}$ is a basis for the holomorphic 1-forms on C which is dual to $\{\gamma_1, \dots, \gamma_g\}$ w.r.t. the de-Rham pairing (i.e. $\int\limits_{\gamma_j} \theta_i = \delta_{ij}$), $c \in C$ is a point. We call (C, B, Ω, c) an *Abel-Jacobi data* (A.J. data).

Given an A.J. structure we have $\int\limits_{\delta_j} \theta_i = \pi_{ij}$ and $\pi = (\pi_{ij}) \in \mathfrak{H}_g$. We denote by $\varphi_c : C \to \mathbb{C}^g/(\pi I)\mathbb{Z}^{2g}$ the *Abel-Jacobi map* with base point c. That is $\varphi_c(P) = \int\limits_c^P {}^t(\theta_1, \dots, \theta_g)$, $P \in C$. Given an A.J. structure, denote Riemann's vector of constants by K_c (see [FK] §VI.2.4). If C is of genus 2 the map $C \to \mathbb{C}^2/(\pi I)\mathbb{Z}^4$ given by $P \mapsto \varphi_c(P) + K_c$ is then independent of c (loc. cit. §VI.3.7).

2) Let $H_1^+ = \left\{\mathrm{Sp}(4, \mathbb{Z}) \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix} : \tau_1, \tau_2 \in \mathfrak{H}\right\}$. It is known that $\pi \notin H_1^+$ if and only if π is obtained from an A.J. structure on some Riemann surface C of genus 2 (see [OU]). The following theorem is well known:

Theorem 0. *Let* $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in \frac{1}{2}\mathbb{Z}^4$ *be an integral characteristic. The function* $\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(z, \tau)$, $\tau \in \mathfrak{H}_2$ *where* τ *corresponds to an A.J. data, vanishes at* $z = 0$ *if and only if* $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ *is an odd characteristic. Moreover, the six points* $(\tau I)\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$, $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ *odd, are precisely the images of the six Weierstrass points of* C *under the Abel-Jacobi embedding* $P \mapsto \varphi_c(P) + K_c$ *associated with the A.J. data.*

Theta functions whose divisor is contained in H_1^+ are (possibly) of particular importance for the construction of units in abelian extensions of quartic C.M. fields (see [DSG]). This explains our interest in the theorems below. We remark that these theorems are also of a certain Diophantine interest because of Riemann's vanishing theorem: The vanishing of $\Theta\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ at τ, which does not lie on H_1^+,

[1] By that we mean: $\gamma_i \cdot \delta_j = \delta_j \cdot \gamma_i = 0$ for $i \neq j$; $\gamma_i \cdot \gamma_j = \delta_i \cdot \delta_j = 0$ for all i, j; $\gamma_i \cdot \delta_i = -\delta_i \cdot \gamma_i = 1$.

is equivalent to saying that under the Abel-Jacobi embedding $P \mapsto \varphi_c(P) + K_c$ (which is independent of c) the curve C passes through the torsion point $(\tau I)\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ in $\mathrm{Jac}(C) \cong \mathbb{C}^2/(\tau I)\mathbb{Z}^4$. Since such Abel-Jacobi embeddings have a canonical definition (they are the embeddings such that $\mathcal{L} = O_{\mathrm{Jac}(C)}(C)$ is symmetric and $e^*_{\mathcal{L}}(0) = 1$ (see [Mum2] for the definition of $e^*_{\mathcal{L}}$)), it makes sense to study the set $C \cap \mathrm{Jac}(C)_{tor}$ for these particular embeddings.

Theorem 1. *Let $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \in Ch(6)$ be a characteristic corresponding to a point of exact order 6. Assume that $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ is 6-odd, i.e., $36\ {}^t\varepsilon \cdot \varepsilon' = 1(\mod 2)$. Then the divisor of $\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)$ is contained in H_1^+.*

Proof. First note that we can uniquely write $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} = \begin{bmatrix}\mu\\ \mu'\end{bmatrix} + \begin{bmatrix}\eta\\ \eta'\end{bmatrix}$ where $\begin{bmatrix}\mu\\ \mu'\end{bmatrix}$ is of exact order 3 and $\begin{bmatrix}\eta\\ \eta'\end{bmatrix}$ is of exact order 2. We easily verify that $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ is odd if and only if $\begin{bmatrix}\eta\\ \eta'\end{bmatrix}$ is an odd characteristic. Let $\tau \in \mathfrak{H}_2 - H_1^+$ and let (C, B, Ω, c) be an A.J. data yielding τ. Choose $P \in C$ a Weierstrass point. Suppose that $\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau) = 0$. Then

1) $\exists\zeta \in C$ such that $\varphi_P(\zeta) + K_P = (\tau I)\begin{bmatrix}\eta\\ \eta'\end{bmatrix}$.

