Hilbert modular varieties of low dimension

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Abstract. We study in detail properties of Hilbert modular varieties of low dimension in positive characteristic p; in particular, the local and global properties of certain stratifications. To carry out this investigation we develop some new tools in the theory of displays, intersection theory on a singular surface and Hecke correspondences at p.

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Contents

1. Introduction

This paper studies Hilbert modular varieties of low dimension. Besides the interesting geometric problems it raises, we also feel that such a detailed study is bound to play a valuable role in future applications to number theory. For example, the Hilbert modular varieties of dimension one are the modular curves that have been studied extensively, and their geometric properties are intimately connected with the theory of modular forms. We consider here mainly the case of dimension 2 and 3.

To carry out this study we had to develop further existing tools and these results are of independent interest. One is intersection theory on a surface with isolated normal singularities, developed in § 7; the other is methods to calculate the universal display of a PEL problem. Regarding the latter, some of the details will appear, under a much more general setting, in a future work [AG4].

Let L be a totally real field of degree g over $\mathbb{Q}$, let $\mathcal{O}_L$ be its ring of integers, let p be a rational prime and let $\mathfrak{M}$ be the moduli space parameterizing abelian varieties of dimension g, in characteristic p, endowed with an action of $\mathcal{O}_L$. Some further conditions are imposed - see § 2. The properties of $\mathfrak{M}$ that we study are mostly defined using the Frobenius morphism on various objects that are $\mathcal{O}_L \otimes \mathbb{F}_p$-modules. For example, the Hodge bundle $\mathbb{E}$ and the cohomology group $H^1(A, \mathcal{O}_A)$ of an abelian variety A. Hence, the analysis is divided according to the prime decomposition of p in $\mathcal{O}_L$. In § 3 we recall the stratifications defined in [AG1, GO] and their main properties.

In § 4 we discuss the singularities of Hilbert modular varieties. We recall the theory of local models, introduced by Deligne-Pappas [DP], de Jong [deJ] and Rapoport-Zink [RZ], and illustrate the results for the Hilbert and Siegel moduli varieties. The singularities in the Hilbert case are local complete intersections.

Given a closed point $x \in \mathfrak{M}^{\mathrm{sing}}$ we determine when is $\widehat{\mathcal{O}}_{\mathfrak{M},x}$ parafactorial. A question of interest here is when the pair $(\mathfrak{M}, \mathfrak{M}^{\mathrm{sing}})$ is parafactorial. This is motivated by the question of whether certain automorphic line bundles, giving rise to Hilbert modular forms, initially defined on the non-singular locus in $\mathfrak{M}$, actually extend to $\mathfrak{M}$. We show that $(\mathfrak{M}, \mathfrak{M}^{\mathrm{sing}})$ is not parafactorial in the presence of ramification. See Theorem 4.4.3, and its corollaries, for applications as indicated.

Section 5 discusses the display of an abelian variety with real multiplication. After some preparatory work, we provide two main theorems. The first, Theorem 5.6.1, gives the universal display with real multiplication. It uses Theorem 5.6.2 that provides a criterion for a display to be universal. Both theorems can be generalized considerably, i.e., to the setting of PEL problems, (hopefully) even with level involving p. Details will appear in [AG4]. The results are applied in the sequel to study the local properties of the strata. See, for instance, § 8.3.1 and § 9.

In § 6 we provide some general results concerning our stratification in the maximally ramified case. This continues our investigation in [AG1]. Some of our results are the following. In § 6.1 we show that each stratum $W_{(j,n)}$ of $\mathfrak{M}$ is quasi–affine and we describe the foliation structure, as defined by Oort [Oo4], on the Newton polygon stratification of $\mathfrak{M}$. In § 6.2 we show that certain of the strata $\mathbb{T}_a$, i.e., where the a-number is greater or equal to a, are connected. In § 6.3 we show (a striking result) that the non-ordinary locus is irreducible for $g \geq 3$.

Section 7 develops intersection theory on a complete surface with isolated normal singularities, building on [RT1, RT2]. Our approach is very concrete and suitable for the calculations we need to perform. This approach can be developed further [Arc]. One of the applications we give is determining in Theorem 8.1.1, for p inert, which automorphic line bundles (yielding Hilbert modular forms of, usually, non-parallel weight) are ample.

Finally, in § 9 we study in some detail Hilbert modular threefolds in the maximally ramified case.

2. Definitions and notations

Let L be a totally real field of degree g over $\mathbb{Q}$ with ring of integers $\mathcal{O}_L$. Let D_L be its different ideal and d_L its discriminant. Let p be a rational prime and $\mathfrak{p}$ a prime of $\mathcal{O}_L$ dividing p. We let $\mathbb{F}_\mathfrak{p}$ denote the residue field $\mathcal{O}_L/\mathfrak{p}$. Let $\mathfrak{a}_1, \dots, \mathfrak{a}_{h^+}$ be ideals of L forming a complete set of representatives for the strict class group $\mathrm{cl}^+(L)$ of L. By an *abelian variety with RM* we shall mean a triple $(A \to S, \iota, \lambda)$ consisting of an abelian scheme A of relative dimension g over a scheme S; an embedding of rings $\iota : \mathcal{O}_L \hookrightarrow \mathrm{End}_S(A)$; an isomorphism of $\mathcal{O}_L$-modules with a notion of positivity $\lambda \colon \mathfrak{a}_i \to M_A := \mathrm{Hom}_{\mathcal{O}_L}(A, A^t)^{\mathrm{symm}}$, where A^t is the dual abelian variety (for some, necessarily unique, i). One imposes the condition $A \otimes_{\mathcal{O}_L} \mathfrak{a}_i \cong A^t$. By a *$\mu_N$-level structure* we mean an embedding of $\mathcal{O}_L$-S-group schemes $\mu_N \otimes_\mathbb{Z} \mathcal{O}_L \to A$. Let $\mathbb{F}$ be the composite of the fields $\mathbb{F}_\mathfrak{p}$ for every $\mathfrak{p}$ dividing p. The moduli problem of abelian varieties with RM over $\mathbb{F}$–schemes and μ_N-level structure is a rigid moduli problem for $N \geq 4$. We let $\mathfrak{M}$ be the moduli space of abelian varieties with RM defined over schemes and level $N \geq 4$ (N prime to p); we let $\mathfrak{N}$ be the moduli space obtained by taking an additional level structure consisting of a connected $\mathcal{O}_L$-group scheme of order p. We refer to [AG1, AG2, DP] for details. Note that our $\mathfrak{N}$ is slightly different from the one appearing in these references (in that we assume the subgroup to be connected). See § 5.1.1 for the definition of a p-divisible group with RM.

The following notation is used: $\mathbb{F}_q$ denotes a field with q elements; $\mathbb{Z}, \mathbb{Q}, \mathbb{Z}_p, \mathbb{Q}_p$ denote the integers, rationals, p-adic integers and p-adic numbers; $\mathbb{W}(k)$ denotes the ring of infinite Witt vectors, with respect to a prime p, over a ring k, and $\mathbb{W}_t(k)$ the truncated Witt vectors $(a_0, \dots, a_{t-1})$. If $C \subset k$ is any subset, we let $\mathbb{W}(C)$ (resp. $\mathbb{W}_t(C)$) denote the vectors in $\mathbb{W}(k)$ (resp. $\mathbb{W}_t(k)$) all whose coordinates belong to C. We denote by ${}^F w$, ${}^V w$ the Frobenius and Verschiebung maps on $\mathbb{W}(k)$, cf. [Zin, pp. 127-8]. For a Dedekind ring R and a prime ideal $\mathfrak{p}$, we let $f_\mathfrak{p} = \dim_{\mathbb{F}_p}(R/\mathfrak{p})$. In the case of $\mathcal{O}_L$, we also let $e_\mathfrak{p}$ be the absolute ramification index of $\mathfrak{p}$ and we define $g_\mathfrak{p} = e_\mathfrak{p} f_\mathfrak{p}$.

For a prime $\mathfrak{p}|p$ of $\mathcal{O}_L$, we let $\mathcal{O}_{L,\mathfrak{p}}$ be the localization of $\mathcal{O}_L$ at the multiplicative set $\mathcal{O}_L \setminus \mathfrak{p}$, we let $\widehat{\mathcal{O}}_{L,\mathfrak{p}}$ be the completion, $L_\mathfrak{p}$ its field of fractions, and $\widehat{\mathcal{O}}_{L,\mathfrak{p}}^{\mathrm{ur}}$ be the ring of integers of the maximal unramified sub-extension of $L_\mathfrak{p}$ over $\mathbb{Q}_p$.

Let k be a perfect field of characteristic p. A p-divisible group over k is called *ordinary* if all its slopes are zero and one. An abelian variety over k is called *ordinary* if its p-divisible group $A(p)$ is; it is called *supersingular* if the slopes of its Newton polygon are all equal $1/2$, equivalently, if it is isogenous to a product of supersingular elliptic curves [Oo1, Thm. 4.2]; it is called *superspecial* if it is isomorphic over $\bar{k}$ to a product of supersingular elliptic curves, equivalently, if $F : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$ is zero [Oo2, Thm. 2]. We denote by $\mathfrak{C}_k$ the category of local artinian k-algebras $(R, \mathfrak{m})$ equipped with an identification $R/\mathfrak{m} = k$. We denote the closure of a set Z in a topological space by Z^c.

Let $\mathfrak{A}_g$ be the Siegel moduli space of principally polarized abelian varieties of dimension g in characteristic p, § 4.2.1 - often with a rigid prime-to-p level n structure that is not explicitly specified; $\mathfrak{X}^{\mathrm{uni}} \to \mathfrak{A}_g$ (or $\mathfrak{X}^{\mathrm{uni}} \to \mathfrak{M}$) will denote the universal object with section e and $\mathbb{E} = e^*\Omega^1_{\mathfrak{X}^{\mathrm{uni}}/\mathfrak{A}_g}$ (or $\mathbb{E} = e^*\Omega^1_{\mathfrak{X}^{\mathrm{uni}}/\mathfrak{M}}$) denotes the Hodge bundle. It is a locally free sheaf of rank g. We let $\omega = \det \mathbb{E}$.

3. Stratification of Hilbert modular varieties

We shall be concerned primarily with the geometry of the moduli space $\mathfrak{M}$. The moduli space $\mathfrak{N}$ will provide us with a 'Hecke correspondence' at p that we shall utilize to study certain strata in $\mathfrak{M}$. Two particular cases will be considered in detail: when p is unramified and when p is maximally ramified, i.e., decomposes as $(p) = \mathfrak{p}^g$ in $\mathcal{O}_L$.

3.1. p unramified.

In this case $\mathcal{O}_L \otimes \mathbb{F}_p \cong \oplus_{\mathfrak{p}|p}\mathbb{F}_{\mathfrak{p}}$ is a sum of fields. Let A/k be a RM abelian variety over a perfect field $k \supseteq \mathbb{F}_p$. It is known that $H^1(A, \mathcal{O}_A)$ is a free $\mathcal{O}_L \otimes_{\mathbb{Z}} k$ module of rank 1. The kernel of Frobenius $F : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$ is a k-subspace of dimension $a = a(A)$. Let us assume that for every $\mathfrak{p}|p$ an embedding $\mathbb{F}_{\mathfrak{p}} \hookrightarrow k$ is given, thus a decomposition $\mathbb{F}_p \otimes k = \oplus_{\mathfrak{p}|p}k$. The action of $\mathbb{F}_{\mathfrak{p}}$ on every $\mathcal{O}_L$-eigenspace of $H^1(A, \mathcal{O}_A)$ is either trivial or is given by x acting as multiplication by x^{p^i} for some $1 \leq i \leq f_{\mathfrak{p}}$. The structure of the $\mathcal{O}_L \otimes k$-module $\mathrm{Ker}(F : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A))$ is therefore uniquely determined by a vector $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p} = (\tau_{\mathfrak{p}})_{\mathfrak{p}|p}(A)$ of sets, with $\tau_{\mathfrak{p}} \subset \{1, \dots, f_{\mathfrak{p}}\}$. There is a natural partial order, induced from inclusion of sets in each component, on the set of possible vectors $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}$.

Given any vector $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}$, where each $\tau_{\mathfrak{p}} \subset \{1, \dots, f_{\mathfrak{p}}\}$, we can define a closed subset $D_{(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}}$ of $\mathfrak{M}$ by the property that for each geometric point $x \in D_{(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}}$ we have $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}(A_x) \geq (\tau_{\mathfrak{p}})_{\mathfrak{p}|p}$. This is a regular subvariety of codimension $\sum_{\mathfrak{p}|p} |\tau_{\mathfrak{p}}|$. For further properties see [Go1, GO].

Consider sets S of the form $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}$ with all $\tau_{\mathfrak{p}} = \emptyset$, except for a single $\mathfrak{p}$ for which $\tau_{\mathfrak{p}}$ is a singleton. For each such set S one can define a Hilbert modular form h_S whose divisor is D_S. Each stratum $D_{(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}}$ is the transversal intersection of the divisors D_S for S as above satisfying $S \leq (\tau_{\mathfrak{p}})_{\mathfrak{p}|p}$. Furthermore, with respect to a suitable cusp, the kernel of the q-expansion map is given by the ideal $(h_S - 1 : S$ a set as above). See [Go2, Thm. 2].

