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Hilbert Modular Forms Modulo p^m: The Unramified Case

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This paper is about Hilbert modular forms on certain Hilbert modular varieties associated with a totally real field L. Let p be unramified in L. We reduce to the inert case and consider modular forms modulo p^m. We study the ideal of modular forms with q-expansion equal to zero modulo p^m, find canonical elements in it, and obtain as a corollary the congruences for the values of the zeta function of L at negative integers. Our methods are geometric and also have applications to lifting of Hilbert modular forms and compactification of certain modular varieties.

1. INTRODUCTION

1.1. *The contents of this paper.* The subject of this paper is the study of Hilbert modular forms on Hilbert modular varieties and some applications. The modular varieties are those parameterizing abelian varieties of dimension g with a given action of the ring of integers of a totally real field L of degree g over $\mathbb{Q}$ and certain level structures, some indigenous to characteristic p. We shall be particularly interested in the case where the domain of the modular form is the modular variety modulo p^m. This allows us to study q-expansions modulo p^m.

The Hilbert modular forms we consider are modular forms in the sense of Katz [12]. Their weights are given by characters of a certain algebraic group over $\mathcal{O}_L$, which is a torus over $\mathcal{O}_L[\mathrm{disc}_L^{-1}]$. Over the complex numbers this just boils down to discussing Hilbert modular forms of possibly non-parallel weight.

We assume a priori that the prime p we are dealing with is non-ramified in L. However, one immediately reduces to the case where the prime is inert. This is a well known principle and we refer the reader to [5] to see how this works. *Assume, henceforth, that p is inert.*

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Denote the graded ring of Hilbert modular forms of μ_N-level ($(N, p) = 1$), defined over $W_m(\mathbb{F})$, by $\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$. We refer the reader to Section 1.2 for precise definitions. In brief: $W_m(\mathbb{F})$ is isomorphic to $\mathcal{O}_L/(p^m)$; a μ_N-level means an $\mathcal{O}_L$-equivariant embedding of $D_L^{-1} \otimes \mu_N$ into the abelian variety.

The main question we ask is: what can one say on the kernel of the q-expansion map on $\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$?

While in characteristic 0 the kernel is trivial, the situation is different in characteristic p. A well-known theorem of P. Swinnerton-Dyer asserts that for $g = 1$ and $m = 1$, the kernel is generated by $E_{p-1} - 1$, where E_{p-1} is an Eisenstein series of weight $p - 1$ (see (2.21) for the definition of E_k for any L), and a well-known theorem of P. Deligne asserts that E_{p-1} modulo p is the Hasse invariant.

Our results are a generalization of these theorems for general totally real fields and any m. One of the psychological shifts one has to make is to completely abandon the method of obtaining relations by reducing from characteristic zero and to work solely modulo p^m. Indeed, the question of whether or not $E_{(p-1)p^r} - 1$ belongs to this kernel depends, for a given r, on the field (and need not hold), and for all $r \gg 0$ is equivalent to Leopoldt's conjecture.

For $m = 1$, that is, modulo p, our results are a direct and precise analog of the above theorems. The complement of the ordinary locus was studied by F. Oort and the author in [7]. It turns out that it canonically decomposes as a union $\bigcup_{i=1}^{g} W_{\{i\}}$ (see Section 1.2).

THEOREM 1 (Theorem 2.1). *Let p be inert in L. There exist Hilbert modular forms $h_1, ..., h_g$, over $\mathbb{F}$, of weights $\chi_g^p \chi_1^{-1}, \chi_1^p \chi_2^{-1}, ..., \chi_{g-1}^p \chi_g^{-1}$ respectively (h_i being of weight $\chi_{i-1}^p \chi_i^{-1}$), such that*

$$(h_i) = W_{\{i\}}.$$

(In particular, the divisor of h_i is reduced.) The q-expansion of h_i at every cusp of $\mathcal{M}^*(\mathbb{F}, \mu_N)$ is 1. Let $h = h_1 \cdots h_g$. Then h is a modular form of weight Norm^{p-1}. *It has q-expansion equal to 1 at every standard cusp and its divisor is reduced, equal to the complement of the ordinary locus.*

We remark that h is up to a sign the Hasse invariant, i.e., the determinant of the Hasse-Witt matrix, and that if $g > 1$ the h_i's *never* lift to characteristic zero!

We then prove (compare Theorem 2.3)

THEOREM 2. *Let p be inert in L. The kernel of the q-expansion map modulo p is the ideal generated by $\{h_1 - 1, ..., h_g - 1\}$.*

Regarding the situation modulo p^m, our results are less complete. Let I_m be the kernel of the q-expansion map modulo p^m. We are able to identify the quotient $\bigoplus_{\chi\in\mathbf{X}} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)/I_m$ and find some canonical elements in I_m that are a generalization of the h_i's. See Theorem 3.8. After adding level structure one can determine the kernel of the q-expansion map modulo p^m completely. See Proposition 3.12.

We provide several applications. One is to construct an explicit compactification of Hilbert modular varieties with μ_p-level, which is non-singular in codimension one. See Theorem 2.9. A second application is to show that there exists a notion of filtration for non-parallel modular forms.

Another application is classical. Let ζ_L be the Dedekind zeta function of L. Recall that by a theorem of C. L. Siegel the values of $\zeta_L(1-k)$, for $k \geqslant 2$ an integer, are rational numbers and are equal to zero if k is odd. From a modern perspective this is quite immediate. There exists an Eisenstein series E_k with rational Fourier coefficients and constant coefficient $2^{-g}\zeta_L(1-k)$. One considers the modular form of weight k given by $E_k - E_k^\sigma$ for an automorphism σ. It turns out that this "rational influence" of the higher coefficients on the constant coefficient can be refined to an "integral influence". This was proved and developed in the case $g=1$ by J.-P. Serre [17], and in general by P. Deligne and K. Ribet in [4], [16]. In truth, our methods are not that far from Deligne-Ribet's methods [4], [16] (who, in turn, follow ideas of N. Katz [9–12] and J.-P. Serre [17]), but our approach is more geometric and is based on [7], [5]. The conclusion of the congruences is clearly in "Serre's style".

Corollary 1 (Corollary 3.11). *Let p be inert in L. Let $k \geqslant 2$.*

(1) *Let $p \neq 2$; if $k \equiv 0 \pmod{p-1}$ then*

$$\mathrm{val}_p(\zeta_L(1-k)) \geqslant -1 - \mathrm{val}_p(k),$$

and $\zeta_L(1-k)$ is p-integral if $k \not\equiv 0 \pmod{p-1}$.

(2) *If $p=2$, then*

$$\mathrm{val}_2(\zeta_L(1-k)) \geqslant g - 2 - \mathrm{val}_2(k).$$

Corollary 2 (Corollary 3.15). *Let p be inert in L. Let $k, k' \geqslant 2$ and $k \equiv k' \pmod{(p-1)\,p^m}$.*

(1) *If $k \not\equiv 0 \pmod{p-1}$ then*

$$(1-p^{g(k-1)})\,\zeta_L(1-k) \equiv (1-p^{g(k'-1)})\,\zeta_L(1-k') \pmod{p^{m+1}}.$$

(2) *If* $k \equiv 0 \pmod{p-1}$ *but* $p \neq 2$, *then*

$$(1-p^{g(k-1)})\,\zeta_L(1-k) \equiv (1-p^{g(k'-1)})\,\zeta_L(1-k') \qquad \pmod{p^{m-1-\mathrm{val}_p(k\cdot k')}}.$$

(3) *If* $p=2$ *then*

$$(1-2^{g(k-1)})\,\zeta_L(1-k) \equiv (1-2^{g(k'-1)})\,\zeta_L(1-k') \pmod{2^{m+g-2-\mathrm{val}_2(k\cdot k')}}.$$

The derivation of the congruences rests on the following Criterion 3.10:

"*Let* $\sum_\chi f_\chi \in I_m$. *Then there exist* a_χ *in some* $W_m(\mathbb{F})$*-algebra such that* $\sum_\chi a_\chi \chi(u) \equiv 0 \pmod{p^m}$ *for all* $u \in (\mathcal{O}_L/(p^m))^\times$ *and* $a_{\mathbf{1}} = f_{\mathbf{1}}$."

It is interesting to note that this criterion allows an inverse in some sense. Given such polynomial relations one obtains relations between values of zeta functions, provided certain restrictions are satisfied.

1.2. *Definitions and notation.* Let L be a totally real field of degree g over $\mathbb{Q}$. Let $\mathcal{O}_L$ be its ring of integers, D_L the different ideal and d_L the discriminant. Let $\mathfrak{c}$ be a fractional ideal of L. Let p a rational prime that is *inert* in L. Let $\mathbb{F}$ be a fixed field of p^g elements. Let $W(\mathbb{F})$ be the ring of infinite Witt vectors over $\mathbb{F}$ and σ its Frobenius automorphism.

All schemes in this paper are over $\mathbb{Z}[d_L^{-1}]$.

- A *HBAS* (Hilbert-Blumenthal abelian scheme) over S is a triple

$$\underline{A} = (A, \iota, \lambda) \tag{1.1}$$

consisting of an abelian scheme $\pi: A \to S$, an embedding of rings $\iota: \mathcal{O}_L \hookrightarrow \mathrm{End}_S(A)$, a polarization $\lambda: (M_A, M_A^+) \to (\mathfrak{c}, \mathfrak{c}^+)$ identifying the $\mathcal{O}_L$-module M_A of symmetric homomorphisms from A to its dual with $\mathfrak{c}$ such that the cone of polarizations M_A^+ is mapped to $\mathfrak{c}^+$. Furthermore, we require that $\mathfrak{t}^*_{A/S}$ be a locally free $\mathcal{O}_L \otimes \mathcal{O}_S$-module of rank 1. In particular, the relative dimension of A is g. Here $\mathfrak{t}_{A/S}$ stands for the locally free sheaf of $\mathcal{O}_S$-modules of rank g given by $\mathrm{Lie}(A/S)$, and $\mathfrak{t}^*_{A/S} = s^*\Omega^1_{A/S}$, where $s: S \to A$ is the identity section, is the dual of $\mathfrak{t}_{A/S}$. We shall employ this notation for a general group scheme $\pi: G \to S$. If π is proper then also $\mathfrak{t}^*_{G/S} = \pi_*\Omega^1_{G/S}$.

By a *non-vanishing differential* on a HBAS $\underline{A}$, we mean an $\mathcal{O}_L \otimes \mathcal{O}_S$ basis to $\mathfrak{t}^*_{A/S}$. Every HBAS possesses a non-vanishing differential Zariski locally on the base.

- A μ_N*-level structure* on a HBAS is a closed immersion of S-group schemes,

$$D_L^{-1} \otimes_{\mathbb{Z}} \mu_N \hookrightarrow A, \tag{1.2}$$

equivariant for the $\mathcal{O}_L$-action. Here $\mathcal{O}_L$ acts canonically on $D_L^{-1} \otimes_{\mathbb{Z}} \mu_N$ from the left. If $p \mid N$ this implies that A is ordinary at every fiber of characteristic p.

- Let $\mathbb{T}$ be the split torus over $W(\mathbb{F})$ associating to a $W(\mathbb{F})$-algebra R the group

$$\mathbb{T}(R) = (\mathcal{O}_L \otimes_{\mathbb{Z}} R)^{\times}. \tag{1.3}$$

Let $\{\sigma_1, ..., \sigma_g\}$ be the embeddings of L into $W(\mathbb{F})$, ordered cyclically with respect to the Frobenius automorphism σ of $W(\mathbb{F}) : \sigma \circ \sigma_i = \sigma_{i+1}$ (the subscripts read mod. g). Once we fix a choice of σ_1, we have a canonical isomorphism

$$\mathcal{O}_L \otimes_{\mathbb{Z}} W(\mathbb{F}) = \bigoplus_{i=1}^{g} W(\mathbb{F}). \tag{1.4}$$

That gives a canonical isomorphism $\mathbb{T} = \mathbb{G}_m^g$ and, in particular, a canonical isomorphism

$$\mathbb{T}(R) = \bigoplus_{i=1}^{g} R^{\times}, \qquad R \in W(\mathbb{F}) - \text{Alg}. \tag{1.5}$$

We let $\chi_1, ..., \chi_g$ denote the projections of $\mathbb{T}$ on its g components.

- Let $\mathbf{X}$ be the group of characters of $\mathbb{T}$. It is the free abelian group on $\chi_1, ..., \chi_g$. We write $\mathbf{X}$ multiplicatively:

$$\mathbf{X} = \{\chi_1^{r_1} \cdots \chi_g^{r_g} : r_i \in \mathbb{Z}\}. \tag{1.6}$$

It is a principal homogeneous space for the group $\mathbb{Z}[\mathbb{Z}/g\mathbb{Z}]$. We denote by $\mathbf{1}$ the trivial character and by "Norm" the product $\chi_1 \cdots \chi_g$.

Let $\mathbf{X}(1)$ be the subgroup of $\mathbf{X}$ generated by the elements $\chi_i^p \chi_{i+1}^{-1}$:

$$\mathbf{X}(1) = \langle \chi_1^p \chi_2^{-1}, \chi_2^p \chi_3^{-1}, ..., \chi_g^p \chi_1^{-1} \rangle. \tag{1.7}$$

It is the subgroup of $\mathbf{X}$ consisting of all characters trivial on $(\mathcal{O}_L/(p))^{\times}$ via

$$(\mathcal{O}_L/(p))^{\times} \hookrightarrow \mathbb{T}(\mathbb{F}) = \bigoplus_{i=1}^{g} \mathbb{F}^{\times}. \tag{1.8}$$

Similarly, we let $\mathbf{X}(m)$ be the subgroup of $\mathbf{X}$ consisting of all characters trivial on $(\mathcal{O}_L/(p^m))^{\times}$. See Section 3.2.

- Let B be a $W(\mathbb{F})$-algebra. Let $\chi \in \mathbf{X}$. A *HMF* (Hilbert modular form) over B, of weight χ, and μ_N-level is a rule,

$$(\underline{A}, \beta, \omega)_{/R} \mapsto f((\underline{A}, \beta, \omega)_{/R}) \in R, \tag{1.9}$$

associating to a HBAS $\underline{A}$ over a B-algebra R, endowed with a μ_N-level β and a non-vanishing differential ω, an element $f((\underline{A}, \beta, \omega)_{/R})$ of R. One requires that $f((\underline{A}, \beta, \omega)_{/R})$ depends only on the R-isomorphism class of $(\underline{A}, \beta, \omega)$, commutes with base-change, and satisfies

$$f((\underline{A}, \beta, \alpha^{-1}\omega)_{/R}) = \chi(\alpha)\, f((\underline{A}, \beta, \omega)_{/R}), \qquad \forall \alpha \in (\mathcal{O}_L \otimes R)^{\times}. \tag{1.10}$$

See [12, Sect. 1.2]. We let $\mathbf{M}(B, \chi, \mu_N)$ denote the B-module of HMFs over B, of weight χ and μ_N-level.