2) $\exists\xi \in C$ such that $\varphi_P(\xi) + K_P = (\tau I)\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} = (\tau I)\begin{bmatrix}\mu\\ \mu'\end{bmatrix} + (\tau I)\begin{bmatrix}\eta\\ \eta'\end{bmatrix}$.

(The equalities hold on the abelian variety $\mathbb{C}^2/(\tau I)\mathbb{Z}^4$). Therefore $\varphi_P(\xi) - \varphi_P(\zeta) = (\tau I)\begin{bmatrix}\mu\\ \mu'\end{bmatrix}$ is of exact order 3. That is, there exists a function f on C such that $(f) = 3\xi - 3\zeta$. However, ζ is a Weierstrass point (by Theorem 0) and 3 is a gap for Weierstrass points on curves of genus 2 ([FK III.5.3). Contradiction. Q.E.D.

Theorem 2. *Let $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \in Ch(4)$ be a characteristic corresponding to a point of exact order 4. Then the divisor of $\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)$ is contained in H_1^+.*

Proof. We start with the following lemma

Lemma. *Let C be a hyperelliptic curve of genus $g \geq 2$. Denote the Weierstrass points of C by $\{P_1, \ldots, P_{2g+2}\}$.*

1) *Every point in $\mathrm{Jac}(C)[2]$ is of the form $P(J) = \sum_{j\in J} P_j - \#J \cdot P_1$ for some $J \subseteq \{2, 3, \ldots, 2g+2\}$ (considered as an element of $\mathrm{Div}^0(C)/\mathrm{Pr}(C) \cong \mathrm{Jac}(C)$). This*

way of writing is unique up to the relations $P(J) = P(\{2,3,\dots,2g+2\} - J)$.

2) Let $g = 2$*, fix some A.J. structure on* C *and let* (τI) *be the resulting period matrix. Identify the integral characteristics with the points of order two of* $\mathrm{Jac}(C) = \mathbb{C}^2/(\tau I)\mathbb{Z}^4$ *by* $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \leftrightarrow (\tau I)\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ *and call a point of order* 2 *in* $\mathrm{Jac}(C)$ *odd/even if the corresponding* $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ *is. Then every point of exact order* 2 *is of the form* $e - K_{P_j}$ *for some odd point of order two* e *and some Weierstrass point* P_j.

Proof (of Lemma). Consider the homomorphism $(\mathbb{Z}/2\mathbb{Z})^{2g+1} \to \mathrm{Jac}(C)[2]$, given by
$(x_1, x_2, \dots, x_{2g+1}) \mapsto \sum x_i P_{i+1} - (\sum x_i)P_1$. By counting it is enough to show that the kernel is contained in $\{0, (1,1,\dots,1)\}$.

In order to show that any element different from $(1,1,\dots,1)$ is not in the kernel, it is enough to prove (by renaming the points if necessary) that for any $n < 2g+1$ the divisor $P_2 + \cdots + P_{n+1} - nP_1$ is not principal. If n is odd we are done because any odd $n < 2g$ is a gap for the Weierstrass point P_1 ([FK] p. 88 Remark). Suppose now that n is even. Then $nP_1 - nP_{n+1}$ is a principal divisor. If $P_2 + \cdots + P_{n+1} - nP_1$ is a principal divisor, so is $P_2 + \cdots + P_{n+1} - nP_1 + nP_1 - nP_{n+1} = P_2 + \cdots + P_n - (n-1)P_{n+1}$ This is a contradiction, because $n-1$ is odd and $< 2g$ and therefore a gap for the Weierstrass point P_{n+1}.

It is a consequence of the proof that $(1,1,\dots,1)$ is in the kernel. Otherwise said, that in any hyperelliptic surface the divisor $P_1 + \cdots + P_{2g+2} - (2g+2)P_1$ is a principal divisor. This is easy to verify: writing the surface as $y^2 = (x-\lambda_1)\cdot\cdots\cdots(x-\lambda_{2g+1})$, $P_1 =$the point above ∞, then $(y) = P_2 + \dots + P_{2g+2} - (2g+1)P_1$.