Example 3.1.1. *For $g = 1$ (so $L = \mathbb{Q}$) the vector $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}(A)$ has a single component and there are only two possibilities. Either $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}(A) = (\emptyset)$, which corresponds to A being an ordinary elliptic curve, or $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}(A) = (\{1\})$, which corresponds to A being supersingular. The locus $D_{(\emptyset)}$ is the whole moduli space (of codimension 0), and the locus $D_{(\{1\})}$ is the supersingular locus (of codimension 1).*

Example 3.1.2. *For $g = 2$ (L is a real quadratic field) we have two cases:*

- p is inert in L. *In this case the possibilities for $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}(A)$ are the vectors of sets $(\emptyset), (\{1\}), (\{2\}), (\{1, 2\})$. The case $(\emptyset)$ corresponds to ordinary abelian surfaces, the cases $(\{1\}), (\{2\})$ to supersingular, but not superspecial abelian surfaces, and the case $(\{1, 2\})$ to superspecial abelian surfaces. The variety $D_{(\emptyset)}$ is the whole moduli space, the varieties $D_1 = D_{(\{1\})}, D_2 = D_{(\{2\})}$ are (usually reducible) divisors, and $D_{(\{1,2\})} = D_1 \cdot D_2$ is the finite set of superspecial points. We also know that each D_i is a disjoint union of non-singular rational curves and that D_1 and D_2 intersect transversely. See Figure 3.1. See [BG] for details.*

- p is split in L. *In this case the possibilities for $(\tau_{\mathfrak{p}})_{\mathfrak{p}|p}(A)$ are $(\emptyset, \emptyset)$, $(\emptyset, \{1\})$, $(\{1\}, \emptyset)$, and $(\{1\}, \{1\})$. The case $(\emptyset, \emptyset)$ corresponds to ordinary abelian surfaces, the cases $(\emptyset, \{1\})$ and $(\{1\}, \emptyset)$ to non-ordinary (but not supersingular) abelian surfaces (they are in fact simple abelian surfaces), and the case $(\{1\}, \{1\})$ to superspecial abelian surfaces.*

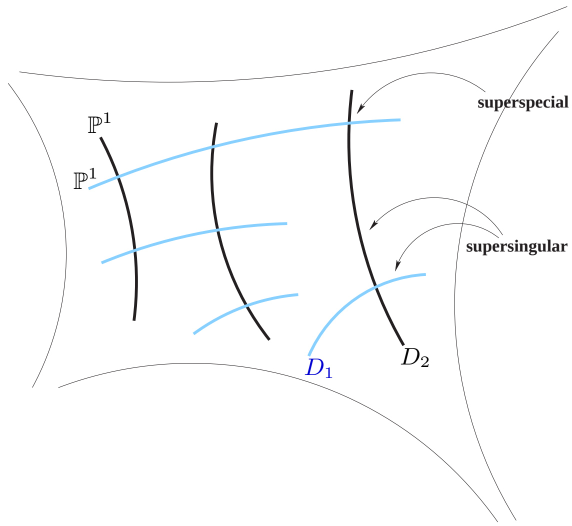


Figure 3.1: Hilbert modular surface - inert case.

In this case, the divisors $D_1 = D_{(\emptyset,\{1\})}$, $D_2 = D_{(\{1\},\emptyset)}$ *are also each a disjoint union of non-singular curves but, in contrast with the situation of inert prime, we have no real information on these curves: they are not the reduction of Shimura curves, we do not know their genera. We do know, however, that* D_1 *and* D_2 *intersect transversely and that* $D_1 . D_2 = D_{(\{1\},\{1\})}$ *is precisely the set of superspecial points, and in* § 8.2 *we provide an argument that suggests that the components of the* D_i *have usually genus* 2.

3.2. p maximally ramified.

Let $k \supseteq \mathbb{F}_\mathfrak{p}$ be a field. In this case $\mathcal{O}_L \otimes k \cong k[T]/(T^g)$, where T may be chosen to be an Eisenstein element of the discrete valuation ring $\mathcal{O}_L \otimes \mathbb{Z}_p$. It is known that $H^1_{\rm dR}(A/k)$ is a free $k[T]/(T^g)$-module of rank 2 [Rap, Lem. 1.3]. We have a sequence of $k[T]/(T^g)$ modules

$$0 \longrightarrow H^0(A, \Omega^1_{A/k}) \longrightarrow H^1_{\rm dR}(A/k) \longrightarrow H^1(A, \mathcal{O}_A) \longrightarrow 0.$$

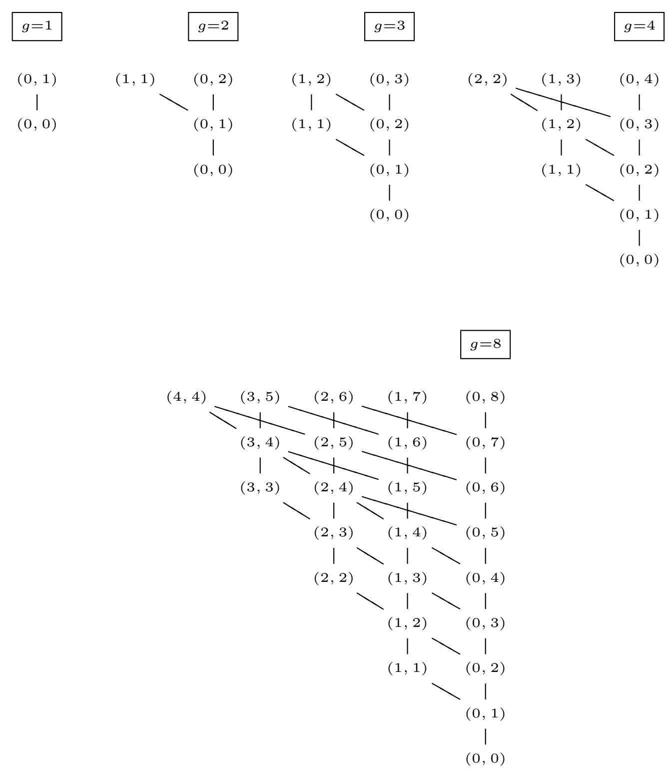
We let $i = i(A), j = j(A)$ be the elementary divisors of $H^0(A, \Omega^1_{A/k})$, normalized so that $j \leq i$. Note that $j = g - i$. Thus, there is a $k[T]/(T^g)$-basis α, β to $H^1_{\rm dR}(A/k)$ such that

$$H^0(A, \Omega^1_{A/k}) \cong (T^i)\alpha \oplus (T^j)\beta.$$

An easy calculation shows that $a(A) \geq 2j$ and we let $n := n(A) = a(A) - j(A)$. Then $j \leq n \leq g - j$.

We let $J = \{(j,n) \in \mathbb{Z}^2 : 0 \leq j \leq n \leq g - j\}$. For every $(j,n) \in J$ one proves [AG1, §5] that there is a locally closed subvariety $W_{(j,n)}$ of $\mathfrak{M}$, whose geometric points parameterize the abelian varieties A with RM such that $(j(A), n(A)) = (j,n)$. We know [AG1, Thm. 10.1] that $W_{(j,n)}$ is pure dimensional, non-singular of dimension $g-(j+n)$, that the Newton polygon is constant on $W_{(j,n)}$, consisting of two slopes $(n/g, (g-n)/g)$ with equal multiplicities (unless $n \geq g/2$ and then the Newton polygon has one slope equal to $1/2$), and that the collection $\{W_{(j,n)} : (j,n) \in J\}$ is a stratification of $\mathfrak{M}$. The description of the order defined by "being in the closure" is complicated to write, but is easy to describe pictorially. We provide the graphs for $g = 1,2,3,4$ and 8 in Diagram A. The convention is that if a point a is above a point b in the graph, and a is connected to b by a strictly decreasing path, then the strata corresponding to a is in the closure of the strata corresponding to b.

Diagram A:



We know that $W^c_{(1,1)} = \cup_{(j,n), j\geq 1} W_{(j,n)}$ is the singular locus of $\mathfrak{M}$, and, in a sense, j is a measure for severity of the singularities. More precisely, put $S_j := W^c_{(j,j)} = \cup_{(s,t), s\geq j} W_{(s,t)}$, then, by [DP, §4]

$$S_{j+1} = S_j^{\text{sing}}. \tag{3.1}$$

We provide a diagram for the case $g = 2$; See Figure 3.2. The lower part of the

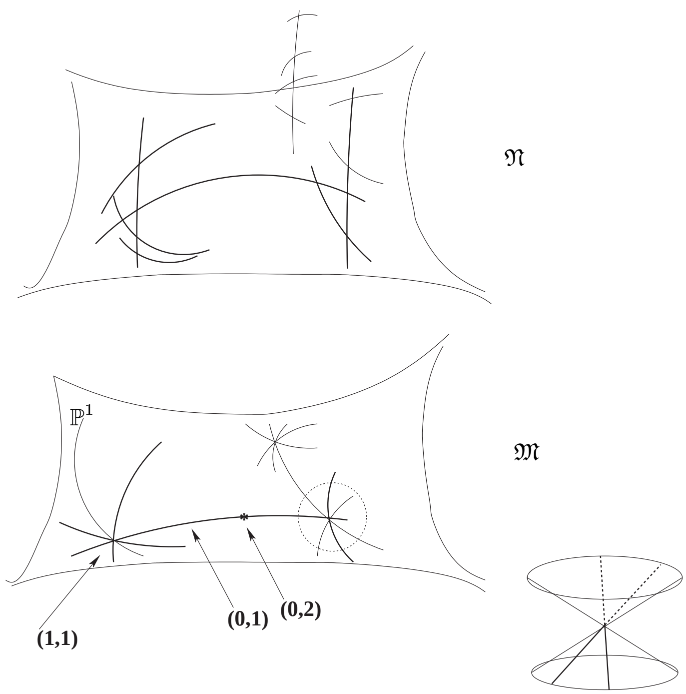
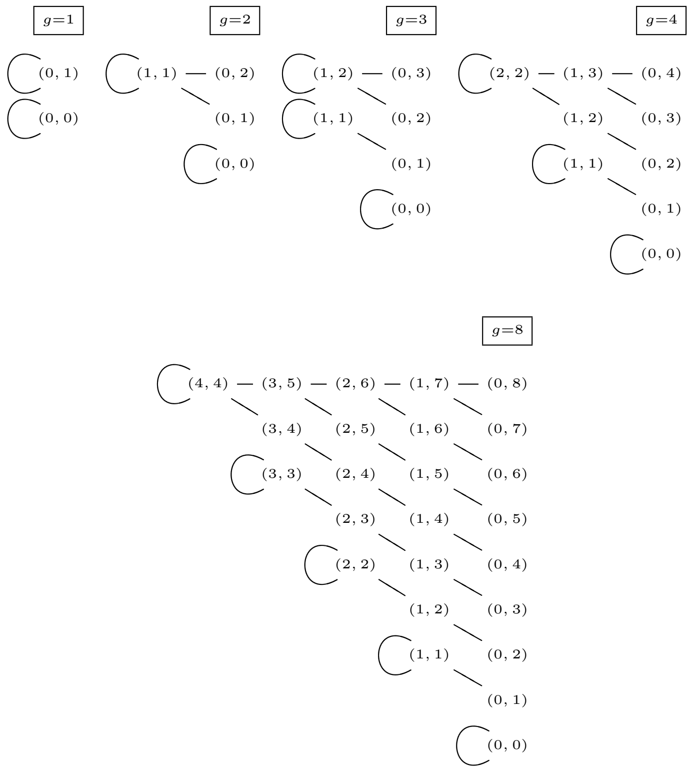


Figure 3.2: Hilbert modular surface - ramified case.

diagram depicts the modular surface $\mathfrak{M}$ with a description of the local structure around a point of type $(1, 1)$. The completion of the local ring is a cone, and the supersingular locus, equal to $W^c_{(0,1)}$, has $p + 1$ branches at such a point.

One of the main tools used in [AG1] is the correspondence defined by the moduli space $\mathfrak{N}$ and its two projections π_1, π_2 to $\mathfrak{M}$. In fact, over $W^c_{(1,1)}$ the morphisms π_i are $\mathbb{P}^1$-bundles. In 3.2 we provide a picture for $g = 2$; in this case $\pi_1, \pi_2 \colon \mathfrak{N} \to \mathfrak{M}$ are blow–ups at the points of type $(1, 1)$ and the $p+1$–branches of the locus $W^c_{(0,1)}$ get separated; cf. Proposition 8.3.1. We can trace the invariants of the image $\pi_2\pi_1^{-1}(x)$ of a point x of type (j, n) under this correspondence. Again, the formal description is cumbersome and we content ourselves with providing Diagram B, referring the reader to [AG1] for more details. The convention is that the invariants along $\pi_2\pi_1^{-1}(x)$ of a point x of type (j, n) are the pairs (j', n'), connected to, and in distance one from the pair (j, n) (whether above or below; a loop is considered distance 1).

Diagram B:



4. Background on the singularities of Hilbert modular varieties

4.1. Cusps.

Let $\mathfrak{X}^{\mathrm{uni}} \to \mathfrak{M}$ be the universal abelian scheme with RM and let $e : \mathfrak{M} \to \mathfrak{X}^{\mathrm{uni}}$ be the identity section. The *Hodge bundle* $\mathbb{E}$ is a locally free sheaf of rank g over $\mathfrak{M}$ defined by $e^*\Omega^1_{\mathfrak{X}^{\mathrm{uni}}/\mathfrak{M}}$. Let $\omega = \det \mathbb{E}$; it is an ample invertible sheaf on $\mathfrak{M}$. This follows from the ampleness of ω on $\mathfrak{A}_g$, cf. [FC, V.2 Thm. 2.3] and from the finiteness of the morphism $\mathfrak{M} \to \mathfrak{A}_g$. The *Satake compactification* $\mathfrak{M}^{\mathcal{S}}$ of $\mathfrak{M}$ is defined as $\mathrm{Proj}(\oplus_{n=0}^{\infty}\Gamma(\mathfrak{M}, \omega^n))$; it is a projective normal variety and $\mathfrak{M}^{\mathcal{S}} \setminus \mathfrak{M}$ is a finite set of points, called cusps. Though ω extends to the Satake compactification, we do not know if the Hodge bundle itself extends.