- In [7], a stratification of Hilbert modular varieties in characteristic p was obtained by means of a *type*, assuming p is inert and principal polarization. (In [5], the reader can find how to define this stratification under less restrictions.) We recall that for every HBAS $\underline{A}$ over a perfect field k containing $\mathbb{F}$ there is associated a type $\tau(\underline{A})$, which is a subset of $\{1, ..., g\}$. It simply encodes the structure of the Dieudonné module of the α-group of $\underline{A}$, $\alpha(\underline{A})$, as an $\mathcal{O}_L \otimes k$-module. For k a perfect field this α-group is $\mathrm{Ker}(F) \cap \mathrm{Ker}(\mathrm{Ver})$. In this case, the Dieudonné module $\mathbb{D}(\alpha(A))$ of $\alpha(\underline{A})$ is a k-vector space, of dimension between 0 and g, on which $\mathcal{O}_L \otimes k$ acts. As $\mathbb{D}(\alpha(A))$ is contained in the Dieudonné module of the kernel of Frobenius, i.e., in the relative cotangent space, it follows that $\mathbb{D}(\alpha(A))$ is a sub-sum of $\bigoplus_{i=1}^{g} k = \mathcal{O}_L \otimes k$. The type $\tau(A)$ is defined by the identity

$$\mathbb{D}(\alpha(A)) = \bigoplus_{i \in \tau(A)} k. \tag{1.11}$$

For every subset τ of $\{1, ..., g\}$, one lets W_τ be the closed reduced sub-scheme of the moduli space, universal for the property "the type contains τ". It has codimension $|\tau|$. We have $W_\tau \cap W_\sigma = W_{\tau \cup \sigma}$. For a rigid level structure, W_τ is regular.

LEMMA 1.1. *Let $N \geqslant 4$. The moduli problem of HBAS with μ_N-level over $\mathbb{Z}[d_L^{-1}]$-schemes is rigid.*

Proof. Let $\underline{A}$ be a HBAS. Let D be the centralizer of L in $\mathrm{End}(A) \otimes \mathbb{Q}$. It is known that D is either L, a CM field such that $D^+ = L$, or a quaternion algebra over L that is ramified everywhere at ∞. See [2], Lemma 6.

Let $\mathcal{O}_D = D \cap \mathrm{End}(\underline{A})$. If $\xi \in \mathcal{O}_D$ is an automorphism of A preserving the polarization, then $\xi\xi^* = 1$, where $*$ is the unique positive involution of D. Hence, ξ is of finite order. It follows that the field $L(\xi)$ is either L, or a CM field whose totally real subfield is L, and that ξ is a root of unity of order n. The case of $L(\xi) = L$ is just the case of $\xi = \pm 1$ and is easily dispensed with. We assume that $L(\xi) \neq L$. Hence, $[L(\xi) : \mathbb{Q}] = 2g$. Equivalently, $1 < \phi(n)$, $\phi(n) \mid 2g$ and $L \cap \mathbb{Q}(\xi) = \mathbb{Q}(\xi)^+$.

If ξ preserves a μ_N-level structure, it follows that $N^g \mid \deg(1-\xi)$. Hence, n is a prime power. Say $n = \ell^r$, ℓ a prime. Then $\deg(1-\xi) = \ell^{2g/\phi(n)}$. Since $\phi(n) > 1$, this is divisible by a g-th power if and only if $\phi(n) = 2$. On the other hand, $\phi(n) = \ell^{r-1}(\ell - 1)$. This implies $r = 1$ and $\ell = 3$, or $r = 2$ and $\ell = 2$. Both imply $N < 4$. ∎

- Let B be a $W(\mathbb{F})$-algebra. We let $\mathcal{M}(B, \mu_N)$ be the moduli space over $\mathrm{Spec}(B)$ of HBAS with μ_N-level. It is the base change to $\mathrm{Spec}(B)$ of $\mathcal{M}(W(\mathbb{F}), \mu_N)$. We let $\mathcal{M}^*(B, \mu_N)$ denote its minimal Satake compactification. We let $\mathcal{M}(B, \mu_N)^{ord}$ be the ordinary locus of $\mathcal{M}(B, \mu_N)$—the base change of $\mathcal{M}(W(\mathbb{F}), \mu_N)$ from which the non-ordinary locus of $\mathcal{M}(\mathbb{F}, \mu_N)$ was deleted. We let $\mathcal{M}^*(B, \mu_N)^{ord}$ be the ordinary locus of $\mathcal{M}^*(B, \mu_N)$. Note that if $p \mid N$ we have $\mathcal{M}(B, \mu_N) = \mathcal{M}(B, \mu_N)^{ord}$ and $\mathcal{M}^*(B, \mu_N) = \mathcal{M}^*(B, \mu_N)^{ord}$. The morphism $\mathcal{M}(B, \mu_N)^{ord} \to \mathcal{M}^*(B, \mu_N)^{ord}$ is an open immersion whose complement consists of finitely many sections over $\mathrm{Spec}(B)$—the cusps.

For every $(N_1, N_2) = 1$, with $N_1 \geqslant 4$, N_2 a power of p, and p nilpotent in B, the map

$$\mathcal{M}(B, \mu_{N_1 N_2})^{\mathrm{ord}} \to \mathcal{M}(B, \mu_{N_1})^{\mathrm{ord}} \tag{1.12}$$

is an étale Galois covering with Galois group canonically isomorphic to $(\mathcal{O}_L/(N_2))^\times$ and $\mathcal{M}^*(B, \mu_{N_1})^{\mathrm{ord}}$ is the quotient of $\mathcal{M}^*(B, \mu_{N_1 N_2})^{\mathrm{ord}}$ by the action of $(\mathcal{O}_L/(N_2))^\times$.

- Let A be a commutative ring with 1. Let M, M' be finitely generated free abelian groups, $N = \mathrm{Hom}(M, \mathbb{Z})$ and $N' = \mathrm{Hom}(M', \mathbb{Z})$. Let $\mathbb{G}_m = \mathrm{Spec}(A[q, q^{-1}])$. We consider the torus

$$\begin{aligned} G(M) &:= \mathrm{Spec}(A[M]) \\ &= \mathrm{Spec}(A[x^m : m \in M]/(x^0 - 1, x^m x^{m'} - x^{m+m'}\ \forall m, m' \in M)). \end{aligned} \tag{1.13}$$

As a functor on schemes over A we may identify it with the functor $N \otimes \mathbb{G}_{m/A}$, where

$$(N \otimes \mathbb{G}_{m/A})(R) := N \otimes_{\mathbb{Z}} R^\times, \qquad R \in A-\mathrm{Alg}. \tag{1.14}$$

One verifies that

$$\mathrm{Lie}(G(M)/A) = N \otimes \mathrm{Lie}(\mathbb{G}_m/A) = N \otimes A \cdot q\frac{\partial}{\partial q}, \tag{1.15}$$

and hence,

$$\mathrm{t}^*_{G(M)/A} = M \otimes \mathrm{t}^*_{\mathbb{G}_m/A} = M \otimes A \cdot \frac{dq}{q}. \tag{1.16}$$

See [1], Exposé II. In the last isomorphism $m \otimes a \cdot \frac{dq}{q}$ corresponds to $ax^{-m}dx^m$.

Let $\phi: M \to M'$ be a homomorphism. It induces a homomorphism of group schemes $\Phi: G(M') \to G(M)$, whose effect on functions is $x^m \mapsto x^{\phi(m)}$. The induced map

$$\Phi^*: \mathrm{t}^*_{G(M)/A} \to \mathrm{t}^*_{G(M')/A} \tag{1.17}$$

is given, innocently enough, by $(dx^m/x^m) \mapsto (dx^{\phi(m)}/x^{\phi(m)})$. Alternately, $m \otimes a \cdot \frac{dq}{q} \mapsto \phi(m) \otimes a \cdot \frac{dq}{q}$.

Consider now the case $M = M' = \mathcal{O}_L$ and $\phi = [\alpha]$, the map of multiplication by an element $\alpha \in \mathcal{O}_L$. That is, we consider the group scheme $D_L^{-1} \otimes \mathbb{G}_m$ over A, which is the torus

$$\mathrm{Spec}(A[\mathcal{O}_L]) = \mathrm{Spec}(A[x^m : m \in \mathcal{O}_L]/(x^0 - 1,\ \ x^m x^{m'} - x^{m+m'}\ \forall m, m' \in \mathcal{O}_L)). \tag{1.18}$$

Thus, $[\alpha]$ acts on functions by $x^m \mapsto x^{\alpha m}$. The identification of $\mathrm{t}^*_{D_L^{-1} \otimes \mathbb{G}_m/A}$ with $\mathcal{O}_L \otimes A \cdot \frac{dq}{q}$ agrees with the action of $\mathcal{O}_L$. In particular, the differential $1 \otimes \frac{dq}{q}$ generates $\mathrm{t}^*_{D_L^{-1} \otimes \mathbb{G}_m/A}$ as an $\mathcal{O}_L \otimes A$-module.

Let N be prime to p. Given a HBAS with μ_{Np^n}-level, say $(\underline{A}, \beta_N \times \beta_{p^n})$, we define

$$[\alpha](\underline{A}, \beta_N \times \beta_{p^n}) = (\underline{A}, \beta_N \times (\beta_{p^n} \circ [\alpha])). \tag{1.19}$$

We let $(\mathcal{O}_L/(p^n))^\times$ act on functions f on $\mathcal{M}(B, \mu_{Np^n})$ by

$$([\alpha]\, f)(\underline{A}, \beta_N \times \beta_{p^n}) = f([\alpha](\underline{A}, \beta_N \times \beta_{p^n})). \tag{1.20}$$

2. MOD p

Let $N \geqslant 4$ and prime to p. Recall that $\mathcal{M}^*(B, \mu_N)$ denotes the base change to B of the *whole* moduli space of HBAS with μ_N-level compactified at infinity. For B an $\mathbb{F}$-algebra, we let $W_{\{i\}}$ be the closed reduced sub-scheme of $\mathcal{M}^*(B, \mu_N)$ where the type contains i. See above and [7] for more details.

THEOREM 2.1. *There exist HMFs $h_1, \ldots, h_g$, over $\mathbb{F}$, of weights $\chi_g^p \chi_1^{-1}$, $\chi_1^p \chi_2^{-1}, \ldots, \chi_{g-1}^p \chi_g^{-1}$ respectively (h_i being of weight $\chi_{i-1}^p \chi_i^{-1}$), such that*

$$(h_i) = W_{\{i\}}. \tag{2.1}$$

(*In particular, the divisor of h_i is reduced.*) *The q-expansion of h_i at every standard cusp of $\mathscr{M}^*(\mathbb{F}, \mu_N)$ is* 1. *Let $h = h_1 \cdots h_g$. Then h is a modular form of weight* Norm^{p-1}. *It has q-expansion equal to* 1 *at every cusp and its divisor is reduced, equal to the complement of the ordinary locus.*

We refer the reader to [5] for complete details and discussion of the partial Hasse invariants h_i. For completeness, we sketch the proof of the theorem. The following lemma follows immediately from the discussion in [7].

LEMMA 2.2. *Let $\underline{A}$ be a HBAS over a perfect field k containing* $\mathbb{F}$. *Assume that $\underline{A}$ is not ordinary. Then the p-divisible group $\underline{A}(p)$ of $\underline{A}$ is local and a universal display over* $\mathrm{Spec}(k[[t_1, ..., t_g]])$ *for its infinitesimal deformations as a HBAS is given by*

$$\Phi = \begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix}. \tag{2.2}$$

Here A, B, C and D are $g \times g$ matrices that are Teichmüller lifts to $W(k[[t_1, ..., t_g]])$ of the display $\Phi_0 = \left(\begin{smallmatrix} A \,(\mathrm{mod}\; p) & B \,(\mathrm{mod}\; p) \\ C \,(\mathrm{mod}\; p) & D \,(\mathrm{mod}\; p)\end{smallmatrix}\right)$ of $\underline{A}$, and can be chosen to be of the form

$$A = \begin{pmatrix} & & & a_1 \\ a_2 & & & \\ & \ddots & & \\ & & a_g & \end{pmatrix} \tag{2.3}$$

(*Similarly for B, C, D.*) *The matrix T is diagonal, with diagonal elements $T_1, ..., T_g$, where T_i is the Teichmüller lift of t_i.*

Let

$$e_1, ..., e_g \tag{2.4}$$

be the idempotents of $\mathcal{O}_L \otimes \mathbb{F}$. Given $(\underline{X}, \omega)_{/R}$ we get a basis $\{e_1\omega, ..., e_g\omega\}$ for $\mathfrak{t}^*_{X/R}$. Let $\{\eta_1, ..., \eta_g\}$ be the basis of $\mathfrak{t}_{X/R}$ dual to that basis. Let F be the Frobenius morphism. It is induced by a choice of prime-to-p $\mathcal{O}_L$-polarization that identifies $\mathfrak{t}_{X/R}$ with $H^1(X, \mathcal{O}_X)$. Put

$$h_i((\underline{X}, \omega)) = F\eta_{i-1}/\eta_i. \tag{2.5}$$

One verifies that indeed $F\eta_{i-1}$ is a multiple of η_i and that h_i is a modular form of weight $\chi_{i-1}^p \chi_i^{-1}$. See [5]. Moreover, if $R = k$ is a perfect field, by the theory of displays the matrix $A+TC$ modulo p gives the action of Frobenius on the tangent space of the universal local deformation of $\underline{X}$. One finds that $a_i \pmod p$ is, up to a unit of the base, $h_i(\underline{X}, \omega)$, and that $a_i + T_i c_i \pmod p$ is, up to a unit of the base, h_i of the universal deformation with some choice of a non-vanishing differential on it. On the other hand, one can prove that $a_i = 0$ if and only if $i \in \tau(\underline{X})$. We see that $(h_i) = W_{\{i\}}$. ∎

The divisor of the total Hasse invariant h is precisely the non-ordinary locus. It is also well known that the line bundle whose sheaf of sections are modular forms of parallel weight 1 is ample. It follows that $\mathcal{M}^*(\mathbb{F}, \mu_{Np^n})^{ord}$ is affine for $n \geqslant 0$. Let R_{Np^n} denote the ring of regular functions on $\mathcal{M}^*(\mathbb{F}, \mu_{Np^n})^{ord}$. Since $\mathcal{M}^*(\mathbb{F}, \mu_{Np^n})^{ord}$ is normal and the cusps are zero dimensional, if $g > 1$ the ring R_{Np^n} is also the ring of regular functions on $\mathcal{M}(\mathbb{F}, \mu_{Np^n})^{ord}$.