2) We have to prove that every point of order 2 is of the form $e - K_Q$ for some $Q \in \{P_1, \dots, P_6\}$ and an odd point e. These e's are precisely the points $\varphi_Q(P_i) + K_Q$, $1 \le i \le 6$. Thus, it is enough to prove that every point of exact order 2 is of the form $P_i - P_j$ for some $i \ne j$. There are 15 such differences. They all give different two torsion points else we have $P_i - P_j = P_k - P_\ell \implies P_i + P_\ell - 2P_r = P_j + P_k - 2P_r$. Separating cases and using part 1) of the Lemma one concludes the proof.

Returning to the proof of the theorem, we assume the notation of the Lemma and let $P = P_1$, $B = \mathrm{Jac}(C)[4] - \mathrm{Jac}(C)[2]$. We need to prove that $B \cap (\varphi_P(C) + K_P) = \emptyset$. If $x \in B \cap (\varphi_P(C) + K_P)$ we know that for some $Q \in \{P_1, \dots, P_6\}$ and some odd point e of exact order 2:

(i) $2x = e - K_Q$.

(ii) $x - K_Q \in \varphi_Q(C)$ (using $\varphi_P(C) + K_P = \varphi_Q(C) + K_Q$ see [FK] VI3.7.).

Now, since K_Q is a point of order 2 we get $2(x - K_Q) = 2x = e - K_Q \in \varphi_Q(C)$ (using e odd $\Rightarrow e \in \varphi_Q(C) + K_Q$). Moreover, using the symmetry of the theta divisor $\varphi_Q(C) + K_Q$ (as follows from (5)) we conclude that since $x \in \varphi_Q(C) + K_Q$ we get also $-x \in \varphi_Q(C) + K_Q = \varphi_Q(C) - K_Q$. That is, $-(x - K_Q) \in \varphi_Q(C)$.

We have shown that if $x \in B \cap (\varphi_P(C) + K_P)$ then $\varphi_Q(C) \supseteq \{0, x - K_Q, 2(x - K_Q), 3(x - K_Q)\}$. However, this implies that the intersection $\varphi_Q(C) . T_{x-K_Q}(\varphi_Q(C)) \geq 4$ which contradicts the fact that the theta divisor defines a principal polarization and is, therefore, of self intersection 2 (it is easy to prove that $\varphi_Q(C) \neq T_{x-K_Q}(\varphi_Q(C))$ and therefore the intersection is proper). Q.E.D.

Theorem 3. 1) *There exists a constant* $\lambda \in \mathbb{Q}(\zeta_{36})^\times$ *such that*

$$(*) \qquad \lambda \prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in A^-} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)^2 = \prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in E} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)^{96}$$

where $A^- = \left\{\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} : -\frac{1}{2} \leq \varepsilon_i,\ \varepsilon_i' < \frac{1}{2},\ i = 1, 2,\ \begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\right.$ *is of exact order* 6 *in* $\mathbb{R}^4/\mathbb{Z}^4$ *and* $36\, {}^t\varepsilon \cdot \varepsilon' \equiv 1 \pmod 2\}$ *and* $E = \left\{\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} : \varepsilon_i, \varepsilon_i' \in \left\{0, \frac{1}{2}\right\},\ i = 1, 2,\ 4\, {}^t\varepsilon \cdot \varepsilon' \equiv 0 \pmod 2\right\}$.

2) *There exists a constants* $\mu \in \mathbb{Q}(\zeta_{16})^\times$ *such that*

$$\mu \prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in B} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)^2 = \prod_{\left[\begin{smallmatrix}\varepsilon\\ \varepsilon'\end{smallmatrix}\right] \in E} \Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)^{48}$$

where $B = \left\{\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} : -\frac{1}{2} \leq \varepsilon_i, \varepsilon_i' < \frac{1}{2},\ i = 1, 2,\ \begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\right.$ *is of exact order* 4 *in* $\mathbb{R}^4/\mathbb{Z}^4\}$.

Proof. 1) By Theorem 1 we know that the support of the divisor of both sides is contained in H_1^+. We claim that $\Theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(\tau)^2$, $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix} \in Ch(k)$, is a modular form of weight 1 and some high level. Indeed, the transformation formula (7) shows that we have to fulfill two conditions: (i) $\zeta\left(\gamma, \begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\right)^2 = 1$ (ii) $K(\gamma)^2 = 1$. Clearly (i) will hold if $\gamma \in \Gamma_2(k^2)$. One can prove that $K(\gamma)^2$ is trivial on $\Gamma_2(4)$ (see [Mum1] p. 197 Corollary (5.9)). Thus, (i), (ii) will hold $\gamma \in \Gamma_2(4k^2)$. We conclude that both sides are modular forms of weight equal to $10 \times 96 \times 1/2 = 480 = (3^4 - 1) \cdot 6 \times 2 \times 1/2 =$ (number of $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ of exact order 6 and odd) $\times 2 \times 1/2$.