The set $\mathfrak{M}^R = \mathfrak{M} \setminus \mathfrak{M}^{\mathrm{sing}}$ is the largest open set S over which the Hodge bundle is a locally free $\mathcal{O}_L \otimes \mathcal{O}_S$-module. One has $\mathfrak{M}^R = \mathfrak{M}$ if and only if p is unramified

[DP, Thm. 2.2]. Let k be a big enough finite field so that $\mathcal{O}_L \otimes k$ is a direct sum of local artinian rings with residue field k. Let $I \lhd \mathcal{O}_L \otimes k$ be an ideal and let $I\mathbb{E}$ be the sub-sheaf of $\mathbb{E}$ corresponding to I, defined over $\mathfrak{M}^R \otimes k$. In general $I\mathbb{E}$ does not extend as a locally free sheaf to the cusps. We illustrate the obstruction below for $g = 2$ and p split.

Example 4.1.1. *1. If $p = \mathfrak{p}_1 \cdots \mathfrak{p}_g$ is a product of split primes, then the Hodge bundle is a direct sum $\mathbb{E} = \mathbb{E}_{\mathfrak{p}_1} \oplus \cdots \oplus \mathbb{E}_{\mathfrak{p}_g}$ of line bundles over $\mathfrak{M}$. Since we shall refer to that case later, we introduce the simpler notation $\mathbb{E} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_g$.*

Assume, to fix ideas, that $g = 2$. If $\mathbb{L}_1$, say, extended to the cusp as an invertible sheaf, then so would $\mathbb{L}_1^{p-1} \otimes \omega^{(p-1)n}$ for every n. Recall that we have two Hilbert modular form h_1, h_2 in this situation (the divisor of h_1 being $D_{(\{1\},\emptyset)}$, of h_2 being $D_{(\emptyset,\{1\})}$). The Hilbert modular form $h_1(h_1h_2)^n$ is a section of $\mathbb{L}_1^{p-1} \otimes \omega^{(p-1)n}$ and is not a cusp form. Since the compactification of $\mathfrak{M}$ is normal and the cusps are of codimension 2, $h_1(h_1h_2)^n$ will extend to a section of the extension of $\mathbb{L}_1^{p-1} \otimes \omega^{(p-1)n}$ to the compactification. Usual base-change arguments, using the vanishing of $H^1(\mathfrak{M}^{\mathcal{S}}, \mathbb{L}_1^{p-1} \otimes \omega^{(p-1)n})$ for large enough n, show that the mod p Hilbert modular form $h_1(h_1h_2)^n$ will lift to a Hilbert modular form in characteristic 0, which is not a cusp form and has non-parallel weight $((p-1)(n+1), (p-1)n)$. This is a contradiction, see [Fre, I, Rmk. 4.8].

2. If p is an inert prime in $\mathcal{O}_L$ then $\mathcal{O}_L \otimes \mathbb{F}_{p^g} = \oplus_{i=1}^g \mathbb{F}_{p^g}$, and the Hodge bundle is again a direct sum of line bundles $\mathbb{E} = \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_g$ over $\mathfrak{M}$.

3. If $p = \mathfrak{p}^g$ is maximally ramified, we get a quotient line bundle $\mathbb{L}$ of $\mathbb{E}$ defined over $\mathfrak{M}^R$. We remark that in this case the complement of $\mathfrak{M}^R$ is of codimension 2 in $\mathfrak{M}$ [DP] and it is not a priori clear whether or not $\mathbb{L}$ can be extended to a line bundle on $\mathfrak{M}$. We shall discuss this problem in § 4.4.

4.2. Local models.

Many of the results we stated above require a detailed understanding of the local (infinitesimal) structure of the moduli space $\mathfrak{M}$. Such information may be obtained by the technique of local models. The theory of local models constructs for a moduli space $\mathfrak{B}$ of abelian varieties another scheme $\mathfrak{B}^{\text{loc}}$, typically a flag variety, such that for every geometric point $x \in \mathfrak{B}$, there exists a geometric point $y \in \mathfrak{B}^{\text{loc}}$ and an isomorphism of completed local rings

$$\widehat{\mathcal{O}}_{\mathfrak{B},x} \cong \widehat{\mathcal{O}}_{\mathfrak{B}^{\text{loc}},y}.$$

Below, we shall use the following notation for Grassmann varieties. Let k be an algebraically closed field, B be a k-algebra, and let $a < b$ be positive integers. Assume that a ring homomorphism $B \to M_b(k)$ is given. Assume also that a bilinear alternating pairing $\langle\ ,\ \rangle$ on k^b is given. We shall use $\text{Grass}(a, b)$

(resp. Grass($\langle\ ,\ \rangle, a, b$); resp. Grass($B, \langle\ ,\ \rangle, a, b$)) to denote the Grassmannian of a-dimensional subspaces of k^b (resp. isotropic; resp. isotropic and B-invariant). Often implicit in the notation Grass($B, \langle\ ,\ \rangle, a, b$) is a connection between the pairing and the action of B, e.g., the elements of B are self-adjoint with respect to the pairing.

4.2.1. The idea of local models. Let $\mathfrak{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g in characteristic p. We shall assume that on $\mathfrak{A}_g$, or $\mathfrak{M}$, there is a given rigid prime to p level structure, which we omit from the notation. Given a point $x \in \mathfrak{M}$, or $x \in \mathfrak{A}_g$, one can trivialize the locally free sheaf H^1_{dR} in a Zariski open neighborhood U of x. Then, the locally free, locally direct summand of rank g given by the Hodge bundle $\mathbb{E}$, provides a morphism $U \to \mathrm{Grass}(\langle\ ,\ \rangle, g, 2g)$ (resp. $\mathrm{Grass}(\mathcal{O}_L, \langle\ ,\ \rangle, g, 2g)$), where the Grassmannian is of isotropic g-dimensional (and $\mathcal{O}_L$-invariant) subspaces of a $2g$-dimensional space with a perfect alternating pairing. The idea of local models is to show that this is an isomorphism on the level of completed local rings. There is a shortcoming to this result in that the morphism is not canonical and therefore it is not a priori clear how to define the strata coming from the moduli space on the local model (even in an infinitesimal neighborhood of a point). The crystalline theory makes this morphism somewhat more canonical. But, in fact, the proof that this is an isomorphism on the completed local rings often requires an auxiliary scheme and a dimension count.

Let $f\colon A \to S$ be an abelian scheme and let $\mathbb{D}^*(A)$ be the associated Grothendieck-Messing crystal, defined on the nilpotent crystalline site of S [Gro, §V.4]. This crystal is defined by

$$\mathbb{D}^*(A) = R^1 f_{\mathrm{crys},*}(\mathcal{O}_{A_{\mathrm{crys}}}).$$

The value of this crystal on S is the de Rham sheaf $\mathbb{D}^*(A)_S = R^1 f_*(\Omega^\bullet_{A/S})$, hence it provides us with a locally free direct summand of rank g, $\mathbb{E}_A \subset \mathbb{D}^*(A)_S$, which is $f_*\Omega_{A/S}$. The crucial theorem here is due to Grothendieck [Gro, p. 116].

Theorem 4.2.1. *Let $S \hookrightarrow S'$ be a nilpotent thickening with a divided powers structure. The filtered Dieudonné functor gives an equivalence of categories between*

1. *the category of abelian schemes over S', and*
2. *the category of pairs $(A, \mathbb{E})$, where A is an abelian scheme over S and $\mathbb{E} \subset \mathbb{D}^*(A)_{S'}$ is a locally free direct summand which lifts $\mathbb{E}_A \subset \mathbb{D}^*(A)_S$. Morphisms are homomorphisms $f : A_1 \to A_2$ such that the induced morphism $f^* : \mathbb{D}^*(A_2)_{S'} \to \mathbb{D}^*(A_1)_{S'}$ satisfies $f^*(\mathbb{E}_2) \subset \mathbb{E}_1$.*

Let S be the spectrum of an algebraically closed field k. Let $S \subset S'$ be a PD thickening such that S' is a local artinian k–algebra. Let $A' \to S'$ be the trivial deformation of A over S' for which $\mathbb{D}^*(A)_{S'} = H^1_{\mathrm{dR}}(A/S) \otimes_k \mathcal{O}_{S'}$. Then, given any other deformation A'' of A to S', the canonical isomorphism $H^1_{\mathrm{dR}}(A''/S') \cong$

$H^1_{\rm dR}(A/k)\otimes_k \mathcal{O}_{S'}$ provides us with a submodule $\mathbb{E}_{A''}\subset H^1_{\rm dR}(A)\otimes_k\mathcal{O}_{S'}$ lifting $\mathbb{E}_A\subset H^1_{\rm dR}(A)$. Thus we get a morphism from the functor of deformations over the nilpotent crystalline site of S to the functor $\mathrm{Grass}(g, H^1_{\rm dR}(A/k))$.

Let T to be the spectrum of $\widehat{\mathcal{O}}_{\mathfrak{M},x}$ and let $f:\mathfrak{X}^{\rm uni}\to T$ be the universal object. Trivialize $R^1f_*(\Omega^\bullet_{\mathfrak{X}^{\rm uni}/T})\cong\widehat{\mathcal{O}}^{2g}_{\mathfrak{M},x}$ with respect to a basis horizontal for the Gauss–Manin connection. Considering the submodule $\mathbb{E}_{\mathfrak{X}^{\rm uni}/T}\subset R^1f_*(\Omega^\bullet_{\mathfrak{X}^{\rm uni}/T})$, we obtain a morphism $T\to\mathrm{Grass}(\mathcal{O}_L\otimes k,\langle\ ,\ \rangle,g,2g)$. Similar constructions can be made with endomorphism and polarization structures. Using this map and the crystalline theory, one obtains [DP, Thm. 3.3], [deJ] the following theorem (recall the tacit assumption of rigid level structure):

Theorem 4.2.2. *1. In the Siegel case, there is an isomorphism*

$$\widehat{\mathcal{O}}_{\mathfrak{A}_g,x}\cong\widehat{\mathcal{O}}_{G,y},$$

where G is the Grassmannian variety $\mathrm{Grass}(\langle\ ,\ \rangle,g,2g)$ *that parameterizes g-dimensional isotropic subspaces of $H^1_{\rm dR}(A/k)$ and y is the point corresponding to the Hodge filtration $H^0(A,\Omega^1_{A/k})\subset H^1_{\rm dR}(A/k)$.*

2. In the Hilbert case, there is an isomorphism

$$\widehat{\mathcal{O}}_{\mathfrak{M},x}\cong\widehat{\mathcal{O}}_{G,y},$$

where G is the Grassmannian variety $\mathrm{Grass}(\mathcal{O}_L\otimes k,\langle,\ ,\ \rangle,g,2g)$ *that parameterizes g-dimensional isotropic $\mathcal{O}_L$-invariant subspaces of $H^1_{\rm dR}(A/k)$ and y is the point corresponding to the Hodge filtration $H^0(A,\Omega_{A/k})\subset H^1_{\rm dR}(A/k)$.*

Remark 4.2.3. *The theorem holds, for a suitably formulated Grassmannian problem, without the restriction to characteristic p. See* [DP, deJ]

4.3. Examples.

We only consider deformations in characteristic p.

4.3.1. The Siegel case. Let V be a $2g$-dimensional vector space, let $\Gamma\subset V$ be a g-dimensional subspace of V and choose a complementary subspace $W\subset V$ such that $V=\Gamma\oplus W$. Then an affine chart of $\mathrm{Grass}(g,V)$ about Γ is given by $\mathrm{Hom}(\Gamma,W)$. Given $t\in\mathrm{Hom}(\Gamma,W)$ we associate to it its graph.

Suppose that V has a symplectic pairing and Γ is isotropic. Choose a basis $a_1,\dots,a_g$ to Γ and complete it to a standard symplectic basis by $b_1,\dots,b_g$. Take W to be the span of $b_1,\dots,b_g$. We may identify t with a $g\times g$ matrix $(t_{i,j})$ such that $a_j\mapsto a_j+\sum_i t_{i,j}b_i$.

The graph of t is isotropic if and only if for each j,k we have

$$(a_j+\sum_i t_{i,j}b_i)\wedge(a_k+\sum_i t_{i,k}b_i)=0. \qquad (4.1)$$

Since $(a_j + \sum_i t_{i,j} b_i) \wedge (a_k + \sum_i t_{i,k} b_i) = t_{j,k} - t_{k,j}$, Equation (4.1) is equivalent to $(t_{i,j})$ being a symmetric matrix. This is of course in accord with $\mathfrak{A}_g$ (with a rigid level structure prime to p) being a non-singular variety of dimension $g(g+1)/2$.

4.3.2. The Hilbert case. We again consider two cases.

- **The inert case.** In this case we have a decomposition $\mathcal{O}_L \otimes k = \oplus_{i=1}^{g} k$. We denote the projection of $\mathcal{O}_L$ on the i-th component by σ_i. One may assume that $\mathrm{Frob} \circ \sigma_i = \sigma_{i+1}$. We then have

$$H^1_{\mathrm{dR}}(A/k) = \oplus_{i=1}^{g} D(i),$$

where each $D(i)$ is a two dimensional k-vector space with a perfect alternating pairing, on which $\mathcal{O}_L$ acts via σ_i. There is a compatible decomposition

$$H^0(A, \Omega_{A/k}) = \oplus_{i=1}^{g} H(i),$$

where each $H(i)$ is a one dimensional k-vector space on which $\mathcal{O}_L$ acts via σ_i. The Grassmannian is therefore isomorphic to

$$\mathrm{Grass}(1,2)^g \cong (\mathbb{P}^1_k)^g.$$

Note that the completed local ring of every point x on $\mathfrak{M}$ is isomorphic to the completed power series ring $k[\![t_1, \dots, t_g]\!]$, where t_i is canonical up to an element of $k[\![t_i]\!]^\times$.