THEOREM 2.3. *Let $N \geqslant 4$ and let p be inert in L.*

1. *There exists a natural surjective homomorphism*

$$r: \bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(\mathbb{F}, \chi, \mu_N) \to R_{Np}, \tag{2.6}$$

whose kernel I is precisely the kernel of the q-expansion map. The ideal I is graded by $\mathbf{X}/\mathbf{X}(1)$ and

$$I = (h_i - 1 : i = 1, \ldots, g). \tag{2.7}$$

2. *Under the isomorphism $\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong R_{Np}$ provided above, we have*

$$\bigoplus_{\chi \in \mathbf{X}(1)} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong R_N. \tag{2.8}$$

Proof. Let $\pi: (\underline{A}^u, \beta^u) \to \mathcal{M}(\mathbb{F}, \mu_{Np})$ be the universal object. Let

$$\Omega = \mathrm{t}^*_{(\underline{A}^u, \beta^u) \to \mathcal{M}(\mathbb{F}, \mu_{Np})} \tag{2.9}$$

be the relative cotangent bundle at the origin. Via β^u we get an isomorphism

$$\Omega \cong \mathrm{t}^*_{\mathcal{D}_L^{-1} \otimes \mu_p \to \mathrm{Spec}(\mathbb{F})} \otimes_{\mathbb{F}} \mathcal{O}_{\mathcal{M}(\mathbb{F}, \mu_{Np})}. \tag{2.10}$$

Hence Ω has a canonical generator ω_{can}: The image of $(1 \otimes \frac{dq}{q}) \otimes 1$. The idempotents $\{e_1, \ldots, e_g\}$ (see (2.4)) give a decomposition

$$\Omega = \bigoplus_{i=1}^{g} \Omega(\chi_i), \omega_{\mathrm{can}} = \bigoplus_{i=1}^{g} a(\chi_i). \tag{2.11}$$

In the case $g=1$ the line bundles $\Omega(\chi_i)$ and the sections $a(\chi_i)$ naturally extend to $\mathcal{M}^*(\mathbb{F}, \mu_{Np})^{ord}$ as follows from the existence of a universal generalized elliptic curve over $\mathcal{M}^*(\mathbb{F}, \mu_{Np})^{ord}$. Given any $\chi \in \mathbf{X}$, $\chi = \chi_1^{r_1} \cdots \chi_g^{r_g}$, we put

$$\Omega(\chi) = \bigotimes_{i=1}^{g} \Omega(\chi_i)^{\otimes r_i}, \; a(\chi) = \bigotimes_{i=1}^{g} a(\chi_i)^{\otimes r_i}. \tag{2.12}$$

Clearly $a(\chi)$ is a canonical section of $\Omega(\chi)$ (ω_{can} is non-vanishing!).

Let $f \in \mathbf{M}(\mathbb{F}, \chi, \mu_N)$. We write f also for the pull-back of f to $\mathcal{M}(\mathbb{F}, \mu_{Np})$ ($\mathcal{M}^*(\mathbb{F}, \mu_{Np})$ if $g=1$). Let

$$r(f) = f/a(\chi). \tag{2.13}$$

We extend the definition linearly and obtain a ring homomorphism

$$\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(\mathbb{F}, \chi, \mu_N) \to R_{Np}. \tag{2.14}$$

It can be interpreted as follows. Given $(\underline{A}, \beta_N \times \beta_p)_{/R}$, we have

$$r\left(\sum f_\chi\right)((\underline{A}, \beta_N \times \beta_p)) = \sum f_\chi\left(\underline{A}, \beta_N, (\beta_p^*)^{-1}\left(1 \otimes \frac{dq}{q}\right)\right). \tag{2.15}$$

From Equation (2.15) we can conclude two facts:

- The map,

$$\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(\mathbb{F}, \chi, \mu_N) \to R_{Np}, \tag{2.16}$$

is $W(\mathbb{F})^\times$-equivariant, where $\alpha \in W(\mathbb{F})^\times$ acts on $f \in \mathbf{M}(\mathbb{F}, \chi, \mu_N)$ by $[\alpha] f = \chi(\alpha) f$. Indeed $r([\alpha] f)(\underline{A}, \beta_N \times \beta_p) = \chi(\alpha)\, r(f)(\underline{A}, \beta_N \times \beta_p) = \chi(\alpha)\, f(\underline{A}, \beta_N, (\beta_p^*)^{-1} 1 \otimes \frac{dq}{q}) = f(\underline{A}, \beta_N, \alpha^{-1} \cdot (\beta_p^*)^{-1} 1 \otimes \frac{dq}{q}) = r(f)(\underline{A}, \beta_N \times \beta_p \circ [\alpha]) = [\alpha](r(f))(\underline{A}, \beta_N \times \beta_p)$.

- Let B be a $W(\mathbb{F})$-algebra. Let Std be the standard cusp of $\mathcal{M}^*(B, \mu_{Np^n})$. It is the Tate object $D_L^{-1} \otimes \mathbb{G}_m/\underline{q}(\mathfrak{c}^{-1})$, with its canonical $\mathcal{O}_L$-action and polarization (see [12] for details), and with its visible μ_{Np^n}-level structure and non-vanishing differential. Evaluation at that object is a q-expansion map.

Taking again $B = \mathbb{F}$ and $n=1$ and employing (2.15), we see, using the theory of toroidal compactifications [2], that the following diagram commutes:

$$\begin{tikzcd} \bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(\mathbb{F}, \chi, \mu_N) \arrow[r] \arrow[dr, "q\text{-expansion}"'] & R_{Np} \arrow[d] \\ & \mathcal{O}_{\mathcal{M}^*(\mathbb{F}, \mu_{Np}), \mathrm{Std}} \end{tikzcd} \tag{2.17}$$

It follows that I is the kernel of the q-expansion map.

The group $\mathbf{X}/\mathbf{X}(1)$ is naturally identified with the group of $\overline{\mathbb{F}}$-valued characters of $(\mathcal{O}_L/(p))^\times$—the Galois group of $\mathcal{M}^*(\mathbb{F}, \mu_{Np})^{ord} \to \mathcal{M}^*(\mathbb{F}, \mu_N)^{ord}$. Note that since $(\mathcal{O}_L/(p))^\times$ is of order prime to p, we have

$$R_{Np} = \bigoplus_{\psi \in \mathbf{X}/\mathbf{X}(1)} R_{Np}^{\psi}, \tag{2.18}$$

where $f \in R_{Np}^{\psi}$ if for every α we have $[\alpha] f = \psi(\alpha) f$.

Given such f, choose some lift χ of ψ to $\mathbf{X}$ and define first a meromorphic modular form g in $\mathbf{M}(\mathbb{F}, \chi, \mu_N)$ by

$$g = f \cdot a(\chi). \tag{2.19}$$

In terms of points,

$$g(\underline{A}, \beta_N, \omega) = f(\underline{A}, \beta_N \times \beta_p) \cdot \chi\left(\frac{(\beta_p^*)^{-1}\left(1 \otimes \dfrac{dq}{q}\right)}{\omega}\right), \tag{2.20}$$

for any μ_p-level β_p. This shows that g is indeed of μ_N-level. Clearly, $r(g) = f$ and g has no poles on the ordinary locus. It follows that $g' = g \cdot h^k$ is a holomorphic modular form for $k \gg 0$. Here h is the total Hasse invariant from Theorem 2.1.

Because I is the kernel of the q-expansion, it follows that for every i, $h_i - 1$ belongs to I. In particular:

- $r(h) = 1$ and hence $r(g') = f$ and the map r is therefore surjective.
- $(h_1 - 1, ..., h_g - 1) \subseteq I$.

We next show that $I = (h_1 - 1, ..., h_g - 1)$. Suppose that $r(\sum_{i=i}^{m} f_i) = 0$. By multiplying by various $h_j - 1$ we may assume that f_i is of weight ψ_i and for $i \neq j$ we have $\psi_i \neq \psi_j \pmod{\mathbf{X}(1)}$. But, since the map r is $W(\mathbb{F})^\times$-equivariant, it follows that each $r(f_i) = 0$, because they fall into different summands of (2.18). However, on each $\mathbf{M}(\mathbb{F}, \chi, \mu_N)$ the map r and q-expansion map are injective. It follows that each $f_i = 0$.

To conclude the proof it only remains to prove part 2. But this follows immediately from Equation (2.18) and the fact that I is generated by elements with weights in $\mathbf{X}(1)$. ∎

Remark 2.4. Let $R = \bigoplus_{\gamma \in \Gamma} R_\gamma$ be a ring graded by an abelian group Γ. Let Γ_0 be a subgroup of Γ. Let J be an ideal generated by elements in $\bigoplus_{\gamma \in \Gamma_0} R_\gamma$. Then J is an ideal graded by Γ/Γ_0: Let $\delta \in \Gamma$. If a finite sum $\sum_{\gamma \in \Gamma} f_\gamma \in J$, then $\sum_{\gamma \in \delta + \Gamma_0} f_\gamma \in J$.

Although the following corollary will be superseded by Corollary 3.15 below, we include it to demonstrate the principle of deriving congruences between zeta values from modular forms, as well as to set notation.

Corollary 2.5. *Let L be a totally real field. Let p be a rational prime that is unramified in L. Let $k \geqslant 2$.*

1. *If $k \not\equiv 0 \pmod{p-1}$ then $\zeta_L(1-k)$ is p-integral.*
2. *If $k \not\equiv 0 \pmod{p-1}$ and $k \equiv k' \pmod{p-1}$ then $\zeta_L(1-k) \equiv \zeta_L(1-k') \pmod{p}$.*

Proof. There exists an Eisenstein series of parallel weight k (i.e., weight Norm^k)

$$E_k = 1 + 2^g \zeta_L(1-k)^{-1} \sum c_{k-1,\alpha} q^\alpha, \tag{2.21}$$

where α runs over a lattice depending on the cusp at which the q-expansion is created and the $c_{k-1,\alpha}$ are sums of $(k-1)$-powers of certain rational integers depending on α and the cusp but not on k. More precisely, under appropriate choices, the q-expansion on a component of the moduli space has coefficients

$$c_{k-1,\alpha} = \begin{cases} \sigma_{k-1}((\alpha)\,\mathfrak{a}^{-1} D_L) & \alpha \in (\mathfrak{a} D_L^{-1})^+ \\ 0 & \text{otherwise,} \end{cases} \tag{2.22}$$

where for any integral ideal $\mathfrak{b}$ we let $\sigma_{k-1}(\mathfrak{b}) = \sum_{\mathcal{O}_L \supset \mathfrak{c} \supset \mathfrak{a}} \mathbf{N}(\mathfrak{c})^{k-1}$. See [5] and (3.51). We let

$$E_k^* = 2^{-g} \zeta_L(1-k) \cdot E_k. \tag{2.23}$$

If $2^{-g} \zeta_L(1-k)$ is not p-integral, then $E_k - 1 \equiv 0 \pmod{p}$. If $k \not\equiv 0 \pmod{p-1}$ then $\mathrm{Norm}^k \neq \mathbf{1} \pmod{\mathbf{X}(1)}$. This and the fact that I is graded by $\mathbf{X}/\mathbf{X}(1)$, imply that $1 \in I$, which is a contradiction.

Assume that $k \not\equiv 0 \pmod{p-1}$. Then $\alpha := 2^{-g}(\zeta_L(1-k') - \zeta_L(1-k))$ belongs to $\mathbb{Z}_p$. Because the coefficients $c_{k-1,\alpha} \pmod{p}$ depend only on $k \pmod{p-1}$ we have

$$E_k^* - E_{k'}^* - \alpha \equiv 0 \qquad \pmod{p}. \tag{2.24}$$

But, using the grading, this implies that $\alpha \pmod p$ belongs to I. That is, $\alpha \equiv 0 \pmod p$. Hence,

$$\zeta_L(1-k) \equiv \zeta_L(1-k') \qquad (\mathrm{mod}\ p). \tag{2.25}$$ ∎

The following corollary identifies, via the map r, certain subrings of $\bigoplus_\chi \mathbf{M}(\mathbb{F}, \chi, \mu_N)$ and R_{Np}.

COROLLARY 2.6. *Let H be the kernel of the Norm map $(\mathcal{O}_L/(p))^\times \to (\mathbb{Z}/(p))^\times$. Let $R_{Np}^{\|}$ be the ring of regular functions of the scheme $\mathcal{M}^*(\mathbb{F}, \mu_{Np})/H$. We have isomorphisms*

$$\bigoplus_{k=0}^{\infty} \mathbf{M}(\mathbb{F}, \mathrm{Norm}^{k(p-1)}, \mu_N)/(h-1) \cong R_N, \tag{2.26}$$

$$\bigoplus_{k=0}^{\infty} \mathbf{M}(\mathbb{F}, \mathrm{Norm}^{k}, \mu_N)/(h-1) \cong R_{Np}^{\|}. \tag{2.27}$$

Proof. Let $\mathbf{X}^{\|} \subset \mathbf{X}$ be the characters trivial on H. Clearly, $\mathbf{X}^{\|} = \langle \mathrm{Norm}, \mathbf{X}(1) \rangle$. It follows immediately from the theorem that

$$\bigoplus_{\chi \in \mathbf{X}(1)} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong R_N, \qquad \bigoplus_{\chi \in \mathbf{X}^{\|}} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong R_{Np}^{\|}. \tag{2.28}$$

Thus, the assertion is that

$$\bigoplus_{\chi \in \mathbf{X}(1)} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong \bigoplus_{k=0}^{\infty} \mathbf{M}(\mathbb{F}, \mathrm{Norm}^{k(p-1)}, \mu_N)/(h-1), \tag{2.29}$$

and

$$\bigoplus_{\chi \in \mathbf{X}^{\|}} \mathbf{M}(\mathbb{F}, \chi, \mu_N)/I \cong \bigoplus_{k=0}^{\infty} \mathbf{M}(\mathbb{F}, \mathrm{Norm}^{k}, \mu_N)/(h-1). \tag{2.30}$$

In both cases the inclusion $\supset$ is clear. Thus, the claim amounts to that for any element $\chi \in \mathbf{X}^{\|}$ (resp. $\mathbf{X}(1)$) we may find suitable non-negative r_i's such that $\chi \cdot (\chi_1^p \chi_2^{-1})^{r_1} \cdots (\chi_g^p \chi_1^{-1})^{r_g}$ is a power of Norm. This is clear. ∎

The notion of filtration plays an important role in theory of elliptic modular forms, e.g., in the weight part of Serre's conjecture. The following corollary yields an analogous filtration on Hilbert modular forms.