Now, the action of $\Gamma_2(1)$ on the integral characteristics takes odd/even characteristics to odd/even characteristics and it acts transitively on the set of odd/even integral characteristics. Under the isomorphism $\left(\frac{1}{3}\mathbb{Z}/\mathbb{Z}\right)^4 \times \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^4 \to \left(\frac{1}{6}\mathbb{Z}/\mathbb{Z}\right)^4$, $\left(\begin{bmatrix} \mu \\ \mu' \end{bmatrix}, \begin{bmatrix} \eta \\ \eta' \end{bmatrix}\right) \mapsto \begin{bmatrix} \mu \\ \mu' \end{bmatrix} + \begin{bmatrix} \eta \\ \eta' \end{bmatrix} = \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ we find that $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ is odd if and only if $\begin{bmatrix} \eta \\ \eta' \end{bmatrix}$ is. The inverse isomorphism $\left(\frac{1}{6}\mathbb{Z}/\mathbb{Z}\right)^4 \to \left(\frac{1}{3}\mathbb{Z}/\mathbb{Z}\right)^4 \times \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^4$ is just $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \mapsto \left(\begin{bmatrix} -2\varepsilon \\ -2\varepsilon' \end{bmatrix}, \begin{bmatrix} 3\varepsilon \\ 3\varepsilon' \end{bmatrix}\right)$. Now, under this isomorphism the action of $\gamma \in \mathrm{Sp}(4,\mathbb{Z})$, $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \mapsto \gamma^* \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ comes out $\left(\begin{bmatrix} \mu \\ \mu' \end{bmatrix}, \begin{bmatrix} \eta \\ \eta' \end{bmatrix}\right) \mapsto \left({}^t\gamma^{-1} \begin{bmatrix} \mu \\ \mu' \end{bmatrix}, \gamma^* \begin{bmatrix} \eta \\ \eta' \end{bmatrix}\right)$ (in the first coordinate we have the usual *linear* action). If follows that the action of $\Gamma_2(1)$ on $Ch(6)$ preserves the even/odd characteristics. Therefore, we may conclude from the transformation formula that the divisor of both sides of (*) is $\Gamma_2(1)$ invariant and thus has the same multiplicity on every component of H_1^+. Let

$$f(\tau) = \Bigg(\prod_{\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in A^-} \Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\tau)^2 \Bigg) \Big/ \Bigg(\prod_{\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in E} \Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\tau)^{96} \Bigg).$$

Then one of the functions $f(\tau)$, $1/f(\tau)$ is a holomorphic function on $\Gamma_2(n)/\mathfrak{H}_2$. Therefore, by the Koecher principle ([FC] p. 146) $f(\tau)$ is constant. It is easy to see that both sides of (*) have q-expansions with coefficients in $\mathbb{Q}(\zeta_{36})$. Hence, we can choose λ such that the equality holds and λ is in $\mathbb{Q}(\zeta_{36})^\times$.

2) The proof is the same (using Theorem 2). Q.E.D.

Remark. In complete contrast with the high dimensional cases the one dimensional analogues of Theorem 3 exist in abundance. For example, let n be any odd number. Define

$$\chi_1(n) = \left\{ \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} : -\frac{1}{2} \leq \varepsilon < \frac{1}{2},\ -\frac{1}{2} \leq \varepsilon' < \frac{1}{2},\ \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \text{ is } 2n\text{-even and of order } n \text{ or } 2n \right\}$$

and for $n \neq 1$ define

$$\chi_{-1}(n) = \left\{ \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} : -\frac{1}{2} \leq \varepsilon < \frac{1}{2},\ -\frac{1}{2} \leq \varepsilon' < \frac{1}{2},\ \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \text{ is } 2n\text{-odd and of order } n \text{ or } 2n \right\}$$

Let $a_1(n) = \#\chi_1(n) = 3\phi(n)(2n - \phi(n))$, $a_{-1}(n) = \#\chi_{-1}(n) = \phi(n)(2n - \phi(n))$. Let

$$T_1(n)(\tau) = \prod_{\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in \chi_1(n)} \Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}^2 (\tau), \quad T_{-1}(n)(\tau) = \prod_{\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in \chi_{-1}(n)} \Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}^2 (\tau).$$

Both T_1 and T_{-1} are modular forms (of weights $a_1(n)$, $a_{-1}(n)$ respectively) for high enough level.