- **The maximally ramified case.** In this case

$$H^1_{\mathrm{dR}}(A/k) \cong k[T]/(T^g) \oplus k[T]/(T^g).$$

The Grassmannian $\mathrm{Grass}(\mathcal{O}_L \otimes k, \langle\ ,\ \rangle, g, 2g)$ is that of parameterizing isotropic g-dimensional subspaces that are $\mathcal{O}_L$-invariant. One can show [DP] that one can replace the k-valued pairing, for which the action of $\mathcal{O}_L$ is self-adjoint, by a $k[T]/(T^g)$-valued pairing, which is $k[T]/(T^g)$-linear.

Given A/k we can find a basis α, β of $H^1_{\mathrm{dR}}(A/k)$ such that

$$H^0(A, \Omega_{A/k}) = (T^i)\alpha \oplus (T^j)\beta, \qquad \alpha \wedge \beta = 1,$$

where $j = j(A)$, $i = g - j$, $i \geq j$. This determines i, j uniquely. We choose the complementary subspace to be

$$\bigoplus_{s=0}^{i-1} T^s k\alpha \oplus \bigoplus_{s=0}^{j-1} T^s k\beta.$$

The deformations f of $H^0(A, \Omega_{A/k})$ in $H^1_{\mathrm{dR}}(A/k)$ that are $\mathcal{O}_L$-linear are determined as follows. Under f,

$$T^i\alpha \mapsto T^i\alpha + \sum_{s=0}^{i-1} a_s T^s\alpha + \sum_{s=0}^{j-1} b_s T^s\beta, \quad T^j\beta \mapsto T^j\beta + \sum_{s=0}^{i-1} c_s T^s\alpha + \sum_{s=0}^{j-1} d_s T^s\beta.$$

We write that in shorthand notation as

$$T^i\alpha \mapsto T^i\alpha + a\alpha + b\beta, \qquad T^j\beta \mapsto T^j\beta + c\alpha + d\beta,$$

with

$$\begin{aligned} a &= \textstyle\sum_{s=0}^{i-1} a_s T^s, & b &= \textstyle\sum_{s=0}^{j-1} b_s T^s, \\ c &= \textstyle\sum_{s=0}^{i-1} c_s T^s, & d &= \textstyle\sum_{s=0}^{j-1} d_s T^s. \end{aligned}$$

To have an isotropic subspace we must require

$$(T^i\alpha + a\alpha + b\beta) \wedge (T^j\beta + c\alpha + d\beta) = 0.$$

This is equivalent to

$$ad - bc + aT^j + dT^i = 0.$$

It then follows that the $\mathcal{O}_L \otimes k$-span of $T^i\alpha + a\alpha + b\beta, T^j\beta + c\alpha + d\beta$ is a g-dimensional isotropic $\mathcal{O}_L$-invariant subspace.

Example 4.3.1. *$j = 0$ (non-singular points). In this case $i = g$. We get immediately $b = d = 0$ and hence also $a = 0$. It follows that $c = \sum_{s=0}^{g-1} c_s T^s$ is unobstructed and we conclude that the completed local ring is isomorphic to*

$$k[\![c_0, \dots, c_{g-1}]\!].$$

Example 4.3.2. *$g = 2$, $i = j = 1$. In this case we find the equation*

$$a_0 d_0 - b_0 c_0 + a_0 T + d_0 T = 0.$$

We get the relations $a_0 = -d_0$ and $a_0 d_0 - b_0 c_0 = 0$. This gives that the completed local ring is isomorphic to

$$k[\![a_0, b_0, c_0]\!]/(a_0^2 + b_0 c_0).$$

Example 4.3.3. *$g = 3$, $j = 1, i = 2$. We have*

$$\begin{aligned} a &= a_0 + a_1 T, & b &= b_0 \\ c &= c_0 + c_1 T, & d &= d_0. \end{aligned}$$

with the equation

$$(a_0 d_0 - b_0 c_0) + (a_0 + a_1 d_0 - b_0 c_1)T + (a_1 + d_0)T^2 = 0.$$

This yields $d_0 = -a_1, a_0 = a_1^2 + b_0 c_1$ and that the completed local ring R is isomorphic to

$$k[\![a_1, b_0, c_0, c_1]\!]/(a_1^3 + a_1 b_0 c_1 + b_0 c_0),$$

which is 3-dimensional with a tangent cone at the origin defined by $b_0c_0 = 0$. *The singular locus of* $\mathrm{Spec}(R)$ *is given by* $b_0 = c_0 = 0$ *(which implies* $a_1 = 0$*) and is hence one dimensional, isomorphic to* $\mathrm{Spec}(k[\![c_0]\!])$.

4.4. Singular points.

Using the local models one can show [DP, Thm. 2.2] that $\mathfrak{M}$ is singular if and only if p is ramified in $\mathcal{O}_L$ and that the singular locus is of codimension 2. However, the singularities are local complete intersections, hence Cohen-Macaulay and so normal, by Serre's criterion. We remark that, in particular, the completed local rings are domains, i.e., the moduli space is locally (formally) irreducible.

In local commutative algebra a property which is subtle and of interest is the property of parafactoriality. The definition is motivated by its relation to factoriality and representability of the local Picard functor of invertible sheaves. For this we refer the interested reader to the references below and to [Lip1]. A noetherian local ring $(R, \mathfrak{m})$ is called *parafactorial* if it is of depth at least 2 and if $\mathrm{Pic}(R - \{\mathfrak{m}\}) = 0$. A global definition follows:

Definition 4.4.1. *Let* (X, Z) *be a pair consisting of a ringed space* X *and a closed subset* Z. *Let* $U = X \setminus Z$. *One says that* (X, Z) *is* parafactorial *if, for every open set* V *of* X, *the restriction functor* $M \mapsto M_{|U\cap V}$, *from the category of invertible* $\mathcal{O}_V$*-modules to the category of invertible* $\mathcal{O}_{U\cap V}$*-modules, is an equivalence of categories.*

We refer the reader to [EGA IV, §21.13], [SGA 2, Exp. XI] for details. In particular, [EGA IV, §21.13.8] gives the equivalence of the definitions for local rings.

Lemma 4.4.2. *Let* k *be a field. Let* R *be the ring*

$$k[\![a_0, \ldots, a_{g-2}, b_0, c_0, \ldots, c_{g-2}, d_0, x_1, \ldots, x_N]\!]/ \\ (a_0d_0 - b_0c_0, a_{g-2} + d_0, \{a_id_0 + a_{i-1} - b_0c_i : 1 \leq i \leq g-2\}). \quad (4.2)$$

The closed set $\mathrm{Spec}(R)^{\mathrm{sing}}$ *is defined by the ideal* (a_0, b_0, c_0, d_0) *of* R. *The pair* $(\mathrm{Spec}(R), \mathrm{Spec}(R)^{\mathrm{sing}})$ *is not parafactorial.*

Proof. First, one proves that R is isomorphic to the ring

$$k[\![b_0, c_0, \ldots, c_{g-2}, d_0, x_1, \ldots, x_N]\!]/ \\ \big(b_0c_0 - d_0b_0c_1 + d_0^2b_0c_2 - d_0^3b_0c_3 + \ldots + (-1)^{g-2}d_0^{g-2}b_0c_{g-2} + (-1)^{g-2}d_0^g\big), \quad (4.3)$$

cf. proof of Lemma 6.3.4. Then, a direct application of the Jacobi criterion gives that $\mathrm{Spec}(R)^{\mathrm{sing}}$ is defined by the ideal (a_0, b_0, c_0, d_0).

Let U_{ab} (resp. U_{cd}) denote the open set where either a_0 or b_0 (resp. c_0 or d_0) are not zero. Note that $U := U_{ab} \cup U_{cd} = \mathrm{Spec}(R) - \mathrm{Spec}(R)^{\mathrm{sing}}$. We consider the closed subscheme given on $U_{ab} \cup U_{cd}$ by (b_0, d_0). Note that by Equation (4.2), this closed subscheme is an irreducible reduced Weil divisor D_0 on $U_{ab} \cup U_{cd}$, automatically locally principal. Now, consider the closed subscheme D of $\mathrm{Spec}(R)$ defined by the same ideal (b_0, d_0). There is unique extension of D_0 as a Weil divisor to $\mathrm{Spec}(R)$ which is just D (because $\mathrm{Spec}(R)^{\mathrm{sing}}$ has codimension 2). If the pair $(\mathrm{Spec}(R), \mathrm{Spec}(R)^{\mathrm{sing}})$ is parafactorial then the invertible sheaf $\mathcal{O}_U(D_0)$ extends to an invertible sheaf $\mathscr{F}$ over $\mathrm{Spec}(R)$. By [Har, Prop. II.6.15] $\mathscr{F} \cong \mathcal{O}(D')$, where D' is locally principal and, without loss of generality, $D'|_U = D_0$ and so $D' = D$. Thus, it remains to prove that D is not locally principal. We follow the argument of [Har, II 6.5.2].

Assume that D is locally principal. Let $\mathfrak{m}_R$ be the maximal ideal of R. Then D is given by a unique equation in $\mathfrak{m}_R/\mathfrak{m}_R^2$. But $\mathfrak{m}_R/\mathfrak{m}_R^2$ is just the k-vector space with basis $b_0, c_0, \dots, c_{g-2}, d_0, x_1, \dots, x_N$. On the other hand, clearly D is given in $\mathfrak{m}_R/\mathfrak{m}_R^2$ by b_0, d_0. Contradiction. □

Theorem 4.4.3. *Assume that p ramifies in $\mathcal{O}_L$. Then the pair $(\mathfrak{M}, \mathfrak{M}^{\mathrm{sing}})$ is not parafactorial. In fact, there is an invertible subsheaf $\mathbb{L}$ of the Hodge bundle that does not extend to any open set strictly containing $\mathfrak{M}^R = \mathfrak{M} - \mathfrak{M}^{\mathrm{sing}}$.*

If p is maximally ramified, then $\mathbb{L} = \mathfrak{p}^{g-1}\mathbb{E}$, and $\mathbb{L}$ extends to an invertible sheaf over $\mathfrak{N}$.

Proof. Assume that $(p) = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$ in $\mathcal{O}_L$ and that $e_1 > 1$. Let k be an algebraically closed field of characteristic p. Write $\mathcal{O}_L \otimes k = \oplus_{\ell=1}^r \oplus_{m=1}^{f(\mathfrak{p}_\ell/p)} k_m[T]/(T^{e_\ell})$, with $k_m = k$ for all m.

Consider a k-rational point x on $\mathfrak{M}$ with the property that $H^0(A_x, \Omega_{A_x/k}) = \oplus_{\ell=1}^r \oplus_{m=1}^{f(\mathfrak{p}_\ell/p)} U_{\ell,m}$ with $U_{\ell,m} = k_m[T]/(T^{e_\ell})$, except for $U_{1,1}$, which is taken to be the $k_m[T]/(T^{e_1})$ module given by $(T) \oplus (T^{e_1 - 1})$. The closure of the collection of such points is a closed subscheme Z of $\mathfrak{M}$. Cf. 3.2, 4.3.2.

The completed local ring S of x is by the theory of local models isomorphic to $\widehat{\otimes}_{\ell=1}^r \widehat{\otimes}_{m=1}^{f(\mathfrak{p}_\ell/p)} R_{\ell,m}$, with $R_{\ell,m}$ a power series ring over k, except for $R_{1,1}$, which is isomorphic to the ring R in Equation (4.2) with $g = e_1$ and $N = 0$. That is, the ring S is itself of the form given in Equation (4.2) with $Z = \mathrm{Spec}(S)^{\mathrm{sing}}$.

Suppose that $(\mathfrak{M}, \mathfrak{M}^{\mathrm{sing}})$ is parafactorial. Recall that over $\mathfrak{M}^R = \mathfrak{M} - \mathfrak{M}^{\mathrm{sing}}$ the relative cotangent space of the universal abelian scheme is a locally free $\left[\oplus_{\ell=1}^r \oplus_{m=1}^{f(\mathfrak{p}_\ell/p)} k_m[T]/(T^{e_\ell})\right] \otimes_k \mathcal{O}_{\mathfrak{M}^R}$ module. Consider the invertible sheaf $\mathbb{L}$ defined on $\mathfrak{M}^R$ by the ideal $\oplus_{\ell=1}^r \oplus_{m=1}^{f(\mathfrak{p}_\ell/p)} I_{\ell,m}$ with $I_{\ell,m}$ equal to 0, except for $I_{1,1}$, which is equal to $T^{e_1-1}k_1[T]/(T^{e_1})$.

Since the pair $(\mathfrak{M}, \mathfrak{M}^{\mathrm{sing}})$ is parafactorial, it follows that the invertible sheaf $\mathbb{L}$ can be extended from $U := \mathrm{Spec}(S) - Z$ to $\mathrm{Spec}(S)$.

The de Rham sheaf corresponds under the theory of local models to a free $\mathcal{O}_L \otimes_k S$ module with generators α, β, and $\mathbb{L}$ is then the submodule generated over U by $T^{e_1-1}a_0\alpha + T^{e_1-1}b_0\beta$ and $T^{e_1-1}c_0\alpha + T^{e_1-1}d_0\beta$.