COROLLARY 2.7. *Given a q-expansion $b(q)$ which is a q-expansion of some HMF of μ_N-level at, say, the standard cusp, there exists a unique*

HMF f_0 such that the set of all modular forms with q-expansion $b(q)$ is the set

$$\left\{f_0 \cdot \prod_{i=1}^{g} h_i^{a_i} : a_i \geqslant 0\right\}. \tag{2.31}$$

We call the weight of f_0 the filtration *of the q-expansion $b(q)$.*

Proof. If f and g have the same q-expansion then $r(f)=r(g)$, and vice versa. We are given that $b(q)$ is a q-expansion of some Hilbert modular form of weight, say, χ. Let f' be a function on $\mathcal{M}^*(\mathbb{F}, \mu_{Np})$ such that $f' \in R^{\chi}_{Np}$ and in the local ring of the appropriate cusp $f' = b(q)$. Then all the meromorphic modular forms having q expansion $b(q)$ are of the form $f' \cdot a(\chi) \cdot \prod h_i^{a_i}$ where the $a_i \in \mathbb{Z}$. But the divisor of h_i is the reduced effective divisor $W_{\{i\}}$. Therefore, there is a choice a_1^* , ..., a_g^* such that $f_0 = f' \cdot a(\chi) \cdot \prod h_i^{a_i^*}$ is holomorphic and non-vanishing on some component of every $W_{\{i\}}$. It follows that every other holomorphic form with the same q-expansion is a multiple $f_0 \cdot \prod_{i=1}^{g} h_i^{a_i}$ with $a_i \geqslant 0$. ▮

We remark that certain variants are possible. For example, for a q-expansion arising from a HMF of parallel weight one can define its "parallel filtration".

The modular forms $a(\chi)$ have other interesting applications. We now discuss how they may be used to construct a compactification with nice properties of $\mathcal{M}^*(\mathbb{F}, \mu_{Np})^{ord}$—the Satake compactification of the moduli space of HBAS over $\mathbb{F}$-algebras together with μ_{Np}-level.

LEMMA 2.8. *We have an equality of modular forms on $\mathcal{M}(\mathbb{F}, \mu_{Np})^{ord}$:*

$$a(\chi_i)^{p^g-1} = h_{i+1}^{p^{g-1}} h_{i+2}^{p^{g-2}} \cdots h_{i-1}^{p} h_i . \tag{2.32}$$

Proof. Indeed, both sides are modular forms on $\mathcal{M}(\mathbb{F}, \mu_{Np})^{ord}$ of the same weight, namely $\chi_i^{p^g-1}$, and the same q-expansion, namely, 1.

Let, therefore,

$$b_i = a(\chi_i)^{p^g-1}, \tag{2.33}$$

be the modular form on $\mathcal{M}(\mathbb{F}, \mu_N)$ of weight $\chi_i^{p^g-1}$ and q-expansion 1. We fix i and consider the scheme

$$\mathcal{M}' = \mathcal{M}(\mathbb{F}, \mu_N)[b_i^{1/(p^g-1)}]. \tag{2.34}$$

We explain our notation and terminology:

The map of global sections

$$\Gamma(\mathscr{M}(\mathbb{F}, \mu_N), \Omega(\chi_i)) \to \Gamma(\mathscr{M}(\mathbb{F}, \mu_N), \Omega(\chi_i^{p^g - 1})) \tag{2.35}$$

is induced from a morphism of schemes over $\mathscr{M}(\mathbb{F}, \mu_N)$

$$\alpha\colon \Omega(\chi_i) \to \Omega(\chi_i^{p^g - 1}), \tag{2.36}$$

given locally by taking $(p^g - 1)$-powers along the fiber. We define $\mathscr{M}' = \mathscr{M}(\mathbb{F}, \mu_N)[b_i^{1/(p^g - 1)}]$ to be the fiber product with respect to the maps α and b_i:

$$\mathscr{M}' = \Omega(\chi_i) \underset{\Omega(\chi_i^{p^q - 1})}{\times} \mathscr{M}(\mathbb{F}, \mu_N). \tag{2.37}$$

Let $p_2\colon \mathscr{M}' \to \mathscr{M}(\mathbb{F}, \mu_N)$ be the projection and consider the line bundles $p_2^*\Omega(\chi_i)$ and $p_2^*\Omega(\chi_i^{p^g - 1})$ on $\mathscr{M}'$. Let s^u be the tautological section

$$s^u\colon \mathscr{M}' \to p_2^*\Omega(\chi_i), \tag{2.38}$$

and let $p_2^*b_i$ be the induced section

$$p_2^*b_i\colon \mathscr{M}' \to p_2^*\Omega(\chi_i^{p^g - 1}). \tag{2.39}$$

The equation

$$(s^u)^{p^g - 1} = p_2^* b_i \tag{2.40}$$

holds on $\mathscr{M}'$. In fact $\mathscr{M}'$ has the following universal property: Given a scheme $f\colon S \to \mathscr{M}(\mathbb{F}, \mu_N)$ and $s \in \Gamma(S, f^*\Omega(\chi_i))$ such that $s^{p^g - 1} = f^*b_i$, there exists a unique morphism $g\colon S \to \mathscr{M}'$ over $\mathscr{M}(\mathbb{F}, \mu_N)$ such that $s = g^*s^u$. We leave the verification of this fact to the reader.

One also sees easily that $(\mathscr{O}_L/(p))^\times$, identified with $\mathbb{F}^\times$, acts faithfully on $\mathscr{M}'$. The morphism $\mathscr{M}' \to \mathscr{M}(\mathbb{F}, \mu_N)$ is $(\mathscr{O}_L/(p))^\times$-equivariant and exhibits $\mathscr{M}(\mathbb{F}, \mu_N)$ as the quotient for this action.

We conclude from Lemma 2.8 and the universal property the existence of an $(\mathscr{O}_L/(p))^\times$-equivariant open immersion

$$\mathscr{M}(\mathbb{F}, \mu_{Np}) \to \mathscr{M}'. \tag{2.41}$$

Note the identity

$$a(\chi_{i-1})^p\, a(\chi_i)^{-1} = h_i. \tag{2.42}$$

We have $a(\chi_i) = a(\chi_{i-1})^p/h_i$. A priori this is a meromorphic modular form on $\mathscr{M}'$. But raising both sides of the equation to the $p^g - 1$ power, and using Lemma 2.8, we find it must be holomorphic. It follows that $\mathscr{M}'$ does not depend on i. Finally, we let $\mathscr{M}$ be the scheme obtained from $\mathscr{M}'$ and $\mathscr{M}^*(\mathbb{F}, \mu_{Np})$ by glueing along $\mathscr{M}(\mathbb{F}, \mu_{Np})$.

THEOREM 2.9. *There exists a scheme $\mathscr{M}$ and a proper morphism $f: \mathscr{M} \to \mathscr{M}^*(\mathbb{F}, \mu_N)$, an open immersion $\mathscr{M}^*(\mathbb{F}, \mu_{Np})^{ord} \to \mathscr{M}$, and a faithful $(\mathcal{O}_L/(p))^\times$ action extending the one on $\mathscr{M}^*(\mathbb{F}, \mu_{Np})^{ord}$ such that f exhibits $\mathscr{M}^*(\mathbb{F}, \mu_N)$ as the quotient by this action. In particular, f is finite.*

The scheme $\mathscr{M}$ is defined by the equation

$$s^{p^g-1} = h_{i+1}^{p^{g-1}} h_{i+2}^{p^{g-2}} \cdots h_{i-1}^{p} h_i \tag{2.43}$$

and is independent of i. The map f is ramified precisely along the complement of the ordinary locus, and is totally ramified there. The singular locus of $\mathscr{M}$ is of pure codimension 2 and is the pre-image of $\bigcup_{i \neq j} W_{\{i,j\}}$.

Proof. The theorem follows from the discussion above; One has to also note that since the divisor of the modular form (h_i) is reduced and equal to $W_{\{i\}}$, Equation (2.43) becomes an Eisenstein polynomial in the local ring of every component of $W_{\{i\}}$, for every i. A similar local calculation yield the identification of the singular locus. ∎

Remark 2.10. One of the reasons to introduce $\mathscr{M}$ is that certain notions regarding modular forms are better formulated on $\mathscr{M}$. For example, the notion of filtration is translated into the notion of order of vanishing along the divisors $W_{\{i\}}$ in $\mathscr{M}$ (Cf. [8]). The problem of existence of modular forms of a specified weight, or filtration, can be viewed as a "Riemann–Roch problem" on $\mathscr{M}$. The theta operators θ_i defined by Katz [12] can be viewed as the operators taking $f \in R^{\psi}_{Np}$ to $(df)_i/KS(a(\chi_i^2))$. Here $\underline{\omega} = \pi_* \Omega^1_{(\underline{A}^U, \beta^U_{Np})/\mathscr{M}(\mathbb{F}, \mu_{Np})}$ is the relative cotangent space at the origin of the universal object, $KS: \underline{\omega}^{\otimes_{\mathcal{O}_L} 2} \to \Omega^1_{\mathscr{M}(\mathbb{F}, \mu_{Np})^{ord}/\mathbb{F}}$ is the Kodaira–Spencer isomorphism, $\otimes_{\mathcal{O}_L}$ means the second tensor power as an $\mathcal{O}_L \otimes \mathcal{O}_{\mathscr{M}(\mathbb{F}, \mu_{Np})^{ord}}$ line bundle, and $(df)_i$ is the χ_i^2 isotypical component. These ideas will be pursued in a future work.

3. MOD p^m

3.1. *Construction of modular forms.* Assume that $N \geqslant 4$ and, as before, p is inert in L. Following Katz [9], we let

$$T_{m,n} = \begin{cases} \mathscr{M}^*(W_m(\mathbb{F}), \mu_{Np^n})^{ord} & g = 1 \\ \mathscr{M}(W_m(\mathbb{F}), \mu_{Np^n})^{ord} & g > 1, \end{cases} \qquad T^*_{m,n} = \begin{cases} \mathscr{M}^*(W_m(\mathbb{F}), \mu_{Np^n})^{ord} & g = 1 \\ \mathscr{M}^*(W_m(\mathbb{F}), \mu_{Np^n})^{ord} & g > 1, \end{cases} \tag{3.1}$$

where $W_m(\mathbb{F})$ is the ring of Witt vectors of length m over $\mathbb{F}$. For every n, the morphism $T_{m,n} \to T_{m,0}$ is étale Galois with Galois group equal to $(\mathcal{O}_L/(p^n))^\times$. For every m, n, the morphisms $T_{m,n} \to T_{m+1,n}$ and $T^*_{m,n} \to T^*_{m+1,n}$ are closed immersions and $T_{m,n} = T_{m+1,n} \otimes W_m(\mathbb{F})$. The scheme $T^*_{m,n}$ is an affine scheme because the invertible sheaf of modular forms of parallel weight is ample and has a global section (some lift of h^n) whose divisor is the non-ordinary locus, and $T_{m,n}$ is smooth over $W_m(\mathbb{F})$, for every m, n. We let $V_{m,n}$ be the ring of regular functions of $T_{m,n}$ (equivalently, $T^*_{m,n}$). Note that $V_{1,1} = R_{Np}$ and $V_{1,0} = R_N$ in the notation of Section 2. The schemes $T_{m,n}$ and the rings $V_{m,n}$ all fit into the following commutative diagrams:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \\ T_{1,2} & \hookrightarrow & T_{2,2} & \hookrightarrow & T_{3,2} & \hookrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ T_{1,1} & \hookrightarrow & T_{2,1} & \hookrightarrow & T_{3,1} & \hookrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ T_{1,0} & \hookrightarrow & T_{2,0} & \hookrightarrow & T_{3,0} & \hookrightarrow & \cdots \end{array} \quad , \quad \begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ V_{1,2} & \twoheadleftarrow & V_{2,2} & \twoheadleftarrow & V_{3,2} & \twoheadleftarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ V_{1,1} & \twoheadleftarrow & V_{2,1} & \twoheadleftarrow & V_{3,1} & \twoheadleftarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ V_{1,0} & \twoheadleftarrow & V_{2,0} & \twoheadleftarrow & V_{3,0} & \twoheadleftarrow & \cdots . \end{array} \tag{3.2}$$

We let

$$T_{m,\infty} = \varprojlim_n T_{m,n}, \qquad T_{\infty,\infty} = \varinjlim_m T_{m,\infty} \tag{3.3}$$

(similarly for $T^*_{m,n}$), and

$$V_{m,\infty} = \varinjlim_n V_{m,n}, \qquad V_{\infty,\infty} = \varprojlim_m V_{m,\infty}. \tag{3.4}$$

LEMMA 3.1. 1. *Fix $1 \leqslant i \leqslant g$. For every $m \leqslant n$ there exist a modular form $a(\chi_i) = a_{m,n}(\chi_i)$ on $T_{m,n}$ of weight χ_i. It has q-expansion equal to* 1 *at the standard cusp* Std.