Proposition. *Let $\alpha, \beta \in \{-1, 1\}$. Let $d = l.c.m.(a_\alpha(n), a_\beta(m))$ ($a_{-1}(1)$ is not allowed). Then there exists a non-zero constant λ such that*

$$\lambda \cdot T_\alpha(n)^{d/a_\alpha(n)} = T_\beta(m)^{d/a_\beta(m)}.$$

In fact $\lambda = \left(\prod_{\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in \chi_\beta(m)} \alpha(\varepsilon, \varepsilon')^{2d/a_\beta(m)} \right) / \left(\prod_{\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in \chi_\alpha(n)} \alpha(\varepsilon, \varepsilon')^{2d/a_\alpha(n)} \right)$, *where* $\alpha(\varepsilon, \varepsilon') = \exp(2\pi i \cdot \varepsilon\varepsilon')$ *if* $\varepsilon \neq -1/2$, *and* $\alpha(\varepsilon, \varepsilon') = \exp(\pi i \cdot \varepsilon') + \exp(-\pi i \cdot \varepsilon')$ *if* $\varepsilon = -1/2$.

Proof. Both sides are modular forms of high enough level, say N. They extend to modular forms over the compactified modular curve of level N, $X(N) = \Gamma(N) \setminus \mathfrak{H}^*$. Moreover, both sides do not vanish on $\Gamma(N) \setminus \mathfrak{H}$ and vanish to the same order at every cusp of $X(N)$ (The last assertion follows from the fact that $\Gamma(1)$ acts transitively on the cusps and that the divisors of both sides are $\Gamma(1)$ invariant as follows from the transformation formula (7)). Therefore either $(T_\alpha(n)^{d/a_\alpha(n)})/(T_\beta(m)^{d/a_\beta(m)})$ or $(T_\beta(m)^{d/a_\beta(m)})/(T_\alpha(n)^{d/a_\alpha(n)})$ are holomorphic and non-vanishing on $X(N)$, hence constant.

To find the precise form of λ we note that the first non-zero coefficient in the q-expansion of $\Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\tau)$ is precisely $\alpha(\varepsilon, \varepsilon')$. This follows immediately from the formula

$$\Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\tau) = \sum_{n=-\infty}^{\infty} \exp(2\pi i(n + \varepsilon) \cdot \varepsilon') \cdot \exp(\pi i(n + \varepsilon)^2 \tau).$$

Since the first non-zero coefficient of $T_\alpha(n)^{d/a_\alpha(n)}$ is the product of the first non-zero coefficients of all the theta functions appearing in its definition, the assertion follows.

Q.E.D.

The following theorem complements Theorem 1:

Theorem 4. *Let* $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in Ch(6)$ *be a* 6*-even non-integral characteristic. Then the divisor of* $\Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau)$ *is not contained in* H_1^+*.*

Proof. Consider the covering $y^2 = x^6 + 1 \to y^2 = x^2 + 1 \cong \mathbb{P}^1$ $((x, y) \mapsto (x^3, y))$. This map is really dividing by the automorphism $(x, y) \mapsto (e^{2\pi i/3}x, y)$, which has the two fixed points $Q = (0, 1)$, $P = (0, -1)$. This shows that image of $Q - P$ in $\mathrm{Jac}(y^2 = x^6 + 1)$ is of order 3. On the other hand the covering $y^2 = x^6 + 1 \to y^2 = x^3 + 1$ $((x, y) \mapsto (x^2, y))$which comes from the involution $(x, y) \mapsto (-x, y)$, shows that $Q + P$ (= the ramification points) is a divisor of a holomorphic differential on $y^2 = x^6 + 1$ (the pull-back of a non-zero holomorphic differential on $y^2 = x^3 + 1$). Therefore, using [FK] VI.3.6, we conclude that $-2K_P = \varphi_P(Q + P) = \varphi_P(Q)$. Hence $6(\varphi_P(Q) + K_P) = 6\varphi_P(Q) - 3\varphi_P(Q) = 0$. Choose any A.J. structure on $y^2 = x^6 + 1$ and let (τI) be the period matrices obtained from this structure. Let $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ be a characteristic such that $(\tau I)\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} = \varphi_P(Q) + K_P$. Note that $\varphi_P(Q) + K_P$ is not of order 2, hence $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in Ch(6)$, non-integral and even (using Theorem 1). We conclude that $\Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau) = 0$ and we have $\tau \notin H_1^+$.