Let U_{ab} (resp. U_{cd}) denote the open set where either a_0 or b_0 (resp. c_0 or d_0) are not zero. We have a trivialization of $\mathbb{L}$ over U_{ab} ($T^{e_1-1}a_0\alpha + T^{e_1-1}b_0\beta$ is a basis) and over U_{cd} ($T^{e_1-1}c_0\alpha + T^{e_1-1}d_0\beta$ is a basis). Note that on S we have the relation $a_0d_0 = b_0c_0$. The transition function between the trivializations is $d_0/b_0 = c_0/a_0$.

Let D be the divisor on Spec(S) defined by the ideal (b_0, d_0). The divisor D is defined on U_{ab} by b_0 and on U_{cd} by d_0 and so has the same transition function as $\mathbb{L}$. Parafactoriality implies that D must be locally principal, cf. the proof of Lemma 4.4.2. But we have shown in that proof that this is not the case.

To show that $\mathbb{L}$ cannot be extended outside $\mathfrak{M}^R$ we argue as follows: Let K be a closed set that contains every point x as constructed above (where we allow a different choice of ℓ, m as long as $e_\ell > 1$). Such points are dense in $\mathfrak{M}^{\rm sing}$ as follows from [DP, §4]. Therefore, $K \supset \mathfrak{M}^{\rm sing}$. Hence, if U is an open set strictly containing $\mathfrak{M}^R$ then U contains such a point x. But we have shown that $\mathbb{L}$ cannot be extended as an invertible sheaf over the completed local ring of x.

Assume now that p is maximally ramified. The first claim was already proven. To prove the second claim, consider the Lie algebra of the subgroup defining the moduli problem $\mathfrak{N}$. It provides us with a locally free quotient sheaf $\mathbb{H}$ of the Hodge bundle $\mathbb{E}$ over $\mathfrak{N}$. We claim that when we restrict $\mathbb{H}$ to $\mathfrak{N}^R = \mathfrak{M}^R$ then $\mathbb{H}$ is isomorphic to $\mathbb{L}$. This follows from the fact that over $\mathfrak{M}^R$ the Hodge bundle $\mathbb{E}$ has a canonical filtration $\mathbb{E} \supset \mathfrak{p}\mathbb{E} \supset \cdots \supset \mathfrak{p}^{g-1}\mathbb{E} \supset 0$, with successive graded pieces being isomorphic under multiplication by T. □

Corollary 4.4.4. *Assume that p is maximally ramified in $\mathcal{O}_L$. The section of the morphism $\mathfrak{N} \to \mathfrak{M}$, $\underline{A} \mapsto (\underline{A}, T^{g-1}\mathrm{Ker}(F_A))$, defined on $\mathfrak{M}^R$, does not extend to any open set strictly containing $\mathfrak{M}^R$.*

Proposition 4.4.5. *Let x be a (scheme theoretic) point of $\mathfrak{M}$ of codimension at least 4. Then the local ring of x is parafactorial. If p is unramified in $\mathcal{O}_L$, the local ring of x is parafactorial for any x.*

Proof. Let x be a (scheme theoretic) point of $\mathfrak{M}$. By [SGA 2, Exp. XI, Cor. 3.7], to show that the local ring $\mathcal{O}_{\mathfrak{M},x}$ is parafactorial it is enough to show that $\widehat{\mathcal{O}}_{\mathfrak{M},x}$ is parafactorial. If x is of codimension at least 4, the ring $\widehat{\mathcal{O}}_{\mathfrak{M},x}$ is of dimension ≥ 4 and is a complete intersection by the theory of local models (see [DP, Prop. 4.4]). It follows from [SGA 2, Exp. XI, Thm. 3.13] that it is parafactorial.

It is known that a regular noetherian local ring of dimension at least 2 is parafactorial - a result due to Auslander-Buchsbaum - cf. [SGA 2, Thm. 3.13], [EGA IV, §21.13.9 (ii)]. □

The parafactoriality of the completed local rings of closed points on $\mathfrak{M}$ is completely covered by the results above except for the situation $g = 3$ and $(p) = \mathfrak{p}^2\mathfrak{q}$. In this case, the completed local ring of any non-singular point is of the form $k[\![x, y, z]\!]/(z^2 + xy)\widehat{\otimes}k[\![t]\!]$. Such a ring is parafactorial [Bou, III, Prop. 1.2].

5. The display of an abelian variety with RM

We wish to study the local deformation theory of abelian varieties with RM in characteristic $p > 0$. In this paper we only study equi-characteristic deformations. Our main tools are the theory of local models and the theory displays, both available in the arithmetic setting as well. One thus hopes that the methods below will extend to the arithmetic setting.

Let $x \in \mathfrak{M}$ be a k-valued point, where k is an algebraically closed field of characteristic p. The theory of local models allows us to determine the ring $\widehat{\mathcal{O}}_{\mathfrak{M},x}$, and even the behavior of the strata $\mathcal{S}_j$, but falls short of describing the behavior of the strata $W_{(j,n)}$.

As we shall explain, the local deformation theory factors according to the prime ideals dividing p in $\mathcal{O}_L$ and that allows us, essentially, to assume that the p-divisible group $A_x(p)$ is either ordinary, or local-local. The first case is studied very effectively using Serre-Tate coordinates but is of no interest to us in this paper. In order to study the second case, we make use of the theory of displays as reformulated and developed by Zink [Zin].

Our main idea, which is similar to [Zin, §2.2], is the following. Suppose, for simplicity, that the abelian variety A_x has a local-local p-divisible group. Then, the display associated to the abelian scheme $A \to \mathrm{Spec}(\widehat{\mathcal{O}}_{\mathfrak{M},x})$, whose fiber over the closed point is A_x, is *universal* with respect to the problem of deformations over local artinian k-algebras $(R, \mathfrak{m})$ with $R/\mathfrak{m} = k$ of the polarized $\mathcal{O}_L$-display associated to A_x. Indeed, the universality is one of Zink's main results.

On the other hand, the theory of local models provides us with a concrete model R for $\widehat{\mathcal{O}}_{\mathfrak{M},x}$, which is the completion of the local ring of a point on a suitable Grassmann variety. We view the universal display $\mathscr{P}^{\mathrm{uni}}$ as lying over R. We explicitly construct a display $\mathscr{P}$ over R that we want to show is universal. By the universal property, $\mathscr{P}$ is obtained from $\mathscr{P}^{\mathrm{uni}}$ by base change coming from a unique map $\varphi\colon R \to R$. At least over $R/\mathfrak{m}_R^2$, the Hodge filtrations defined by $\mathscr{P}^{\mathrm{uni}}$ and $\mathscr{P}$ produce two maps (that are unique) $\psi_1, \psi_2\colon R \to R$, coming from the interpretation of R as a completed local ring of a point on a Grassmannian, and the crystalline nature of displays. One gets a commutative diagram $\varphi \circ \psi_1 = \psi_2$. We then argue that, in fact, ψ_1 and ψ_2 are isomorphisms, hence so is φ. The universality of $\mathscr{P}$ ensues.

We next discuss the connection to a well known result that gives the universal display for the Siegel case [Oo3, pp. 412-414],[Zin, Eqn. (86)]. Let $(X, \lambda)/k$ be a principally polarized abelian variety over an algebraically closed field k of characteristic $p > 0$ and choose a symplectic basis for the display of X to yield a matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, as explained in [Zin, pp. 128-9]. Let $R := k[\![t_{ij} : i, j = 1, \cdots, g]\!]/(t_{ij} - t_{ji})$ be the completed local ring provided by the theory of local models (cf. § 4.3.1). It is identified, non-canonically, with the completed local ring of the k-rational point of $\mathfrak{A}_g$ corresponding to (X, λ) (the usual choice of auxiliary rigid level structure prime to p is required for that). Let T_{ij} be the Teichmüller lift of t_{ij} and let T be the square $g \times g$ matrix (T_{ij}). The universal display for the universal infinitesimal equi-characteristic deformation of (X, λ) is then given by

$$\begin{pmatrix} A + TC & B + TD \\ C & D \end{pmatrix}. \tag{5.1}$$

Note, for example, that $A + TC$ is the "universal Hasse-Witt matrix" and thus the non-ordinary locus is infinitesimally defined by the equation $\det(A + TC) = 0 \pmod{p}$. This determinant can be interpreted as the Hasse invariant - a Siegel modular form of weight $p - 1$ that vanishes exactly along the non-ordinary locus.

Equation (5.1) is a red herring of a sort. In that expression the Hodge filtration "seems constant"; namely, in the specified basis $e_1, \dots, e_{2g}$, with respect to which the display is given, the kernel of Frobenius modulo p is the span of $e_{g+1}, \dots, e_{2g}$. As such, its behavior is exactly the opposite of the behavior expected from the crystalline theory and the theory of local models.

However, consider the automorphism of the underlying module of the display provided by $\left(\begin{smallmatrix} I & T \\ 0 & I \end{smallmatrix}\right)$ and write $\left(\begin{smallmatrix} A+TC & pB+TpD \\ C & pD \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & T \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} A & pB \\ C & pD \end{smallmatrix}\right)$. One checks that with respect to a suitable basis (see below) the Frobenius operator is given by

$$\begin{pmatrix} A & pB \\ C & pD \end{pmatrix}\begin{pmatrix} I & T^\sigma \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & AT^\sigma + pB \\ C & CT^\sigma + pD \end{pmatrix}. \tag{5.2}$$

The kernel of the Frobenius operator modulo p is now spanned by the columns of the matrix $\left(\begin{smallmatrix} -T \\ I \end{smallmatrix}\right)$, which indeed has the "desired maximal variation" dictated by the local model. The point is, the basis in which Equation (5.1) is given is not horizontal with respect to the Gauss-Manin connection, whereas the basis in which Equation (5.2) is written is, at least over $R/\mathfrak{m}_R^2$. As will become apparent from the discussion below (§ 5.6), this is enough to conclude that this display is a universal display.

We make all this more precise. Consider the composition $\phi \circ \tau$ of two operators, ϕ being a σ-linear map and τ being a linear automorphism. Here the operators are operating on the underlying module of the display of the special fibre, extended trivially to a display over R. We take ϕ to be the Frobenius operator and τ the automorphism expressed in a basis $\mathcal{B}$ by $\left(\begin{smallmatrix} I & T \\ 0 & I \end{smallmatrix}\right)$. Let $[\phi \circ \tau]_{\mathcal{B}}$ be the expression of $\phi \circ \tau$ as a matrix with respect to the basis $\mathcal{B}$. Then $[\phi \circ \tau]_{\mathcal{B}} = \left(\begin{smallmatrix} A & AT^\sigma + pB \\ C & CT^\sigma + pD \end{smallmatrix}\right)$. Let $\mathcal{C}$ be the basis $\tau^{-1}(\mathcal{B})$. Then $[\phi \circ \tau]_{\mathcal{C}} = [\tau]_{\mathcal{B}}[\phi]_{\mathcal{B}} = \left(\begin{smallmatrix} A+TC & pB+TpD \\ C & pD \end{smallmatrix}\right)$.

Furthermore, let I_R be the augmentation ideal of $\mathbb{W}(R)$, $I_R =^V \mathbb{W}(R)$. Let K be the kernel of $\phi \pmod{I_R}$ so $\tau^{-1}(K)$ is the kernel of $\phi \circ \tau \pmod{I_R}$. Let $[K]_{\mathcal{B}}$ be the set of coordinate vectors expressing K in the basis $\mathcal{B} \pmod{I_R}$. Then we have $[\tau^{-1}(K)]_{\mathcal{C}} = [K]_{\mathcal{B}}$, but of course $[\tau^{-1}K]_{\mathcal{B}} = \left(\begin{smallmatrix} I & -T \\ 0 & I \end{smallmatrix}\right)[K]_{\mathcal{B}}$.

According to Zink's theory (see [Zin, Thm. 44] and §5.1.3 below) the display $\mathscr{P}_0$ over k determined by $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ gives a crystal $D_{\mathscr{P}_0}$ over the nilpotent crystalline site of $\mathrm{Spec}(k)$. To conclude our discussion, it remains to show that there is a display $\mathscr{P}$ over R, whose Frobenius operator is given by (5.2), such that the isomorphism from $\widehat{\mathscr{P}} \pmod{\mathfrak{m}_R^2}$ to $\widehat{\mathscr{P}_0} \otimes \mathbb{W}(R/\mathfrak{m}_R^2)$, dictated by the crystalline theory, is simply the identity. In essence, that follows from the fact that the operator $\left(\begin{smallmatrix} I & T^\sigma \\ 0 & I \end{smallmatrix}\right)$ is the identity when reduced modulo I_R and then modulo $\mathfrak{m}_R^2$.

5.1. Recall.

In this section we review the theory of displays, developed in [Zin], discussing a variant where a real multiplication is considered. Having in mind applications to local models, we recall the connection between displays and crystals as developed in [Zin].