2. *The $a(\chi_i) = a_{m,n}(\chi_i)$ are compatible in the following sense:*

a. Under the map $f: T_{m,n+n'} \to T_{m,n}$ we have

$$f^* a_{m,n}(\chi_i) = a_{m,n+n'}(\chi_i). \tag{3.5}$$

b. Under the map $f: T_{m,n} \to T_{m+m',n}$, where $m+m' \leqslant n$, we have

$$f^* a_{m+m',n}(\chi_i) = a_{m,n}(\chi_i). \tag{3.6}$$

Proof. Let $(\underline{A}^u, \beta_N^u \times \beta_{p^n}^u) \to T_{m,n}$ be the universal object. Note that

$$\mathfrak{t}^*_{D_L^{-1} \otimes \mu_{p^n} \to W_m(\mathbb{F})} \cong \mathcal{O}_L \otimes \mathfrak{t}^*_{\mu_{p^n} \to W_m(\mathbb{F})}. \tag{3.7}$$

(See the discussion in Section 1.2.) The invariant differentials $\mathfrak{t}^*_{\mu_{p^n} \to W_m(\mathbb{F})}$ are contained in

$$\Omega^1_{\mu_{p^n} \to W_m(\mathbb{F})} = W_m(\mathbb{F})[q]/(q^{p^n} - 1, p^n q^{p^n - 1}) \cdot dq. \tag{3.8}$$

The differential $\omega = q^{p^n - 1}\, dq$ is invariant and $p^n \omega = 0$. Thus, $m \leqslant n$ if and only if $\mathfrak{t}^*_{D_L^{-1} \otimes \mu_{p^n} \to W_m(\mathbb{F})}$ is a free $\mathcal{O}_L \otimes W_m(\mathbb{F})[q]/(q^{p^n} - 1)$ module of rank 1. Since we assume that $m \leqslant n$, it follows as in the proof of Theorem 2.3 that the relative cotangent space of $(\underline{A}^u, \beta_N^u \times \beta_{p^n}) \to T_{m,n}$ is a free $\mathcal{O}_L \otimes \mathcal{O}_{T_{m,n}}$ module of rank 1 with a *canonical* generator ω_{can}—"the pull-back of $(1 \otimes \frac{dq}{q}) \otimes 1$".

Let $\{e_1, ..., e_g\}$ be the idempotents as in (2.4). Let

$$a(\chi_i) = e_i \cdot \omega_{\mathrm{can}}. \tag{3.9}$$

It is a modular form of weight χ_i. The compatibility assertions are easily reduced to the following simple observations:

- The canonical map

$$D_L^{-1} \otimes \mu_{p^n/W_m(\mathbb{F})} \hookrightarrow D_L^{-1} \otimes \mu_{p^{n+n'}/W_m(\mathbb{F})} \tag{3.10}$$

induces an isomorphism of the relative cotangent spaces.

- The canonical map

$$D_L^{-1} \otimes \mu_{p^n/W_{m+m'}(\mathbb{F})} \hookrightarrow D_L^{-1} \otimes \mu_{p^n/W_m(\mathbb{F})} \tag{3.11}$$

induces an isomorphism $\mathfrak{t}^*_{D_L^{-1} \otimes \mu_{p^n} \to W_{m+m'}(\mathbb{F})} \bigotimes_{W_{m+m'}(\mathbb{F})} W_m(\mathbb{F}) \cong \mathfrak{t}^*_{D_L^{-1} \otimes \mu_{p^n} \to W_m(\mathbb{F})}$. ∎

The following corollary follows immediately:

COROLLARY 3.2. *Let $\chi = \chi_1^{r_1} \cdots \chi_g^{r_g} \in \mathbf{X}$. Define for $m \leqslant n$*

$$a(\chi) = a(\chi_1)^{r_1} \cdots a(\chi_g)^{r_g}. \tag{3.12}$$

Then the $a(\chi)$ are "independent of (m, n)" and define a modular form $a(\chi)$ on $T_{\infty,\infty}$. This modular form is of weight χ and has q-expansion 1 *at the standard cusp* Std *of $T^*_{\infty,\infty}$.*

The group $(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$ acts as automorphisms of $T^*_{m,n}$. This action is given on $T_{m,n}$ in terms of points:

$$[\alpha](\underline{A}, \beta_N \times \beta_{p^n}) \mapsto (\underline{A}, \beta_N \times (\beta_{p^n} \circ [\alpha])). \tag{3.13}$$

Of course the action factors through $(\mathcal{O}_L/(p^n))^\times$. We let

$$[\alpha]\colon T^*_{m,n} \to T^*_{m,n} \tag{3.14}$$

denote the automorphism induced by α. The morphism $[\alpha]$ induces an automorphism of modular forms (a diamond operator). This may be seen as follows: The modular forms of weight χ are sections of $\Omega(\chi)$ (see (2.11), (2.12)). Let $\mathrm{pr}\colon T_{m,n} \to T_{m,0}$ be the natural projection. Then "$\mathrm{pr}^*\Omega(\chi) = \Omega(\chi)$". Indeed, $(\underline{A}^u, \beta_N^u \times \beta_{p^n}^u) \cong (\underline{A}^u, \beta_N^u) \times_{T_{m,0}} T_{m,n}$. But $[\alpha]^*\mathrm{pr}^* = (\mathrm{pr} \circ [\alpha])^* = \mathrm{pr}^*$. Moreover, the formula for the action on a modular form f is

$$([\alpha] f)(\underline{A}, \beta_n \times \beta_{p^n}, \omega) = f(\underline{A}, \beta_N \times (\beta_{p^n} \circ [\alpha]), \omega). \tag{3.15}$$

LEMMA 3.3. *Let $\alpha \in (\mathcal{O}_L/(p^m))^\times$. Let $a(\chi)$ be the modular form on $T_{m,n}$ constructed above. Then*

$$[\alpha]\, a(\chi) = \chi(\alpha)^{-1}\, a(\chi). \tag{3.16}$$

Let $c(\chi) = c_m(\chi)$ be the minimal non-negative integer such that

$$p^{c(\chi)}(1-\chi)(t) \equiv 0 \qquad (\mathrm{mod}\ p^m), \quad \forall t \in (\mathcal{O}_L/(p^m))^\times. \tag{3.17}$$

Then $p^{c(\chi)} a(\chi)$ is invariant under $(\mathcal{O}_L/(p^m))^\times$, and in particular, $a(\chi)$ is invariant under $(\mathcal{O}_L/(p^m))^\times$ if and only if χ is the trivial map $(\mathrm{mod}\ p^m)$.

Proof. Let $\chi = \chi_1^{r_1} \cdots \chi_g^{r_g}$. In terms of points we have

$$a(\chi)(\underline{A}, \beta_n \times \beta_{p^n}, \omega) = \prod_{i=1}^{g} \left(e_i \cdot (\beta_{p^n}^*)^{-1} \left(1 \otimes \frac{dq}{q}\right) \Big/ e_i \cdot \omega \right)^{r_i}. \tag{3.18}$$

The assertion (3.16) and the rest of the Lemma follow easily. ∎

Let $\mathbf{X}(m)$ be the characters in $\mathbf{X}$ that are trivial on $(\mathcal{O}_L/(p^m))^\times$ under the composition

$$(\mathcal{O}_L/(p^m))^\times \hookrightarrow (\mathcal{O}_L \otimes W_m(\mathbb{F}))^\times = \mathbb{T}(W_m(\mathbb{F})) \xrightarrow{\chi} \mathbb{G}_m(W_m(\mathbb{F})) = W_m(\mathbb{F})^\times. \tag{3.19}$$

We shall discuss $\mathbf{X}(m)$ further below. For now, note that $\mathbf{X}(m+1) \subset \mathbf{X}(m)$, and if j is the maximal non-negative integer such that $\chi \in \mathbf{X}(j)$ then

$$c(\chi) = \max\{m - j, 0\}. \tag{3.20}$$

We say that an element χ of $\mathbf{X}(m)$ is *p-positive* if in its expression as

$$\chi = (\chi_g^p \chi_1^{-1})^{r_1} (\chi_1^p \chi_2^{-1})^{r_2} \cdots (\chi_{g-1}^p \chi_g^{-1})^{r_g}, \tag{3.21}$$

every $r_i \geqslant 0$.

COROLLARY 3.4. *Fix an integer $m \geqslant 1$. Let $c(\chi) = c_m(\chi)$ be defined as above.*

1. *For every $\chi \in \mathbf{X}$ there exists a modular form $p^{c(\chi)}a(\chi)$ on $T_{m,0}$ of weight χ ($a(\chi)$ is given by (3.12)). Its q-expansion at every standard cusp is $p^{c(\chi)}$. In particular, for every $\chi \in \mathbf{X}(m)$, the modular form $a(\chi)$ is a modular form of weight χ and q-expansion 1 on $T_{m,0}$.*

2. *Let $\chi \in \mathbf{X}(m)$. The modular form $a(\chi)$ extends to the non-ordinary locus, i.e., it is a modular form over $\mathscr{M}(W_m(\mathbb{F}), \mu_N)$ (and $\mathscr{M}^*(W_m(\mathbb{F}), \mu_N)$ if $g = 1$), if and only if the character $\chi = (\chi_g^p \chi_1^{-1})^{r_1} (\chi_1^p \chi_2^{-1})^{r_2} \cdots (\chi_{g-1}^p \chi_g^{-1})^{r_g}$ is p-positive. Furthermore,*

$$a(\chi) = h_1^{r_1} \cdots h_g^{r_g} \pmod p. \tag{3.22}$$

Proof. It follows from Lemma 3.3 that $p^{c(\chi)}a(\chi)$ is a modular form on $T_{m,0}$, of weight χ, and that its q-expansion at every standard cusp is $p^{c(\chi)}$. This is clear if one thinks of a modular form as in (1.9).

Consider $a(\chi) \pmod p$. It has the same weight and q-expansion as the r.h.s. of Equation (3.22) and that proves the equation. The divisor of $a(\chi)$ on $T_{m,n}$ intersects the special fiber in the divisor of $h_1^{r_1} \cdots h_g^{r_g}$. But according to Theorem 2.1 we have

$$(h_1^{r_1} \cdots h_g^{r_g}) = r_1 W_{\{1\}} + \cdots + r_g W_{\{g\}}. \tag{3.23}$$

Hence, this divisor is effective if and only if each $r_i \geqslant 0$. ∎

3.2. *Digression on* $\mathbf{X}(m)$. We consider now more closely the group $\mathbf{X}(m)$. Let us change notation. Let $G = \langle \sigma \rangle$ be a cyclic group of order g. Let $\mathbb{Z}[G]$ be the group ring of G and $\mathbb{Z}_p[G]$ be the group ring of G over $\mathbb{Z}_p$. The group $W(\mathbb{F})^\times$ is a module over $\mathbb{Z}[G]$, where σ acts as σ—the Frobenius.

- Assume first that $p \neq 2$.

We have

$$W(\mathbb{F})^\times = \mu \times U_1, \tag{3.24}$$

where μ is the cyclic group of order $p^g - 1$ consisting of the roots of unity in $W(\mathbb{F})$, and U_m are the units congruent to 1 modulo (p^m). Clearly, as a $\mathbb{Z}[G]$ module,

$$\mu \cong \mathbb{Z}[G]/(p^g - 1, p - \sigma) = \mathbb{Z}[G]/(p - \sigma). \tag{3.25}$$

By a theorem of Krasner [13, Theorem 17] U_1 is a free $\mathbb{Z}_p[G]$-module of rank 1. Hence,

$$W_m(\mathbb{F})^\times = \mu \times U_1/U_m \cong \mu \times U_1/U_1^{p^{m-1}} \tag{3.26}$$

and it follows that as a $\mathbb{Z}[G]$-module

$$W_m(\mathbb{F})^\times \cong \mathbb{Z}[G]/(p-\sigma) \oplus \mathbb{Z}[G]/(p^{m-1}) \cong \mathbb{Z}[G]/(p^{m-1}(p-\sigma)). \tag{3.27}$$

In other words:

$$\mathbf{X}(m) \cong \langle \chi_1^{p^m} \chi_2^{-p^{m-1}}, \ldots, \chi_g^{p^m} \chi_1^{-p^{m-1}} \rangle. \tag{3.28}$$

Note that these are p-positive generators.

- Assume now that $p=2$. We have

$$W(\mathbb{F})^\times = \mu \times U_1 = \mu \times \{\pm 1\} \times U, \tag{3.29}$$

where μ are the $2^g - 1$ roots of unity and U is a torsion free subgroup of U_1.

Assume that g is odd. Then by [13, Theorem 17] we have

$$U \cong \mathbb{Z}_p[G]. \tag{3.30}$$

Thus, for $m=1$,

$$W_1(\mathbb{F})^\times \cong \mathbb{Z}[G]/(2-\sigma), \tag{3.31}$$

and for $m \geqslant 2$

$$W_m(\mathbb{F})^\times \cong \mathbb{Z}[G]/(2-\sigma) \oplus \mathbb{Z}[G]/(\sigma, 2) \oplus \mathbb{Z}[G]/(2^{m-2}). \tag{3.32}$$

The group $\mathbf{X}(m)$ is thus the intersection of ideals $(2-\sigma) \cap (\sigma, 2) \cap (2^{m-2})$. We have $(2-\sigma) \subset (\sigma, 2)$, $(2^{m-2}) \subset (\sigma, 2)$ if $m > 2$ and $(2^{m-2}) \supset (\sigma, 2)$ if $m=2$. Thus,

$$\mathbf{X}(m) = \begin{cases} (2-\sigma) & m=1 \\ (2^{m-2}(2-\sigma)) & m \geqslant 2. \end{cases} \tag{3.33}$$

In any case $\mathbf{X}(m)$ has naturally chosen p-positive generators,

$$x, x\sigma, \ldots, x\sigma^{g-1}, \tag{3.34}$$

where x is $2-\sigma$ or $2^{m-2}(2-\sigma)$, depending on the case.

If g is even, the situation is more complicated. The decomposition (3.29) still holds, but U can not always be chosen to be a G-module. We allow ourselves simply to remark that $\mathbf{X}(1)$ is the free abelian group generated by $\chi_1^2 \chi_2^{-1}, \ldots, \chi_g^2 \chi_1^{-1}$ and the notion of positivity is the one obtained by identifying $\mathbf{X}(1)$ with $\mathbb{Z}^g$ by sending $\chi_i^2 \chi_{i+1}^{-1}$ to the i-th standard basis element.

The group $\mathbf{X}(m)$ is a sub-lattice and is therefore automatically generated by 2-positive elements. Without going into the details of its structure, we let

$$\psi_1, ..., \psi_g \tag{3.35}$$

be 2-positive generators for it. For the applications we give, the following observation suffices:

Remark 3.5. The character Norm^k belongs to $\mathbf{X}(m)$ if and only if $2^{e(m)} \mid k$, where $2^{e(m)}$ is the exponent of the group $(\mathbb{Z}/(2^m))^\times$. I.e., $e(m) = m-1$ for $m = 1, 2$, and $m-2$ for $m > 2$.