Let us denote the set of even non-integral characteristics $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in Ch(6)$ by A^+. Now, to get that for every $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \in A^+$ we have $\Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}(\tau) = 0$ for some $\tau \notin H_1^+$, it is enough, by the transformation formula (7), to prove that $\Gamma_2(1)$ acts transitively on the set $A^+(\mathrm{mod}\mathbb{Z}^4)$. We have already seen in the proof of Theorem 3 that $\Gamma_2(1)$ acts on $A^+(\mathrm{mod}\mathbb{Z}^4)$. In fact, recall that under the isomorphism $\left(\frac{1}{6}\mathbb{Z}/\mathbb{Z}\right)^4 \to \left(\frac{1}{3}\mathbb{Z}/\mathbb{Z}\right)^4 \times \left(\frac{1}{2}\mathbb{Z}/\mathbb{Z}\right)^4$, $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \mapsto \left(\begin{bmatrix} \mu \\ \mu' \end{bmatrix}, \begin{bmatrix} \eta \\ \eta' \end{bmatrix}\right)$ we find that $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ is odd if and only if $\begin{bmatrix} \eta \\ \eta' \end{bmatrix}$ is and that the action on $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$, $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} \mapsto \gamma^* \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ comes out $\left(\begin{bmatrix} \mu \\ \mu' \end{bmatrix}, \begin{bmatrix} \eta \\ \eta' \end{bmatrix}\right) \mapsto \left({}^t\gamma^{-1}\begin{bmatrix} \mu \\ \mu' \end{bmatrix}, \gamma^* \begin{bmatrix} \eta \\ \eta' \end{bmatrix}\right)$. Let $\begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} = \left(\begin{bmatrix} \mu \\ \mu' \end{bmatrix}, \begin{bmatrix} \eta \\ \eta' \end{bmatrix}\right)$ and $\begin{bmatrix} \alpha \\ \alpha' \end{bmatrix} = \left(\begin{bmatrix} \beta \\ \beta' \end{bmatrix}, \begin{bmatrix} \zeta \\ \zeta' \end{bmatrix}\right)$ be in A^+. Since, by Witt's Theorem, $\mathrm{Sp}(4, \mathbb{Z})$ acts (linearly) transitively on $(\mathbb{Z}/3\mathbb{Z})^4$, we can find $\gamma \in \mathrm{Sp}(4, \mathbb{Z})$ such that $\gamma^* \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} = \left(\begin{bmatrix} \beta \\ \beta' \end{bmatrix}, \begin{bmatrix} \xi \\ \xi' \end{bmatrix}\right)$ and $\begin{bmatrix} \xi \\ \xi' \end{bmatrix}$ is necessarily an odd integral characteristic. We can act now by $\delta \in \Gamma_2(3)$ on $\gamma^* \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix}$ and get $\left(\begin{bmatrix} \beta \\ \beta' \end{bmatrix}, \delta^* \begin{bmatrix} \xi \\ \xi' \end{bmatrix} \right)$. One has only to verify that $\Gamma_2(3)$ acts transitively on the set of the odd integral characteristics (mod $\mathbb{Z}^4$). There are six odd characteristic represented by:

$$\begin{bmatrix} 0 & 1/2 \\ 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 0 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}, \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Denoting them by $1, 2, 3, 4, 5, 6$, respectively, one checks that $\begin{pmatrix} I & 3I \\ 0 & I \end{pmatrix}$ acts as the permutation $(12)(34)$, $\begin{pmatrix} I & 0 \\ 3I & I \end{pmatrix}$ acts as $(15)(36)$ and $\begin{pmatrix} \alpha & 0 \\ 0 & {}^t\alpha^{-1} \end{pmatrix}$, $\alpha = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$, acts as $(12)(36)(45)$. Q.E.D.

Final Remark. Using ramified coverings of elliptic curves gives one a systematic approach for proving vanishing of theta constants $\Theta \begin{bmatrix} \varepsilon \\ \varepsilon' \end{bmatrix} (\tau)$ for $\varepsilon, \varepsilon' \in \mathbb{Q}^2$, $\tau \in \mathfrak{H}_2$, at points $\tau \notin H_1^+$. This will be explained in [Gor]. It seems to suggest that one can not expect theorems like Theorems 1,2 for other characteristics.

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