5.1.1. The deformation theory of abelian varieties is equivalent, by Serre-Tate, to the deformation theory of their p-divisible groups. One wishes to isolate the type of p-divisible groups on which $\mathcal{O}_L$ acts as endomorphisms that arise in this fashion from RM abelian varieties. To illustrate the problem, note that if p splits in $\mathcal{O}_L$ then $\mathcal{O}_L$ acts as endomorphisms of any one dimensional p-divisible group, but does not act on any elliptic curve. To rule out such possibilities we make the following definition:

Definition 5.1.1. *Let B be a finitely generated $\mathbb{Z}_p$-algebra. Let k be a field of characteristic p. Let G be a p-divisible over k on which B acts as endomorphisms. We say that G has* RM *by B if the Dieudonné module of $G \otimes_k k^{\mathrm{alg}}$ is a free $B \otimes_{\mathbb{Z}_p} \mathbb{W}(k^{\mathrm{alg}})$-module of rank 2. We say that G has* RM *by $\mathcal{O}_L$ if it has RM by $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$ in the sense just defined.*

Let R be a local Noetherian ring with residue field k as above. A p-divisible group G over R is said to have RM *by B if B acts as endomorphisms of G and $G \otimes k$ has RM by B in the sense defined above. Ibid. for RM by $\mathcal{O}_L$.*

A polarized p-divisible group with RM over a ring R as above, is a pair (G, λ) where G is a p-divisible group over R with RM by B and $\lambda : G \to G^t$ is a B-linear symmetric isomorphism.

5.1.2. Let R be an $\mathbb{F}_p$–algebra. Let $\mathbb{W}(R)$ be the Witt vectors over R and let I_R be the kernel of the ring homomorphism $\mathbb{W}(R) \to R$ given by projection on the first coordinate.

A polarized display $\mathscr{P}$ over R with real multiplication by $\mathcal{O}_L$, a *RM display* for short, is a quintuple $\left(P, Q, F, V^{-1}, \langle\_,\_\rangle\right)$ consisting of:

1. a projective $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}(R)$-module P of rank 2;
2. a finitely generated $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}(R)$-submodule Q of P such that $I_R P \subset Q \subset P$ and P/Q is a direct summand of the R-module $P/I_R P$;
3. additive maps $F\colon P \to P$ and $V^{-1}\colon Q \to P$, which are linear with respect to $\mathcal{O}_L$ and σ-linear with respect to $\mathbb{W}(R)$, and satisfy $V^{-1}\left({}^V wy\right) = wF(y)$ for any $w \in \mathbb{W}(R)$ and any $y \in P$. One imposes a further nilpotence condition [Zin, Def. 2];

4. an $\mathcal{O}_L \otimes \mathbb{W}(R)$-bilinear map $\langle\_,\_\rangle\colon P \times P \to D_L^{-1} \otimes \mathbb{W}(R)$ satisfying the identity ${}^V\langle V^{-1}(x), V^{-1}(y)\rangle = \langle x, y\rangle$ for every x and y in Q.

Define

$$D_{\mathscr{P}} := P/I_R P, \qquad H_{\mathscr{P}} := Q/I_R P.$$

The filtration $H_{\mathscr{P}} \subset D_{\mathscr{P}}$ is called the *Hodge filtration* of $\mathscr{P}$.

Replacing $\mathcal{O}_L$ with its completion $\mathcal{O}_{L_\mathfrak{p}}$ and D_L with its completion at $\mathfrak{p}$, one gets the notion of a *polarized display with* $\mathcal{O}_{L_\mathfrak{p}}$*–action.*

The main example of a display is the Dieudonné module. Let k be a perfect field of characteristic p and let G be a connected polarized p-divisible group with RM by $\mathcal{O}_L$ over k. Then the Dieudonné module of G, say P, equipped with its Frobenius and Verschiebung morphisms and $\mathcal{O}_L$-bilinear pairing, gives the RM display $(P, VP, F, V^{-1}, \langle\_,\_\rangle)$.

A variant of [Zin, Thm. 9] is the following:

Theorem 5.1.2. *Let R be an excellent local ring or a ring such that R/pR is an algebra of finite type over a field k. Assume that $p = 0$ in R. Then there is a natural equivalence of categories between the category of polarized connected p-divisible groups over R with RM by $\mathcal{O}_L$ (resp. $\mathcal{O}_{L_\mathfrak{p}}$) and the category of displays over R with RM by $\mathcal{O}_L$ (resp. $\mathcal{O}_{L_\mathfrak{p}}$).*

5.1.3. The following is a consequence of [Zin, Thm. 44]. Let $S \to R$ be a surjective homomorphism of rings such that p is nilpotent in S and its kernel $\mathfrak{a}$ is equipped with divided powers. Let $\mathscr{P} := \big(P, Q, F, V^{-1}, \langle\_,\_\rangle\big)$ be an RM display (or a polarized display with $\mathcal{O}_{L_\mathfrak{p}}$–action) over R. Let $\mathscr{P}_i = \big(P_i, Q_i, F_i, V_i^{-1}, \langle\_,\_\rangle_1\big)$, $i = 1, 2$, be RM displays (or polarized displays with $\mathcal{O}_{L_\mathfrak{p}}$–action) over S reducing to $\mathscr{P}$. Let $\widehat{Q}_i$ be the inverse image of Q via $P_i \to P$. Then, V_i^{-1} extends uniquely to $\widehat{Q}_i$. The theorem states that there is a *unique isomorphism,*

$$\alpha\colon \widehat{\mathscr{P}_1} := \big(P_1, \widehat{Q}_1, F_1, V_1^{-1}\big) \longrightarrow \widehat{\mathscr{P}_2} := \big(P_2, \widehat{Q}_2, F_2, V_2^{-1}\big),$$

reducing to the identity on $\mathscr{P}$ and commuting with the $\mathcal{O}_L$-action (or $\mathcal{O}_{L_\mathfrak{p}}$–action). Hence, the sheaf $P\big(\mathrm{Spec}(R) \subset \mathrm{Spec}(S)\big) := P_1$ on the crystalline site (of pd-thickenings with kernel a nilpotent ideal) of $\mathrm{Spec}(R)$ defines a crystal. Analogously, $D_{\mathscr{P}_1}$ and $D_{\mathscr{P}_2}$ are canonically isomorphic. Hence, the sheaf $D\big(\mathrm{Spec}(R) \subset \mathrm{Spec}(S)\big) := D_{\mathscr{P}_1}$ on the crystalline site of $\mathrm{Spec}(R)$ defines a crystal called *the covariant Dieudonné crystal*.

Let A be an abelian variety over R with RM, let G be its p-divisible group and let $\mathscr{P}$ be the associated display. The crystal $D_{\mathscr{P}}$ is canonically isomorphic to the crystal $\mathbb{D}^*(A^t)$. See [Zin, Thm. 6] and [MM, II (1.5)].

5.2. Factorizing according to primes.

5.2.1. The local deformation theory and displays.

Lemma 5.2.1. *Let k be an algebraically closed field of positive characteristic p. Let $x \in \mathfrak{M}$ be a k-valued point. Then,*

1. *the RM p-divisible group $A_x(p)$ factors canonically as the product of the $\mathcal{O}_{L_\mathfrak{p}}$-polarized p-divisible groups, denoted $A_x(\mathfrak{p})$;*
2. *for each $\mathfrak{p}$, the $\mathcal{O}_{L_\mathfrak{p}}$-polarized p-divisible group $A_x(\mathfrak{p})$ is either ordinary or local-local. Its Dieudonné module is a free $\mathcal{O}_{L_\mathfrak{p}} \otimes_\mathbb{Z} \mathbb{W}(k)$-module of rank 2;*
3. *the functor of deformations of $A_x(p)$ on $\mathfrak{C}_k$ as an $\mathcal{O}_L$-polarized p-divisible group is naturally equivalent to the direct product, over $\mathfrak{p}$ dividing p, of the functors of deformations of $A_x(\mathfrak{p})$ on $\mathfrak{C}_k$ as an $\mathcal{O}_{L_\mathfrak{p}}$-polarized p-divisible group.*

One considers RM displays as in §5.1 and polarized displays with $\mathcal{O}_{L_\mathfrak{p}}$ action. It is easy to see that the first category is naturally isomorphic to the direct product of the categories of polarized displays with $\mathcal{O}_{L_\mathfrak{p}}$ action, where $\mathfrak{p}$ runs over primes factor of $p\mathcal{O}_L$.

Under the equivalence of categories stated in Theorem 5.1.2 between deformations of connected p-divisible groups and displays, the decomposition according to primes is respected.

5.2.2. The associated local model. Let D_0 be the $\mathcal{O}_L$-module $\mathcal{O}_L \otimes k \oplus \mathcal{O}_L \otimes k$, let $\langle\ ,\ \rangle \colon D_0 \times D_0 \to \wedge^2_{\mathcal{O}_L} D_0 = \mathcal{O}_L \otimes k$ be the wedge product, and let $H_0 \subset D_0$ be an isotropic $\mathcal{O}_L \otimes k$-submodule of D_0 having dimension g over k. Let R be the complete local ring pro-representing the moduli problem of associating to a local artinian k-algebra $(S, \mathfrak{m}_S)$ an $\mathcal{O}_L \otimes S$-submodule H of $D := D_0 \otimes_k S$, such that H is free as a S-module, is a direct summand of D, is totally isotropic with respect to the pairing $\langle\ ,\ \rangle$, and reduces to H_0 modulo $\mathfrak{m}_S$. The ring R is isomorphic to the completion of the local ring of the point corresponding to (H_0, D_0) in the appropriate Grassmann variety $\mathrm{Grass}(\mathcal{O}_L \otimes k, \langle\ ,\ \rangle, g, 2g)$.

The Grassmann variety $\mathrm{Grass}(\mathcal{O}_L \otimes k, \langle\ ,\ \rangle, g, 2g)$ is canonically isomorphic to the product, over $\mathfrak{p}|p$, of the Grassmann varieties $\mathrm{Grass}(\mathcal{O}_{L,\mathfrak{p}} \otimes k, \langle\ ,\ \rangle_\mathfrak{p}, g_\mathfrak{p}, 2g_\mathfrak{p})$. In particular, writing $D_0 = \oplus_\mathfrak{p} D_0(\mathfrak{p})$, $H_0 = \oplus_\mathfrak{p} H_0(\mathfrak{p})$, using the decomposition $\mathcal{O}_L \otimes k = \oplus_{\mathfrak{p}|p} \mathcal{O}_{L_\mathfrak{p}} \otimes k$, and noting that the pairing decomposes accordingly, we find that $R = \widehat{\otimes}_{\mathfrak{p}|p} R(\mathfrak{p})$, where $R(\mathfrak{p})$ is the completed local ring of the point $(H_0(\mathfrak{p}), D_0(\mathfrak{p}))$ on the Grassmann variety $\mathrm{Grass}(\mathcal{O}_{L,\mathfrak{p}} \otimes k, \langle\ ,\ \rangle_\mathfrak{p}, g_\mathfrak{p}, 2g_\mathfrak{p})$ and the completed tensor product is taken over k.

5.3. The setting in which the theorems are proved.

Using the decomposition above, one sees that the construction of the universal RM display (for deformations of a given RM display over k) may be considered "one prime at a time", and therefore, for notational convenience, one may assume that $p\mathcal{O}_L = \mathfrak{p}^e$. The results in this section will be formulated under this assumption, from which the more general assertions follow immediately.

We set the following notation: $p\mathcal{O}_L = \mathfrak{p}^e$, $f = [\mathcal{O}_L/\mathfrak{p} : \mathbb{F}_p]$. Let $\sigma_1, \ldots, \sigma_f$ denote the embeddings of $\widehat{\mathcal{O}}^{\mathrm{ur}}_{L,\mathfrak{p}} \to \mathbb{W}(k)$, ordered such that ${}^F(\cdot) \circ \sigma_i = \sigma_{i+1}$. Note that $\mathcal{O}_L \otimes \mathbb{W}(k) = \oplus_{i=1}^f B(i)$, where the decomposition is induced by the isomorphism $\mathbb{W}(\mathbb{F}_{p^f}) \otimes_{\mathbb{Z}_p} \mathbb{W}(k) \cong \oplus_{i=1}^f \mathbb{W}(k)$, $a \otimes \lambda \mapsto (\ldots, \sigma_i(a)\lambda, \ldots)$. We also have $\mathcal{O}_L \otimes k = \oplus_{i=1}^f \overline{B}(i)$ with the obvious notation. Note that $\overline{B}(i) \cong k[T]/(T^e)$, where T is the reduction of an Eisenstein element for the extension $\widehat{\mathcal{O}}_{L,\mathfrak{p}}/\widehat{\mathcal{O}}^{\mathrm{ur}}_{L,\mathfrak{p}}$.

For any k-algebra S denote by ${}^F\cdot$ and ${}^V\cdot$ the maps on $\mathcal{O}_L \otimes \mathbb{W}(S)$ given by ${}^F(\ell \otimes w) \mapsto \ell \otimes {}^F w$ and ${}^V(\ell \otimes w) \mapsto \ell \otimes {}^V w$ for all $\ell \in \mathcal{O}_L$ and $w \in \mathbb{W}(S)$.