3.3. *The q-expansion map mod p^m.* In this section we study the kernel of the q-expansion map on Hilbert modular forms modulo p^m and level prime to p. Our results are not complete in the sense that we fail to produce a complete set of generators for the kernel I_m of the q-expansion map. However, see Theorem 3.8 and Remark 3.13. We do obtain enough information on I_m to deduce, after introducing a "technical device", the classical congruences and estimates on values of ζ_L at negative integers. See Corollaries 3.11 and 3.15 below.

We remark that our techniques apply to more general L-functions. But the true difficulty now is in the *construction* of Hilbert modular forms with a q-expansion whose constant term is the desired special value and whose higher coefficients have integrality and congruence properties. For this see [4] and [19].

Definition 3.6. Let $\chi \in \mathbf{X}$ and consider it as a character $\chi: (\mathcal{O}_L/(p^m))^\times \to W_m(\mathbb{F})^\times$. Let

$$V^\chi_{m,m} = \{ f \in V_{m,m} : [\alpha] f = \chi(\alpha) f,\ \forall \alpha \in (\mathcal{O}_L/(p^m))^\times \}. \tag{3.36}$$

Let $V^K_{m,m}$—the "Kummer part" of $V_{m,m}$—be given by

$$V^K_{m,m} = \sum_{\chi \in \mathbf{X}/\mathbf{X}(m)} V^\chi_{m,m}. \tag{3.37}$$

Remark 3.7. Note that if $m > 1$ the inclusion $V^K_{m,m} \hookrightarrow V_{m,m}$ is always a *strict* inclusion and the sum in (3.37) is never a direct sum.

Theorem 3.8. 1. *There exists a natural surjective homomorphism of rings*

$$r: \bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N) \to V^K_{m,m}. \tag{3.38}$$

Let I_m be the kernel of r. Then I_m is equal to the kernel of the q-expansion map.

2. *Let I'_m be the ideal $I_m \cap \bigoplus_{\chi \in \mathbf{X}(m)} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$. The map r induces an isomorphism*

$$\bigoplus_{\chi \in \mathbf{X}(m)} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)/I'_m \cong V_{m,0}. \tag{3.39}$$

3. *If $p \neq 2$, the ideal I_m contains the ideal $\langle a(\chi_1^{p^{m+1}} \chi_2^{-p^m}) - 1, ..., a(\chi_g^{p^{m+1}} \chi_1^{-p^m}) - 1 \rangle$, and if $p = 2$, it contains $\langle a(\psi_1) - 1, ..., a(\psi_g) - 1 \rangle$ (where for g odd we have generators as in* (3.34), *and for g even the generators are as in* (3.35)).

Proof. The proof follows the same lines as the proof of Theorem 2.3. We shall therefore be brief.

The map r is defined as in Theorem 2.3. Namely, if $f \in \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$, we let $r(f) = f/a(\chi)$. Using Corollary 3.2 we see that f and $r(f)$ have the same q-expansion, and since $V_{m,m}$ is irreducible, we conclude that I_m is the kernel of the q-expansion map. Certainly Corollary 3.4 implies that if $p \neq 2$,

$$I_m \supseteq \langle a(\chi_1^{p^{m+1}} \chi_2^{-p^m}) - 1, ..., a(\chi_g^{p^{m+1}} \chi_1^{-p^m}) - 1 \rangle, \tag{3.40}$$

and if $p = 2$,

$$I_m \supseteq \langle a(\psi_1) - 1, ..., a(\psi_s) - 1 \rangle. \tag{3.41}$$

Moreover, one verifies that the map r is $(\mathcal{O}_L \otimes \mathbb{Z}_p)^\times$-equivariant, where $([\alpha] f) = \chi(\alpha) f$ for $f \in \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$, and $([\alpha] f)(\underline{A}, \beta_N \times \beta_{p^n}) = f(\underline{A}, \beta_N \times (\beta_{p^n} \circ [\alpha]))$ for $f \in V_{m,m}$. This shows that the image of r is contained in $V_{m,m}^K$. On the other, a construction as in Theorem 2.3, shows that r is surjective onto $V_{m,m}^K$.

It remains only to note that the equivariance also implies (3.39). ∎

Remark 3.9. For $m > 1$, it is not true that I'_m generates I_m. This has to do again with (3.37) not being a direct sum.

The following Criterion follows directly from the methods of the proof of Theorem 3.38. Weak as it seems, it will suffice to derive the classical congruences between values of ζ_L (and more generally, of suitable L-functions).

CRITERION 3.10. Let $\sum_\chi f_\chi \in I_m$. Then there exist a_χ in some $W_m(\mathbb{F})$-algebra such that

$$\sum_\chi a_\chi \chi(u) \equiv 0 \qquad (\mathrm{mod}\ p^m),\ \forall u \in (\mathcal{O}_L/(p^m))^\times, \tag{3.42}$$

and $a_{\mathbf{1}} = f_{\mathbf{1}}$.

Proof. Consider the relation $\sum_\chi r(f_\chi) = 0$. Evaluate it at a point and let the Galois group act. ∎

COROLLARY 3.11. *Let* $k \geqslant 2$.

1. *Let* $p \neq 2$; *if* $k \equiv 0 \pmod{p-1}$ *then*

$$\operatorname{val}_p(\zeta_L(1-k)) \geqslant -1 - \operatorname{val}_p(k), \tag{3.43}$$

and $\zeta_L(1-k)$ *is p-integral if* $k \not\equiv 0 \pmod{p-1}$.

2. *If* $p = 2$, *then*

$$\operatorname{val}_2(\zeta_L(1-k)) \geqslant g - 2 - \operatorname{val}_2(k). \tag{3.44}$$

Proof. 1. The case $k \not\equiv 0 \pmod{p-1}$ was treated in Corollary 2.5. Assume $k \equiv 0 \pmod{p-1}$. Let E_k be the Eisenstein series as in (2.21). Let

$$\ell = \max\{-\operatorname{val}_p(2^{-g}\zeta_L(1-k)), 0\}. \tag{3.45}$$

If $\ell = 0$ there is nothing to prove. Assume therefore that $\ell > 0$ and consider the congruence

$$E_k - 1 \equiv 0 \qquad (\text{mod } p^\ell). \tag{3.46}$$

Then Criterion 3.10 says that for some a in a $W_\ell(\mathbb{F})$ algebra, the polynomial $a \cdot \operatorname{Norm}(x)^k - 1$ is identically zero on $(\mathcal{O}_L/(p^\ell))^\times$ or, equivalently, the polynomial $ax^k - 1$ is identically zero on $(\mathbb{Z}/p^\ell\mathbb{Z})^\times$—a cyclic group of order $(p-1)p^{\ell-1}$. Taking $x = 1$ we see that $a = 1$. It follows that $p^{\ell-1}$ divides k and, hence, $\operatorname{val}_p(k) \geqslant \ell - 1 \geqslant -\operatorname{val}_p(2^{-g}\zeta_L(1-k)) - 1$.

2. When $p = 2$ one argues the same and obtains that $ax^k - 1$ is identically zero on $(\mathbb{Z}/2^\ell\mathbb{Z})^\times$. Analysis of the structure of this group yields the result. ∎

3.4. *Adding level p-structure.* In this section we briefly discuss modular forms of level μ_N (for $(N, p) = 1$) together with an extra level structure of either the form μ_{p^m}, or the form $\Gamma_0(p)$. The first additional level structure already appeared above as involving the *target* of the q-expansion map modulo p^m. It will now appear in the level of the modular forms themselves. This will clarify the nature of the ideal I_m of Theorem 3.8.

The second level structure is introduced to derive the precise congruences between, say, values of the zeta function, that are needed to construct the p-adic zeta function. The same technique would work for a wide variety of L-functions.

Adding μ_{p^m} *level.* Let us consider the graded ring of modular forms $\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_{Np^m})$ on the scheme $T_{m,m}$. The ring of modular forms on $T_{m,0}$, $\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$, embeds in the ring $\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_{Np^m})$ by pull-back via the canonical projection $T_{m,m} \to T_{m,0}$.

PROPOSITION 3.12. *Let $I_m(Np^m)$ be the kernel of the q-expansion map on $\bigoplus_{\chi\in\mathbf{X}}\mathbf{M}(W_m(\mathbb{F}),\chi,\mu_{Np^m})$. Then*

$$I_m(Np^m)=\langle a(\chi)-1:\chi\in\mathbf{X}\rangle, \tag{3.47}$$

and

$$I_m(N)=I_m(Np^m)\cap\bigoplus_{\chi\in\mathbf{X}}\mathbf{M}(W_m(\mathbb{F}),\chi,\mu_N)\subset I_m(Np^m)^{\mathbf{1}}, \tag{3.48}$$

where $I_m(Np^m)^{\mathbf{1}}$ stands for the elements of $I_m(Np^m)$ invariant under the Galois group $(\mathcal{O}_L/(p^m))^{\times}$.

Proof. First, by Corollary 3.2 indeed $a(\chi)-1$ belongs to $I_m(Np^m)$. Suppose that the q-expansion of $\sum_\chi f_\chi$ is zero. Then we may replace an f_χ by $f_\chi+f_\chi(a_\chi-1)$. Repeating this as necessary we obtain a modular form g of parallel positive weight whose q-expansion is zero. Hence, g is zero. That is $\sum_\chi f_\chi\in\langle a(\chi)-1:\chi\in\mathbf{X}\rangle$. The rest is clear. ∎

Remark 3.13. The proposition above clearly demonstrates the problem of determining $I_m(N)$ explicitly. The elements in $I_m(Np^m)^{\mathbf{1}}$ need not extend to a *holomorphic* modular form on $T_{m,0}$.

Adding $\Gamma_0(p)$ level. By a $\Gamma_0(p)$ level structure on a HBAS $\underline{A}$ we mean a finite flat subgroup scheme $H\subset A[p]$, $\mathcal{O}_L$-invariant and of order p^g. Such a subgroup is automatically isotropic with respect to any $\mathcal{O}_L$-polarization. We refer the reader to [14], [18] and [6] for details. However, it may benefit the exposition to recall some basic facts without proofs.

Let us denote the Satake compactification of the fine moduli scheme representing HBAS with level μ_N and level $\Gamma_0(p)$, over $W_m(\mathbb{F})$-algebras, by S_m ($m\leqslant\infty$). Let us denote by S_m^{ord} the ordinary locus. The scheme S_1 has two "horizontal" components, denoted S_1^F and S_1^V, that correspond to taking as H the kernel of Frobenius or the kernel of Verschiebung, respectively. The natural morphism

$$\pi:S_1\to\mathcal{M}^*(W_m(\mathbb{F}),\mu_N) \tag{3.49}$$

induces an isomorphism, $S_1^F\to\mathcal{M}^*(W_m(\mathbb{F}),\mu_N)$, and a totally inseparable morphism of degree p^g, $S_1^V\to\mathcal{M}^*(W_m(\mathbb{F}),\mu_N)$. The scheme S_1 has many other components parameterized by the type and the geometric fibers of π are stratified by projective spaces.

Consider the restriction of the section $\mathcal{M}^*(W_m(\mathbb{F}),\mu_N)\to S_1^F$ to $T_{1,0}^*$, where as above, $T_{1,0}^*$ stands for the ordinary part of $\mathcal{M}^*(W_m(\mathbb{F}),\mu_N)$. Let $S_m^{F,\,ord}$ be the open subscheme of S_m consisting of ordinary HBAS $\underline{A}$ with H being the connected part $A[p]$. We have the following commutative diagram in which the vertical arrows are isomorphisms:

$$\begin{tikzcd} S_1^F \arrow[d] & S_1^{F,\,ord} \arrow[l, hook'] \arrow[r, hook] \arrow[d] & S_m^{F,\,ord} \arrow[d] & T^*_{m,\,1} \arrow[l] \arrow[dl] \\ \mathscr{M}^*(W_m(\mathbb{F}), \mu_N) & T^*_{1,\,0} \arrow[l, hook'] \arrow[r, hook] & T^*_{m,\,0}. \end{tikzcd} \tag{3.50}$$

Let $\tau = (\tau_1, ..., \tau_g) \in \mathfrak{H}^g$. Consider the modular form

$$E_k^*(\tau) = 2^{-g}\zeta_L(1-k) + \sum_{\nu \in \mathcal{O}_L+} \left(\sum_{\mathfrak{c} \mid (\nu)} \operatorname{Norm}(\mathfrak{c})^{k-1} \right) e^{2\pi i \operatorname{Tr}(\nu\tau)}. \tag{3.51}$$

It is a modular form of weight Norm^k on $\mathrm{SL}_2(\mathcal{O}_L \oplus D_L^{-1})$, a fortiori on $\mathscr{M}^*(\mathbb{C}, \mu_N)$, if the polarization module $\mathfrak{c}$ (see Section 1.2) is chosen to be $\mathcal{O}_L$ with its natural notion of positivity. The coefficient of $e^{2\pi i \operatorname{Tr}(\nu\tau)}$ can also be written as $\sigma_{k-1}((\nu))$, where for every integral ideal $\mathfrak{b}$ we let

$$\sigma_{k-1}(\mathfrak{b}) = \sum_{\mathcal{O}_L \supseteq \mathfrak{c} \mid \mathfrak{b}} \operatorname{Norm}(\mathfrak{c})^{k-1}. \tag{3.52}$$

The function σ_{k-1} is multiplicative:

$$\sigma_{k-1}(\mathfrak{b}\mathfrak{c}) = \sigma_{k-1}(\mathfrak{b})\, \sigma_{k-1}(\mathfrak{c}), \qquad (\mathfrak{b}, \mathfrak{c}) = 1. \tag{3.53}$$

It follows that for every prime ideal $\mathfrak{q}$, an ideal $\mathfrak{b} \subset \mathcal{O}_L$ prime to $\mathfrak{q}$, and any $r \geqslant 0$, we have

$$\sigma_{k-1}(\mathfrak{q}^{r+1}\mathfrak{b}) - q^{f(k-1)}\sigma_{k-1}(\mathfrak{q}^r\mathfrak{b}) = \sigma_{k-1}(\mathfrak{b}), \tag{3.54}$$

where q is the rational prime below $\mathfrak{q}$ and $f = f(\mathfrak{q}/q)$.

Retaining our assumption that p is inert in L, let us put

$$\sigma_{k-1,\,p}(p^r\mathfrak{b}) = \sigma_{k-1}(\mathfrak{b}), \qquad (p, \mathfrak{b}) = 1. \tag{3.55}$$

We then obtain the expansion

$$E_k^\dagger(\tau_1, ..., \tau_g) \overset{\text{def}}{=} E_k^*(\tau_1, ..., \tau_g) - p^{g(k-1)}E_k^*(p\tau_1, ..., p\tau_g) \tag{3.56}$$

$$= (1 - p^{g(k-1)})\, 2^{-g}\, \zeta_L(1-k) + \sum_{\nu \in \mathcal{O}_L+} \sigma_{k-1,\,p}((\nu))\, e^{2\pi i \operatorname{Tr}(\nu\tau)}. \tag{3.57}$$

The point important to us is that all the coefficients (except the constant one) are $(k-1)$ powers of natural numbers that are prime to p. Hence, the following facts hold:

Let $k, k' \geqslant 2$ and $k \equiv k' \pmod{(p-1)p^m}$. Let

$$\ell = \max\{-\mathrm{val}_p(2^{-g}\zeta_L(1-k)), -\mathrm{val}_p(2^{-g}\zeta_L(1-k')), 0\}, \tag{3.58}$$

and put

$$r = \max\{\mathrm{val}_p(k), \mathrm{val}_p(k')\}, \qquad r' = \min\{\mathrm{val}_p(k), \mathrm{val}_p(k')\}. \tag{3.59}$$

Note the following points: (i) If $p \neq 2$ then $0 \leqslant \ell \leqslant 1 + r$; (ii) If $p = 2$ then $0 \leqslant \ell \leqslant r + 2$; (iii) If $k \not\equiv 0 \pmod{p-1}$ then $\ell = 0$. They all follow from Corollary 3.11.

We may further assume, w.l.o.g., that if $p = 2$ then $\mathrm{val}_2(k) \leqslant \mathrm{val}_2(k')$ and that k and k' are even. Let

$$i = \begin{cases} 1 & p \neq 2 \\ 2 & p = 2. \end{cases} \tag{3.60}$$

Let

$$\alpha = p^\ell((1 - p^{g(k-1)})\, 2^{-g}\zeta_L(1-k) - (1 - p^{g(k'-1)})\, 2^{-g}\zeta_L(1-k')). \tag{3.61}$$

Then $\alpha \in \mathbb{Z}_p$ and

$$p^\ell E_k^\dagger - p^\ell E_{k'}^\dagger - \alpha \equiv 0 \qquad (\mathrm{mod}\ p^{m+i+\ell}). \tag{3.62}$$

(The congruence means congruence of q-expansions.)

Now, the point is that $p^\ell E_k^\dagger$, $p^\ell E_{k'}^\dagger$ and α are modular forms over $\mathbb{C}$ of level $\Gamma_0(p)$ having integral q-expansion, hence are modular forms on $S_{m+i+\ell}$, hence on $S_{m+i+\ell}^{F,\,ord}$. Therefore, $p^\ell E_k^\dagger$, $p^\ell E_{k'}^\dagger$ and α are meromorphic modular forms on $T^*_{m+i+\ell,0}$ with poles supported on the complement of the ordinary locus (the poles coming from the singularities of S_m). Criterion 3.10 holds also for meromorphic modular forms and we obtain that there exist a, b such that

$$ap^\ell x^k - bp^\ell x^{k'} - \alpha \equiv 0, \qquad \forall x \in (\mathbb{Z}/(p^{m+i+\ell}))^\times. \tag{3.63}$$

Since for every $x \in (\mathbb{Z}/(p^{m+i}))^\times$ we have $x^k = x^{k'} \pmod{p^{m+i}}$, we deduce that there exists a c in a W_{m+i}-algebra such that $cx^k - \alpha \equiv 0 \pmod{p^{m+i}}$ for every x in $(\mathbb{Z}/(p^{m+i}))^\times$. Taking $x = 1$ we see that the following holds

$$\alpha(x^k - 1) \equiv 0 \qquad (\mathrm{mod}\ p^{m+i}), \qquad \forall x \in (\mathbb{Z}/(p^{m+i}))^\times. \tag{3.64}$$

Remark 3.14. The reader notices that we “lose” information by going from (3.63) to (3.64). We remark that the congruences obtained are “good enough” for the purposes of p-adic interpolation.

We separate cases:

(i) $k \not\equiv 0 \pmod{p-1}$. Then $\ell = 0$, and one gets that $\alpha \equiv 0 \pmod{p^{m+1}}$.

(ii) $k \equiv 0 \pmod{p-1}$ but $p \neq 2$. We observe that

$$\mathrm{val}_p(k) + 1 = \min\{\mathrm{val}_p(x^k - 1) : x \in \mathbb{Z},\ p \nmid x\}. \tag{3.65}$$

We therefore obtain that $\mathrm{val}_p(\alpha) \geqslant m + 1 - (r' + 1) = m - r'$.

(iii) $k \equiv 0 \pmod{p-1}$ and $p = 2$. (We still assume that k is even, since k odd implies that k' is odd and we get $\zeta_L(1-k) = \zeta_L(1-k') = 0$). Observe:

$$\mathrm{val}_2(k) + 2 = \min\{\mathrm{val}_2(x^k - 1) : x \in \mathbb{Z},\ 2 \nmid x\}. \tag{3.66}$$

Therefore, $\mathrm{val}_2(\alpha) \geqslant m + 2 - (r' + 2) = m - r'$.

We observe that $m - r' - \ell \geqslant m - i - (r + r')$. We may therefore sum up the discussion above in

Corollary 3.15. *Let $k, k' \geqslant 2$ and $k \equiv k' \pmod{(p-1)\,p^m}$.*

1. *If $k \not\equiv 0 \pmod{p-1}$ then*

$$(1 - p^{g(k-1)})\,\zeta_L(1-k) \equiv (1 - p^{g(k'-1)})\,\zeta_L(1-k') \qquad \pmod{p^{m+1}}. \tag{3.67}$$

2. *If $k \equiv 0 \pmod{p-1}$ but $p \neq 2$, then*

$$(1 - p^{g(k-1)})\,\zeta_L(1-k) \equiv (1 - p^{g(k'-1)})\,\zeta_L(1-k') \qquad \pmod{p^{m-1-\mathrm{val}_p(k\cdot k')}}. \tag{3.68}$$

3. *If $p = 2$ then*

$$(1 - 2^{g(k-1)})\,\zeta_L(1-k) \equiv (1 - 2^{g(k'-1)})\,\zeta_L(1-k') \quad \pmod{2^{m+g-2-\mathrm{val}_2(k\cdot k')}}. \tag{3.69}$$

4. LIFTING OF q-EXPANSIONS

Proposition 4.1. *Any modular form $f \in \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$ can be lifted to $T_{\infty,\infty}$.*

Proof. Clearly the regular function $f/a(\chi) \in V_{m,m} \subset V_{m,\infty}$ can be lifted to $V_{\infty,\infty}$. Indeed, $V_{m,\infty} = V_{\infty,\infty} \otimes W_m(\mathbb{F})$. On the other hand, by Corollary 3.2, $a(\chi)$ itself lifts to $T_{\infty,\infty}$. ∎

A much more subtle question is that of lifting a modular form $f \in \mathbf{M}(W_m(\mathbb{F}), \chi, \mu_N)$ to a modular form in $\mathbf{M}(W(\mathbb{F}), \chi, \mu_N)$. For example, take $m=1$. The modular forms h_i do not lift, because any non-cusp form of *finite* level must have parallel weight. Or, any modular form of finite level must have non-negative weights. This does not contradict Proposition 4.1. The level there is *infinite*. The following theorem says, heuristically, that the h_i's are the prototype of modular forms that can not be lifted. The geometric explanation for this phenomenon is that the line bundle $\Omega(\chi)$, for χ not a multiple of Norm, does not extend to a line bundle over the minimal compactification, though it does extend to a line bundle over any smooth toroidal compactification.

THEOREM 4.2. *Let B be any $W(\mathbb{F})$-algebra and let $B_m = B \otimes W_m(\mathbb{F})$. Let I_m be the kernel of the q-expansion map as in Theorem* 3.8. *The map*

$$\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(B, \chi, \mu_N) \to \bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(B_1, \chi, \mu_N)/I_1 \tag{4.1}$$

is surjective. The map

$$\bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(B, \chi, \mu_N)^{\text{cusp}} \to \bigoplus_{\chi \in \mathbf{X}} \mathbf{M}(B_m, \chi, \mu_N)^{\text{cusp}}/I_m \tag{4.2}$$

is surjective.

Proof. The proof uses the following lemma:

LEMMA 4.3 ([15], Proposition 6.11). *If $f \in \mathbf{M}(B_1, \chi, \mu_N)$ has some q-expansion in which the constant term is non-zero then $\chi \in \mathbf{X}(1)$.*

Thus, if f is not a cusp form then for a suitable $g \in I_1$ we have that $f+g$ is of weight Norm^k for some $k>0$, which we may take as large as needed.

Let us put $T^S = \mathcal{M}^*(W(\mathbb{F}), \mu_N)$—the moduli space of HBAS over $W(\mathbb{F})$-algebras with μ_N-level with its Satake compactification. Recall the notation (2.12). It is well know that $\Omega(\text{Norm})$ extends to T^S and that $\Omega(\text{Norm})$ is an ample line bundle (our level is rigid). It follows that for k large enough every section of $\Omega(\text{Norm}^k)$ can be lifted. We may therefore restrict our attention to cusp forms.

Let $D \hookrightarrow T^S$ be the cusps and $T^0 = T^S - D$. Let T^{tor} be a smooth toroidal compactification. We have a commutative diagram

$$\begin{tikzcd} T^0 \arrow[r] \arrow[dr] & T^{\text{tor}} \arrow[d, "b"] \\ & T^S. \end{tikzcd} \tag{4.3}$$

The map b is proper and the other two maps are open immersions. Let D^{tor} be the pre-image of D.

LEMMA 4.4. *There exists a quasi-coherent sheaf $\mathscr{S}(\chi)$ on T^S whose global sections are cusp forms of weight χ.*

Theorem 4.2 follows immediately from Lemma 4.4. For k large enough all the higher cohomology of $\mathscr{S}(\chi)\otimes\Omega(\mathrm{Norm}^k)$ vanishes and there are thus no obstructions to lifting. It remains to prove the lemma:

There exists a semi-abelian variety with real multiplication

$$(\mathscr{A}, \beta_N)\xrightarrow{\pi} T^{\mathrm{tor}}. \tag{4.4}$$

Let $\Omega = \mathrm{t}^*_{(\mathscr{A},\beta_N)\to T^{\mathrm{tor}}}$ and define $\Omega(\chi)$ as usual (on T^0 this agrees with our previous definition). Let $\mathscr{I}$ be the ideal sheaf defining D^{tor}. Let

$$\mathscr{S}(\chi)=\pi_*(\Omega(\chi)\otimes\mathscr{I}). \tag{4.5}$$

The sheaf $\mathscr{S}(\chi)$ is quasi-coherent sheaf on T^S. We need only show that its global sections are cusp forms. The map from $\Gamma(T^S, S(\chi))=\Gamma(T^{\mathrm{tor}}, \Omega(\chi)\otimes\mathscr{I})$ to $\Gamma(T^0, \Omega(\chi))\subset\mathbf{M}(W(\mathbb{F}), \chi, \mu_N)$, given by restriction, is clearly injective. It has image contained in the cusp forms. Indeed, if $f\in\Gamma(T^S, S(\chi))$ and $\tilde{f}$ its image, then the q-expansion of $\tilde{f}$ is none-other then f viewed as an element of the structure sheaf of the completion of T^{tor} along $\mathscr{I}$. For this one needs to choose a particular trivialization of $\Omega(\chi)$ in a neighborhood of the component of D^{tor} under consideration. See [3], Main Theorem.

Conversely, a cusp form $\tilde{f}$, viewed as a section of $\Gamma(T^0, \Omega(\chi))$, or $\Gamma(T^0, S(\chi))$ extends to an a priori meromorphic section f of $\Gamma(T^S, S(\chi))$, whose expression as an element of the structure sheaf of the completion of T^{tor} along $\mathscr{I}$ has zero constant coefficient. That just means that locally around D^{tor} it belongs to $\mathscr{I}$. See loc. cit. (x). ∎

Remark 4.5. The point of Theorem 4.1 is that it says that every HMF modulo p, say f, can lifted to characteristic zero, in the sense that its *q-expansion* can be lifted. I.e., though often one can not lift the modular form f itself, there *does* exist a modular form g of characteristic zero and weight equal to the weight of f modulo $\mathbf{X}(1)$, whose q-expansion is equal to the q-expansion of f modulo p.

Practically the same proof gives the following:

Let f be a modular form over $W_m(\mathbb{F})$ whose constant coefficient in one q-expansion is a unit. Then f has weight in $\mathbf{X}(m)$ and its q-expansion lifts to a q-expansion of a HMF over $W(\mathbb{F})$ of the same level and weight in $\mathbf{X}(m)$. A similar statement holds for cusp forms.

In fact the method of the proof allows one to control the difference between the weights of f and the "lift" if one has an effective bound on k such that $H^1(T^S, \mathscr{S}(\chi) \otimes \Omega(\mathrm{Norm}^k)) = 0$.

5. TABULATION OF SOME ZETA VALUES

Remark 5.1. The computations were done using PARI and are subject to the following reservations: (i) My lack of expertise in such calculations. (ii) The validity of a factor being a prime. In particular, almost surely, those huge numerators which are not decomposed at all are composite. (iii) However, the factorization of the denominator is always into primes.

We explain how the data was obtained by giving an example. To obtain $\zeta_{\mathbb{Q}(\sqrt{7})}(-31)$ first raise the real precision of PARI by writing "\p 150." Execute the command "f = zetakinit(x^2 − 7);" (that creates the data that PARI needs in order to calculate values of the zeta function of $\mathbb{Q}(\sqrt{7})$). Writing "x = zetak(f, -31)" gives the real number

$$
\begin{aligned}
x = {} & 85915187317986217088414870447749176723 \\
& 5740853295481011573359732.500490196078 \\
& 4313725490196078431372549019607843137 2 \\
& 5490196078431372549019607843137254999.
\end{aligned}
$$

Note that Corollary 3.43 gives a bound on the denominator of the rational number approximated by x. Thus, one knows that $y = x \times 32!$ must be an integer. Writing "y = x*32!" we get

$$
\begin{aligned}
y = {} & 22606935144296765680860441138044034718 \\
& 24035695198359578560196240784639781684 \\
& 8315501563504230400000.00000000000000 \\
& 0000000000000000000000000000000000025.
\end{aligned}
$$

The command, "factor(round(y)/32!)" yields the value given in the table below.