5.4. Further decomposition of the local model.

For $r = 1, \ldots, f$ let

$$D_0(r) := \overline{B}(r) \oplus \overline{B}(r)$$

and denote by $\langle\ ,\ \rangle \colon D_0(r) \times D_0(r) \to \overline{B}(r)$ the wedge product. Let $H_0(r) \subset D_0(r)$ be an isotropic $\overline{B}(r)$-submodule of $D_0(r)$ having dimension e over k. There exist a basis $\{\alpha(r), \beta(r)\}$ of $D_0(r)$, as a $\overline{B}(r)$-module, such that $\langle \alpha(r), \beta(r) \rangle = 1$ and

$$H_0(r) = (T^{i(r)})\alpha(r) \oplus (T^{j(r)})\beta(r)$$

for uniquely determined integers $e \geq i(r) \geq j(r) \geq 0$ satisfying $i(r) + j(r) = e$. Let $R(r)$ be the complete local ring pro-representing the moduli problem of associating to an object $(S, \mathfrak{m}_S)$ of $\mathfrak{C}_k$ a $\overline{B}(r) \otimes S$-submodule $H(r)$ of $D(r) := D_0(r) \otimes_k S$ such that $H(r)$ is free as a S-module, is a direct summand of $D(r)$, is totally isotropic with respect to the pairing $\langle\ ,\ \rangle$, and reduces to $H_0(r)$ modulo $\mathfrak{m}_S$. Then,

$$\begin{aligned} R(r) \cong k[\![a(r)_0, \ldots, a(r)_{i(r)-1}, b(r)_0, \ldots, b(r)_{j(r)-1},& \\ c(r)_0, \ldots, c(r)_{i(r)-1}, d(r)_0, \ldots, d(r)_{j(r)-1}]\!]/& \\ \big(a(r)d(r) - b(r)c(r) + a(r)T^{j(r)} + d(r)T^{i(r)}\big),& \end{aligned}$$

where $a(r) := a(r)_0 + \ldots + a(r)_{i(r)-1}T^{i(r)-1}$, $b(r) := b(r)_0 + \ldots + b(r)_{j(r)-1}T^{j(r)-1}$, $c(r) := c(r)_0 + \ldots + c(r)_{i(r)-1}T^{i(r)-1}$ and $d(r) := d(r)_0 + \ldots + d(r)_{j(r)-1}T^{j(r)-1}$. The universal flag $H(r) \subset D(r)$ over $R(r)$ is defined by the $\overline{B}(r)$-span of $T^{i(r)}\alpha(r) + a(r)\alpha(r) + b(r)\beta(r)$ and $T^{j(r)}\beta(r) + c(r)\alpha(r) + d(r)\beta(r)$. Note that the Grassmann variety $\mathrm{Grass}(\mathcal{O}_L \otimes k, \langle\ ,\ \rangle, g, 2g)$ decomposes as the product of the Grassmann varieties $\mathrm{Grass}(\overline{B}(r), \langle\ ,\ \rangle_r, e, 2e)$. Hence,

$$R \cong \widehat{\otimes}_{r=1}^f R(r).$$

5.5. The display over the special fiber and its trivial extension.

Let $\mathscr{P}_0 := \big(P_0, Q_0, F_0, V_0^{-1}, \langle\ ,\ \rangle_0\big)$ be a RM display over k with an $\mathcal{O}_L \otimes k$-linear isomorphism of the Hodge filtration $H_{\mathscr{P}_0} \subset D_{\mathscr{P}_0}$ with $H_0 \subset D_0$, compatible with the pairings on P_0 and D_0. Choose a decomposition $P_0 = \oplus_r \big(B(r)\alpha(r) \oplus B(r)\beta(r)\big)$ as $\mathcal{O}_L \otimes \mathbb{W}(k)$-module so that $P_0/pP_0 = D_0$, $Q_0/pP_0 = H_0$ and $\langle \alpha(r), \beta(r)\rangle_0 = 1$. Note that $F_0 = \oplus F_0(r)$, a direct sum of ^{F}-linear maps, and

$$F_0(r)\left[B(r)\alpha(r) \oplus B(r)\beta(r)\right] \subset \left[B(r+1)\alpha(r+1) \oplus B(r+1)\beta(r+1)\right].$$

The matrix of $F_0(r)$ with respect to the bases $\{\alpha(r), \beta(r)\}$ and $\{\alpha(r+1), \beta(r+1)\}$ is of the form

$$F_0(r) := \begin{pmatrix} T^{j(r)} g_{1,1}(r) & T^{i(r)} g_{1,2}(r) \\ T^{j(r)} g_{2,1}(r) & T^{i(r)} g_{2,2}(r) \end{pmatrix}. \tag{5.3}$$

To state the main theorem of this section we need some more notation. Let $\hat{a}(r)_s$, $\hat{b}(r)_t$, $\hat{c}(r)_s$ and $\hat{d}(r)_t$ be the Teichmüller lifts in $\mathbb{W}\big(R(r)\big)$ of $a(r)_s$, $b(r)_t$, $c(r)_s$ and $d(r)_t$ for $1 \le r \le f$, $0 \le s \le i(r)-1$ and $0 \le t \le j(r)-1$. Define $\hat{a}(r) := \sum_{s=0}^{i(r)-1} \hat{a}(r)_s T^s$, $\hat{b}(r) := \sum_{s=0}^{j(r)-1} \hat{b}(r)_s T^s$, $\hat{c}(r) := \sum_{s=0}^{i(r)-1} \hat{c}(r)_s T^s$ and $\hat{d}(r) := \sum_{s=0}^{j(r)-1} \hat{d}(r)_s T^s$; these are elements of $B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}\big(R(r)\big)$. Let

$$n(r) := \hat{a}(r)\hat{d}(r) - \hat{b}(r)\hat{c}(r) + \hat{a}(r)T^{j(r)} + \hat{d}(r)T^{i(r)}.$$

Lemma 5.5.1. *Let $M(r)$ be the maximal ideal of $R(r)$. Then, the element ${}^F n(r)$ lies in $T^e B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}\big(M(r)\big)$. Let*

$$u_r := 1 + T^{-e}\, {}^F n(r).$$

Then u_r is a unit in $B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}\big(R(r)\big)$.

Proof. Note that $T^e B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}\big(M(r)\big)$ is equal to $pB(r) \otimes_{\mathbb{W}(k)} \mathbb{W}\big(M(r)\big)$. Since multiplication by p coincides with the composition of Verschiebung and Frobenius, we conclude that $p\mathbb{W}\big(M(r)\big)$ consists of the Witt vectors $(a_0, a_1, \ldots)$ with $a_0 = 0$ and $a_i \in {}^F M(r)$. The assertion concerning ${}^F n(r)$ follows. Note that $u(r)$ lies in $1 + B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}\big(M(r)\big)$. It is a unit by Lemma 5.5.2. □

Lemma 5.5.2. *Let S be a k–algebra. Let $v \in B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}(S)$. Assume that the image $\overline{v}$ of v via the composition $B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}(S) \to \overline{B}(r) \otimes_{\mathbb{W}(k)} S \to S$ is a unit. Then, v is a unit.*

Proof. Let Norm on $B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}(S)$ (resp. $\overline{B}(r) \otimes_{\mathbb{W}(k)} S$) be the norm as a $\mathbb{W}(S)$-module (resp. a S-module). Then, v (resp. $\overline{v}$) is is a unit if and only if $\mathrm{Norm}(v)$ (resp. $\mathrm{Norm}(\overline{v})$) is a unit. Hence, we may assume $\mathcal{O}_L = \mathbb{Z}$. Let u be an element of $\mathbb{W}\big(S\big)$ such that $uv = 1 - i$ with $i \equiv 0$ in $\mathbb{W}_1(S) = S$. Note that $i^n \equiv 0$ in $\mathbb{W}_n(S)$. Since $\mathbb{W}(S) = \varprojlim \mathbb{W}_n(S)$, we get that the element $z = \sum_n i^n$ exists in $\mathbb{W}(S)$. Hence, $v(uz) = 1$. □

Let

$$F(r)\colon B(r)\otimes_{\mathbb{W}(k)}\mathbb{W}\big(R\big)\oplus B(r)\otimes_{\mathbb{W}(k)}\mathbb{W}(R(r)) \longrightarrow B(r+1)\otimes_{\mathbb{W}(k)}\mathbb{W}(R)\oplus B(r+1)\otimes_{\mathbb{W}(k)}\mathbb{W}(R)$$

be the ^{F}-linear operator whose matrix with respect to the bases $\{\alpha(r),\beta(r)\}$ and $\{\alpha(r+1),\beta(r+1)\}$ is:

$$F(r):=u(r)^{-1}\times \begin{pmatrix} T^{j(r)}g_{1,1}(r)+{}^F(\hat{d}(r))g_{1,1}-{}^F(\hat{b}(r))g_{1,2} & T^{i(r)}g_{1,2}(r)-{}^F(\hat{c}(r))g_{1,1}+{}^F(\hat{a}(r))g_{1,2}\\ T^{j(r)}g_{2,1}(r)+{}^F(\hat{d}(r))g_{2,1}-{}^F(\hat{b}(r))g_{2,2} & T^{i(r)}g_{2,2}(r)-{}^F(\hat{c}(r))g_{2,1}+{}^F(\hat{a}(r))g_{2,2}\end{pmatrix}. \tag{5.4}$$

5.6. The main results on displays.

Theorem 5.6.1. *Let $P:=P_0\otimes_{\mathbb{W}(k)}\mathbb{W}(R)$ and let $\langle\ ,\ \rangle$ be the base change of $\langle\ ,\ \rangle_0$ via $\mathbb{W}(k)\to\mathbb{W}(R)$. Let Q be the inverse image of H via the projection $P\to D$. Let $F\colon P\to P$ be the ^{F}-linear map whose matrix form with respect to the decomposition $P=\oplus_r B(r)\otimes_{\mathbb{W}(k)}\mathbb{W}(R)\alpha(r)\oplus B(r)\otimes_{\mathbb{W}(k)}\mathbb{W}(R)\beta(r)$ is*

$$\begin{pmatrix} 0 & 0 & \dots & 0 & F(g)\\ F(1) & 0 & \dots & 0 & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & F(g-1) & 0\end{pmatrix},$$

with $F(r)$ given in Equation (5.4). Then, there exists a unique ^{F}-linear homomorphism $V^{-1}:Q\to P$ so that $\mathscr{P}:=\big(P,Q,F,V^{-1},\langle\ ,\ \rangle\big)$ is a RM display. Moreover,

1. *its base change via $R/\mathfrak{m}=k$ coincides with $\mathscr{P}_0$ as RM display;*
2. *$(R,\mathscr{P})$ is the universal pro-representing object and the universal RM display for the moduli problem of deforming $\mathscr{P}_0$ to objects of $\mathfrak{C}_k$ as a RM display;*
3. *the projection $P\to D$ identifies $H_{\mathscr{P}}\subset D_{\mathscr{P}}$ with $H\subset D$ compatibly with the pairings on P and D.*

Proof. Let ψ be the map $P\otimes_{\mathbb{Z}}\mathbb{Q}\to P\otimes_{\mathbb{Z}}\mathbb{Q}$ defined as $\operatorname{diag}\big[\psi(1),\dots,\psi(f)\big]$, where the map $\psi(r)$ is defined with respect to the basis $\{\alpha(r),\beta(r)\}$ by the matrix

$$\begin{pmatrix} 1+\hat{a}(r)T^{-i(r)} & \hat{c}(r)T^{-j(r)}\\ \hat{b}(r)T^{-i(r)} & 1+\hat{d}(r)T^{-j(r)}\end{pmatrix}.$$

Note that F is the composition $F_0\circ\psi^{-1}$ of the ^{F}-linear base change of F_0 to $P=P_0\otimes_{\mathbb{W}(k)}\mathbb{W}(R)$ with the inverse of ψ.

One proves that, indeed, F is well defined. One defines $V^{-1}:=\frac{F}{p}$ on $P\otimes\mathbb{Q}$ and one proves that V^{-1} restricted to Q is well defined, it is compatible with $\langle\ ,\ \rangle$ and $V^{-1}(Q)$ spans P. By definition V^{-1} is compatible with F. See [AG4] for details. Claims (1) and (3) follow immediately from the construction. Claim (2) follows from the following theorem. ◻

Theorem 5.6.2. *Let $\mathscr{P} := \big(P, Q, F, V^{-1}, \langle\ ,\ \rangle\big)$ be a RM display over R and let $\tau\colon D_{\mathscr{P}} \to D$ be an isomorphism as $\mathcal{O}_L \otimes R$-modules, compatible with pairings, such that $\tau\big(H_{\mathscr{P}}\big) = H$ and τ is a horizontal map mod $\mathfrak{m}^2$. Here, we consider the connection on $D_{\mathscr{P}} \otimes_R R/\mathfrak{m}^2$ induced by the fact that $D_{\mathscr{P}}$ is a crystal and we consider on $D \otimes_R R/\mathfrak{m}^2$ the connection having $D_0 \subset D$ as horizontal sections. Then, $(R, \mathscr{P})$ is the universal pro-representing object and the universal RM display for the moduli problem of deforming the special fiber $\mathscr{P}_0$ of $\mathscr{P}$ to local artinian k-algebras as RM display.*

Proof. Let $\mathscr{P}^{\mathrm{uni}} := \big(P^{\mathrm{uni}}, Q^{\mathrm{uni}}, F^{\mathrm{uni}}, (V^{\mathrm{uni}})^{-1}, \langle\ ,\ \rangle^{\mathrm{uni}}\big)$ be the universal RM display deforming the special fiber $\mathscr{P}_0$. By the theory of local models [DP, Thm. 3.3] and the equivalence of categories between deformations of displays and of formal p-divisible groups [Zin, Thm. 9] it exists over R.

Let $\phi\colon \mathrm{Spec}(R) \to \mathrm{Spec}(R)$ be the unique homomorphism such that $\mathscr{P} = \phi^*\big(\mathscr{P}^{\mathrm{uni}}\big)$. Since R pro-represents a Grassmannian moduli problem, we get unique maps $\psi_i\colon \mathrm{Spec}\big(R/\mathfrak{m}^2\big) \to \mathrm{Spec}\big(R/\mathfrak{m}^2\big)$, such that $\psi_1^*\,(H \subset D) \cong (H_{\mathscr{P}^{\mathrm{uni}}} \subset D_{\mathscr{P}^{\mathrm{uni}}})$ and $\psi_1^*(D) \cong D_{\mathscr{P}^{\mathrm{uni}}}$ is horizontal, and ψ_2 such that $\psi_2^*\big(H \subset D\big) = \big(H_{\mathscr{P}} \subset D_{\mathscr{P}}\big)$ and $\psi_2^*(D) \cong D_{\mathscr{P}}$ is horizontal. Moreover, $\psi_1 \circ \phi = \psi_2$ - all the maps appearing being canonical. By [DP, Lem. 3.5] the map ψ_1 is an isomorphism. Hence, ϕ is an isomorphism on tangent spaces.