Field: $L = \mathbb{Q}$.

k	$\zeta_{\mathbb{Q}}(1-k)$		k	$\zeta_{\mathbb{Q}}(1-k)$
2	$\frac{-1}{2^2\cdot 3}$		20	$\frac{283\cdot 617}{2^3\cdot 3\cdot 5^2\cdot 11}$
4	$\frac{1}{2^3\cdot 3\cdot 5}$		22	$\frac{-131\cdot 593}{2^2\cdot 3\cdot 23}$
6	$\frac{-1}{2^2\cdot 3^2\cdot 7}$		24	$\frac{103\cdot 2294797}{2^4\cdot 3^2\cdot 5\cdot 7\cdot 13}$
8	$\frac{1}{2^4\cdot 3\cdot 5}$		26	$\frac{-657931}{2^2\cdot 3}$
10	$\frac{-1}{2^2\cdot 3\cdot 11}$		28	$\frac{9349\cdot 362903}{2^3\cdot 3\cdot 5\cdot 29}$
12	$\frac{691}{2^3\cdot 3^2\cdot 5\cdot 7\cdot 13}$		30	$\frac{-1721\cdot 1001259881}{2^2\cdot 3^2\cdot 7\cdot 11\cdot 31}$
14	$\frac{-1}{2^2\cdot 3}$		32	$\frac{37\cdot 683\cdot 305065927}{2^6\cdot 3\cdot 5\cdot 17}$
16	$\frac{3617}{2^5\cdot 3\cdot 5\cdot 17}$		34	$\frac{-151628697551}{2^2\cdot 3}$
18	$\frac{-43867}{2^2\cdot 3^3\cdot 7\cdot 19}$		36	$\frac{26315271553053477373}{2^3\cdot 3^3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37}$

Field: $L = \mathbb{Q}(\sqrt{2})$.
Ideals: Ramified: 2; Split: $7, 17, 23, 31$; Inert: $3, 5, 11, 13, 19, 29$.

k	$\zeta_L(1-k)$		k	$\zeta_L(1-k)$
2	$\frac{1}{2^2\cdot 3}$		20	$\frac{283\cdot 617\cdot 211202599\cdot 51060226589}{2^3\cdot 3\cdot 5^2\cdot 11}$
4	$\frac{11}{2^3\cdot 3\cdot 5}$		22	$\frac{131\cdot 593\cdot 169471\cdot 1358111\cdot 31902217001}{2^2\cdot 3\cdot 23}$
6	$\frac{19^2}{2^2\cdot 3^2\cdot 7}$		24	$\frac{11\cdot 19\cdot 103\cdot 977\cdot 3343\cdot 2294797\cdot 678737272814753}{2^4\cdot 3^2\cdot 5\cdot 7}$
8	$\frac{24611}{2^4\cdot 3\cdot 5}$		26	$\frac{657931\cdot 39944352181\cdot 146669017694031181}{2^2\cdot 3}$
10	$\frac{2873041}{2^2\cdot 3\cdot 11}$		28	$\frac{9349\cdot 362903\cdot 474581\cdot 14048849748204034731603631}{2^3\cdot 3\cdot 5\cdot 29}$
12	$\frac{13\cdot 691\cdot 3031619}{2^3\cdot 3^2\cdot 5\cdot 7}$		30	$\frac{79\cdot 1721\cdot 1190311\cdot 1001259881\cdot 3010773946258042928744719}{2^2\cdot 3^2\cdot 7\cdot 11}$
14	$\frac{11\cdot 151\cdot 78007661}{2^2\cdot 3}$		32	$\frac{37\cdot 89\cdot 683\cdot 39217\cdot 111392753\cdot 305065927\cdot 34033706948594999426699}{2^6\cdot 3\cdot 5\cdot 17}$
16	$\frac{79\cdot 3617\cdot 558366571709}{2^5\cdot 3\cdot 5\cdot 17}$		34	$\frac{11\cdot 37\cdot 59\cdot 151628697551\cdot 943340112506873639105567440995835717}{2^2\cdot 3}$
18	$\frac{43867\cdot 19450718635716001}{2^2\cdot 3^3\cdot 7\cdot 19}$		36	$\frac{59\cdot 14437\cdot 16631\cdot 3657637\cdot 26315271553053477373\cdot 64876981486621133416770347}{2^3\cdot 3^3\cdot 5\cdot 7\cdot 19\cdot 37}$

Field: $L = \mathbb{Q}(\sqrt{5})$.
Ideals: Ramified: 5; Split: $11, 19, 29, 31$; Inert: $2, 3, 5, 7, 13, 17, 23$.

k	$\zeta_L(1-k)$		k	$\zeta_L(1-k)$
2	$\frac{1}{2\cdot 3\cdot 5}$		20	$\frac{283\cdot 617\cdot 564172514549641}{2^2\cdot 3\cdot 5^2\cdot 11}$
4	$\frac{1}{2^2\cdot 3\cdot 5}$		22	$\frac{107\cdot 131\cdot 149\cdot 593\cdot 47058898298437}{2\cdot 3\cdot 5\cdot 23}$
6	$\frac{67}{2\cdot 3^2\cdot 5\cdot 7}$		24	$\frac{103\cdot 1093\cdot 1214221\cdot 2294797\cdot 36228867817}{2^3\cdot 3^2\cdot 5\cdot 7\cdot 13}$
8	$\frac{19^2}{2^3\cdot 3\cdot 5}$		26	$\frac{19\cdot 5839\cdot 657931\cdot 823345533268358047}{2\cdot 3\cdot 5}$
10	$\frac{191\cdot 2161}{2\cdot 3\cdot 5^2\cdot 11}$		28	$\frac{2969\cdot 9349\cdot 362903\cdot 2735340507483319678769}{2^2\cdot 3\cdot 5\cdot 29}$
12	$\frac{691\cdot 1150921}{2^2\cdot 3^2\cdot 5\cdot 7\cdot 13}$		30	$\frac{17\cdot 1721\cdot 13815257\cdot 33847091\cdot 1001259881\cdot 13133142812173}{2\cdot 3^2\cdot 5^2\cdot 7\cdot 11\cdot 31}$
14	$\frac{17\cdot 33446579}{2\cdot 3\cdot 5}$		32	$\frac{37\cdot 131\cdot 683\cdot 305065927\cdot 3389247557\cdot 5539193421920211463}{2^5\cdot 3\cdot 5\cdot 17}$
16	$\frac{457\cdot 3617\cdot 33092833}{2^4\cdot 3\cdot 5\cdot 17}$		34	$\frac{347\cdot 661\cdot 3359\cdot 271805903\cdot 151628697551\cdot 39267702302944517}{2\cdot 3\cdot 5}$
18	$\frac{41\cdot 43867\cdot 317680421579}{2\cdot 3^3\cdot 5\cdot 7\cdot 19}$		36	$\frac{557\cdot 1601\cdot 531342581699\cdot 2615600385513088367\cdot 26315271553053477373}{2^2\cdot 3^3\cdot 5\cdot 7\cdot 13\cdot 19\cdot 37}$

Field: $L = \mathbb{Q}(\sqrt{7})$.
Ideals: Ramified: $2, 7$; Split: $3, 19, 29, 31$; Inert: $5, 11, 13, 17, 23$.

k	$\zeta_L(1-k)$
2	$\frac{2}{3}$
4	$\frac{113}{3\cdot 5}$
6	$\frac{2\cdot 173\cdot 257}{3^2\cdot 7}$
8	$\frac{37040933}{2\cdot 3\cdot 5}$
10	$\frac{2\cdot 13\cdot 4073517757}{3\cdot 11}$
12	$\frac{691\cdot 1355989\cdot 85309877}{3^2\cdot 5\cdot 7\cdot 13}$
14	$\frac{2\cdot 23\cdot 31126933\cdot 500577719}{3}$
16	$\frac{3617\cdot 1494552660374041255373}{2^2\cdot 3\cdot 5\cdot 17}$
18	$\frac{2\cdot 29\cdot 21529\cdot 23801\cdot 43867\cdot 543274577837461}{3^3\cdot 7\cdot 19}$
20	$\frac{283\cdot 617\cdot 12391\cdot 4424992276888190779824023}{3\cdot 5^2\cdot 11}$
22	$\frac{2\cdot 13^2\cdot 131\cdot 593\cdot 773\cdot 829\cdot 6449\cdot 654804091271409612853}{3\cdot 23}$
24	$\frac{73\cdot 103\cdot 2294797\cdot 62951846444926898226001136261791181}{2\cdot 3^2\cdot 5\cdot 7\cdot 13}$
26	$\frac{2\cdot 2857\cdot 657931\cdot 1468739855213\cdot 13049197046499760097510021}{3}$
28	$\frac{439\cdot 9349\cdot 65993\cdot 362903\cdot 26349352999952723053544872812364085479}{3\cdot 5\cdot 29}$
30	$\frac{2\cdot 1721\cdot 1001259881\cdot 11476721593\cdot 1072572990203117012275682682777367140637}{3^2\cdot 7\cdot 11\cdot 31}$
32	$\frac{37\cdot 683\cdot 92413\cdot 305065927\cdot 1566074201\cdot 543869790242280163\cdot 2888308889549447149987}{2^3\cdot 3\cdot 5\cdot 17}$
34	$\frac{2\cdot 13\cdot 151628697551\cdot 205042093447897\cdot 1788619008252178278652191975327334133823661}{3}$

Field: $L = \mathbb{Q}(\zeta_7)^+ = \mathbb{Q}[x]/(x^3 + x^2 - 2x - 1)$
Ideals: Ramified: 7; Split: 13, 29; Inert: 2, 3, 5, 11, 17, 19, 23, 31.

k	$\zeta_L(1-k)$
2	$\frac{-1}{3\cdot 7}$
4	$\frac{79}{2\cdot 3\cdot 5\cdot 7}$
6	$\frac{-7393}{3^2\cdot 7}$
8	$\frac{142490119}{2^2\cdot 3\cdot 5\cdot 7}$
10	$\frac{-1141452324871}{3\cdot 7\cdot 11}$
12	$\frac{691\cdot 10903\cdot 278995143079}{2\cdot 3^2\cdot 5\cdot 7\cdot 13}$
14	$\frac{-1033\cdot 5410539334962035689}{3\cdot 7^2}$
16	$\frac{3617\cdot 19387\cdot 6997171\cdot 399890401961287}{2^3\cdot 3\cdot 5\cdot 7\cdot 17}$
18	$\frac{-97\cdot 43867\cdot 9105835027474306843301627809}{3^3\cdot 7\cdot 19}$
20	$\frac{283\cdot 617\cdot 21766351\cdot 5118351012301487009695100128 9}{2\cdot 3\cdot 5^2\cdot 7\cdot 11}$
22	$\frac{-131\cdot 593\cdot 751\cdot 1657\cdot 95131\cdot 2557424168676190300514101539043}{3\cdot 7\cdot 23}$
24	$\frac{103\cdot 2294797\cdot 400092417143059\cdot 4831713226649233\cdot 8824732711929451}{2^2\cdot 3^2\cdot 5\cdot 7\cdot 13}$
26	$\frac{-29527\cdot 657931\cdot 3306506726170474827689 89\cdot 67834018071999401112773 17}{3\cdot 7}$
28	$\frac{9349\cdot 82471\cdot 362903\cdot 743035325831593\cdot 975522475034052090 7\cdot 5887531329453854793 73}{2\cdot 3\cdot 5\cdot 7^2\cdot 29}$
30	$\frac{-1721\cdot 3373\cdot 1001259881\cdot 118925035288906 09\cdot 18187804159430514026455875465707583598047742 9}{3^2\cdot 7\cdot 11\cdot 31}$
32	$\frac{37\cdot 683\cdot 24847\cdot 38575843\cdot 125089171\cdot 305065927\cdot 1270758367\cdot 14038769171773584013\cdot 31813057640664448263019}{2^4\cdot 3\cdot 5\cdot 7\cdot 17}$
34	$\frac{-103\cdot 367\cdot 151628697551\cdot 30924576265171010083636208263230553718869153969291311796245209292857661 91}{3\cdot 7}$

Field: $L = \mathbb{Q}(\zeta_{11})^+ = \mathbb{Q}[x]/(x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1)$
Ideals: Ramified: 11; Split: 23; Inert: 2, 3, 5, 7, 13, 17, 19, 29, 31.

k	$\zeta_L(1-k)$
2	$\frac{-2^2\cdot 5}{3\cdot 11}$
4	$\frac{2\cdot 71\cdot 11941}{3\cdot 5\cdot 11}$
6	$\frac{-2^2\cdot 5\cdot 521\cdot 4888380551}{3^2\cdot 7\cdot 11}$
8	$\frac{13721\cdot 2520121\cdot 102462575851}{3\cdot 5\cdot 11}$
10	$\frac{-2^2\cdot 5\cdot 98178488021\cdot 1560850707193521481}{3\cdot 11}$
12	$\frac{2\cdot 691\cdot 1607981\cdot 6134561\cdot 29139491\cdot 3791335077949195217 41}{3^2\cdot 5\cdot 7\cdot 11\cdot 13}$
14	$\frac{-2^2\cdot 5\cdot 31\cdot 71\cdot 109841\cdot 4712650115236500312066042412229825266552711}{3\cdot 11}$
16	$\frac{31\cdot 3617\cdot 18131\cdot 42641\cdot 2466915721\cdot 1653690578739888729472018694801115596823523 1}{2\cdot 3\cdot 5\cdot 11\cdot 17}$
18	$\frac{-2^2\cdot 5\cdot 43867\cdot 113011\cdot 835818164077607527662719035981440776856878764991606492392923228841381}{3^3\cdot 7\cdot 11\cdot 19}$
20	$\frac{2\cdot 131\cdot 283\cdot 617\cdot 821\cdot 481951783190606372931457121941057256238988336323490351990340248253504373198746671}{3\cdot 5^2\cdot 11}$
22	$\frac{-2^2\cdot 5\cdot 31\cdot 131\cdot 593\cdot 2111\cdot 9811\cdot 4754681\cdot 150743667211\cdot 7485309344691968588719378106517487509425242700571390702015324593161626701}{3\cdot 11^2\cdot 23}$
24	$\frac{2140902620412862787531604307645176100404841968426079695283278792036337157813694947611346966909345714822325560273519654201}{3^2\cdot 5\cdot 7\cdot 11\cdot 13}$
26	$\frac{-272623514370642870558372772196000774324744469363909470795760720136333301617973028388628650301184844782829144056234056572114064826020}{3\cdot 11}$

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