Let $\mathrm{Gr}(R)$ be the graded ring $\oplus_n \mathfrak{m}^n/\mathfrak{m}^{n+1}$ associated to R. The induced map $\mathrm{Gr}(\phi^\sharp)\colon \mathrm{Gr}(R) \to \mathrm{Gr}(R)$ is then surjective on each graded piece and, hence, by dimension considerations it is injective. Since $\mathrm{Gr}(\phi^\sharp)$ is an isomorphism, we conclude that $\phi^\sharp$ is an isomorphism as well [AtM, Lem. 10.23]. Hence, ϕ is an isomorphism as claimed. ◻

Corollary 5.6.3. *Let p be maximally ramified. Let $x \in \mathfrak{M}$ be a geometric point of type (j, n).*

1. *The deformations to $S_{j'}$, where $j' \leq j$, are parameterized by the closed subscheme defined by the ideal $\langle a_i, b_i, c_i, d_i : 0 \leq i \leq j' - 1\rangle$.*

2. *The deformations to $W_{(j', n')}$, where $j' \leq j$, are parameterized by the closed subscheme of deformations to $S_{j'}$ intersected with the closed subscheme (with the reduced structure) given by the relations $T^{j'+n'} | F^2$.*

6. Some general results concerning strata in the maximally ramified case

6.1. Foliations of Newton polygon strata.

In this section we complete the analysis, started in [AG1], of the strata $\{W_{(j,n)}\}$. For their definition see §3.2. We prove that each stratum $W_{(j,n)}$ is quasi-affine. We proceed as follows. First, by an explicit normalization of the display over the completed local ring of a point of type (j,j), we prove that for every m the p^m-torsion of the universal RM abelian scheme over $W_{(j,j)}$ can be trivialized over a finite cover of $W_{(j,j)}$ (depending on m). Using the "Raynaud trick", we conclude that $W_{(j,j)}$ is quasi-affine. We deduce the quasi-affineness of $W_{(j,n)}$ by showing that it is the image of $W_{(n,n)}$ if $n > \frac{g}{2}$ (resp. $W_{(g-n,n)}$ if $n \leq \frac{g}{2}$) via iterated Hecke correspondences at p. We also describe the analogue of the foliations of the Newton polygon strata introduced by [Oo4] in the Siegel case. Recall that the stratification $\{W_{(j,n)}\}$ refines the Newton polygon stratification; [AG1, Thm. 10.1]. Since the universal RM p-divisible group over $W_{(j,j)}$ is geometrically constant, $W_{(j,j)}$ is the central leaf at any of its points; cf. Definition 6.1.1. The foliation on the loci $W_{(j,n)}$ is then described using the Hecke correspondence linking $W_{(j,n)}$ and $W_{(n,n)}$ if $n > \frac{g}{2}$ (resp. $W_{(g-n,n)}$ if $n \leq \frac{g}{2}$).

Definition 6.1.1. ([Oo4, §2]) *Let $\mathscr{P}$ be a RM display over a perfect field k of positive characteristic p. Let T be a noetherian scheme over k. Let $\underline{A} \to T$ be a RM abelian scheme. Define*

$$\mathscr{C}_{\mathscr{P}}\big(\underline{A} \to T\big)$$

as the subset of T consisting of the geometric points $t \in T$ for which there exists an isomorphism of RM displays between $\mathscr{P} \otimes_{W(k)} W(k(t))$ and the display associated to $\underline{A}_t$.

If $\mathscr{P}$ is the RM display associated to a geometric point $x \in T$, we write $\mathscr{C}_{\underline{A}_x}$ instead of $\mathscr{C}_{\mathscr{P}}$ and we call it the central leaf *at x.*

Note that the Newton polygon and the type, in the sense of [Oo3], of the geometric points of $\mathscr{C}_{\mathscr{P}}$ are those of $\mathscr{P}$ and hence constant. By [Oo4, Thm. 2.2] the set $\mathscr{C}_{\mathscr{P}}$ is a closed subset of the locally closed subscheme of T consisting of the points having the same Newton polygon as $\mathscr{P}$.

Definition 6.1.2. *Let k be a perfect field of characteristic p and let S be a k-algebra. Let $s = t(s)g + r(s)$ with $t(s) \in \mathbb{N}$ and $0 \leq r(s) \leq g-1$. Consider the exact sequence of $\mathbb{W}(S)$-modules*

$$0 \longrightarrow S \xrightarrow{\varphi} \mathbb{W}_{t(s)+1}(S) \longrightarrow \mathbb{W}_{t(s)}(S) \longrightarrow 0.$$

The map φ is the $t(s)$-th power of Verschiebung $r \overset{\varphi}{\mapsto} (0,\dots,0,r)$. It identifies S with the $\mathbb{W}(S)$-module whose additive structure is that of S and multiplication *of* $r \in S$ *by* $a = (a_0, a_1 \ldots) \in \mathbb{W}(S)$ *is given by* $a \cdot r := a_0^{p^t} r$. *The sequence*

$$0 \longrightarrow \mathcal{O}_L \otimes_{\mathbb{Z}} S \stackrel{1\otimes\varphi}{\longrightarrow} \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}_{t(s)+1}(S) \longrightarrow \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}_{t(s)}(S) \longrightarrow 0$$

is an exact sequence of $\mathcal{O}_L \otimes \mathbb{W}(S)$*-modules. Since* S *is of characteristic* p, *we have that* $\mathcal{O}_L \otimes_{\mathbb{Z}} S \cong \mathbb{F}_p[T]/(T^g) \otimes_{\mathbb{F}_p} S$.

Consider the $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}(S)$*-submodule of* $\mathcal{O}_L \otimes_{\mathbb{Z}} S$ *defined by* $I_n := T^n S \oplus \cdots \oplus T^{g-1} S$. *Let*

$$\mathcal{Z}_s(S) := \big(\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}_{t(s)+1}(S)\big)/\big(1 \otimes \varphi(I_{r(s)})\big).$$

By construction we have an exact sequence of $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}(S)$*-modules*

$$0 \longrightarrow S \longrightarrow \mathcal{Z}_{s+1}(S) \longrightarrow \mathcal{Z}_s(S) \longrightarrow 0,$$

where S *is a* $\mathbb{W}(S)$*-module as above and* $\mathcal{O}_L$ *acts on* S *via the quotient* $\mathcal{O}_L/(T)$. *We note that*

$$\mathcal{O}_L \otimes \mathbb{W}(S) = \varprojlim_s \ \mathcal{Z}_s(S).$$

Note that T^g *is equal to* p *up to a unit in* $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p$. *Thus, multiplication by* T^g *on* $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}(S)$ *is, up to a unit, multiplication by* p. *The latter coincides with the composite of Verschiebung and Frobenius on* $\mathbb{W}(S)$. *In particular, if* S *is reduced for every positive integer* i *the kernel of multiplication by* T^i *on* $\mathcal{Z}_{s+i}(S)$ *coincides with the kernel of* $\mathcal{Z}_{s+i}(S) \to \mathcal{Z}_s(S)$.

Remark 6.1.3. *Let* A *be an abelian variety with RM by* $\mathcal{O}_L$ *over a perfect field of characteristic* p, p *totally ramified in* L. *It is proven in* [AG1, Prop. 4.10] *that one can choose an* $\mathcal{O}_L \otimes \mathbb{W}(k)$ *basis* α, β *for the Dieudonné module (or "display the display") of* A *such that, if* A *is not superspecial, Frobenius is given with respect to this basis by a matrix*

$$\begin{pmatrix} T^n & c_3 T^i \\ T^j & 0 \end{pmatrix}, \tag{6.1}$$

with $c_3 \in (\mathcal{O}_L \otimes \mathbb{W}(k))^\times$. *Furthermore, it follows from* [AG1, Prop. 7.2] *that* $W_{(j,j)}$ *is regular of dimension* $g - 2j$ *and that for any geometric point* x *of* $W_{(j,j)}$, *the completed local ring* $\widehat{\mathcal{O}}_{W_{(j,j)},x}$ *is isomorphic to* $k[\![f_j, \ldots, f_{i-1}]\!]$. *Moreover, the isomorphism can be chosen so that Frobenius on the universal display over this ring is of the form*

$$\begin{pmatrix} (1 + w(f_j) + w(f_{j+1})T + \cdots + w(f_{i-1})T^{i-j-1})T^j & c_3 T^i \\ T^j & 0 \end{pmatrix}, \tag{6.2}$$

where $w(f_h)$ *denotes the Teichmüller lift of* f_h.

Remark 6.1.4. *Let* $j \geq g/2$ *be an integer. If* A *is superspecial then, in fact, one can choose the basis for the Dieudonné module of* A *so that the matrix of Frobenius* *is*

$$\begin{pmatrix} 0 & T^i \\ T^j & 0 \end{pmatrix}. \tag{6.3}$$

The locus $W_{(j,g-j)}$ is zero dimensional. Since Frobenius of the Dieudonné module of each of its points has the canonical form described by the matrix in Equation (6.3), it follows that for every $m \in \mathbb{N}$ the $\mathcal{O}_L$-group scheme $A[p^m] \times_{\mathfrak{M}} W_{(j,g-j)}$ is constant.

Proposition 6.1.5. *Let R be an $\mathbb{F}_p$-algebra. Let $\big(P, Q, F, V^{-1}\big)$ be an O_L-display over R such that*

$$F(\alpha) = d\,T^j\alpha + T^j\beta, \qquad F(\beta) = c_3 T^i \alpha, \tag{6.4}$$

where d and c_3 are invertible elements of $\mathcal{O}_L \otimes \mathbb{W}(R)$ and we require $i > j > 0$. Then, there exist ring extensions $R = R_0 \subset \cdots \subset R_s \subset R_{s+1} \subset \cdots$, and elements A_s and B_s in $\mathcal{Z}_s(R_s)$, such that defining the elements of $P \otimes_{\mathcal{O}_L \otimes \mathbb{W}(R)} \mathcal{Z}_s(R_s)$

$$\alpha_s := A_s\alpha + B_s\beta, \qquad \beta_s := \big(dA_s^\sigma - A_s + B_s^\sigma c_3 T^{i-j}\big)\alpha + \big(A_s^\sigma - B_s\big)\beta, \tag{6.5}$$

the following properties hold:

1. *We have*
$$F(\alpha_s) = T^j\alpha_s + T^j\beta_s, \qquad F(\beta_s) = T^i\alpha_s; \tag{6.6}$$
2. *the elements α_s and β_s generate $P \otimes_{\mathcal{O}_L \otimes \mathbb{W}(R)} \mathcal{Z}_s(R_s)$ as a $\mathcal{Z}_s(R_s)$–module;*
3. *A_{s+1} and B_{s+1} map to A_s and B_s respectively, viewing A_s and B_s as lying in $\mathcal{Z}_s(R_{s+1})$ via the inclusion $\mathcal{Z}_s(R_s) \subset \mathcal{Z}_s(R_{s+1})$;*
4. *R_{s+1} is a finite free R_s–module;*
5. *for every $s \in \mathbb{N}$ the extension $R \subset R_s$ satisfies the following universal property. Let S be a reduced R-algebra. Let $\widetilde{\alpha}_{s+i}$ and $\widetilde{\beta}_{s+i}$ be $\mathcal{O}_L \otimes \mathbb{W}(S)$–generators of $P \otimes_{\mathcal{O}_L \otimes \mathbb{W}(R)} \mathcal{Z}_{s+i}(S)$ satisfying (6.6). Then, there exists a unique R–algebra homomorphism $f_s \colon R_s \to S$ such that $f_s(\alpha_s) = \widetilde{\alpha}_{s+i}$ and $f_s(\beta_s) = \widetilde{\beta}_{s+i}$ in $P \otimes_{\mathcal{O}_L \otimes \mathbb{W}(R)} \mathcal{Z}_s(S)$.*

Proof. First of all, we reformulate property (5) in a way which is more convenient for the proof. As remarked above, since S is reduced, the kernel of multiplication by T^i in $\mathcal{Z}_{s+i}(S)$ coincides with the kernel of the reduction map $\mathcal{Z}_{s+i}(S) \to \mathcal{Z}_s(S)$. In particular, it factors via $\mathcal{Z}_s(S)$ and $\mathcal{Z}_s(S)$ embeds in $\mathcal{Z}_{s+i}(S)$ via multiplication by T^i. Thus, property (5) is equivalent to the existence of a unique R–algebra homomorphism $f_s \colon R_s \to S$ such that $T^i f_s(\alpha_s) = T^i\widetilde{\alpha}_{s+i}$ and $T^i f_s(\beta_s) = T^i\widetilde{\beta}_{s+i}$ in $P \otimes_{\mathcal{O}_L \otimes \mathbb{W}(R)} \mathcal{Z}_{s+i}(S)$. This is the actual identity we verify below.

Put formally

$$\alpha_s := A_s\alpha + B_s\beta, \qquad \beta_s := G_s\alpha + H_s\beta. \tag{6.7}$$

Then $F(\alpha_s) = A_s^\sigma F(\alpha) + B_s^\sigma F(\beta) = dA_s^\sigma T^j\alpha + A_s^\sigma T^j\beta + B_s^\sigma c_3 T^i\alpha$. Since $T^j\alpha_s + T^j\beta_s = A_s T^j\alpha + B_s T^j\beta + T^j\beta_s$, the first equality of (6.6) gives that $A_s T^j\alpha + B_s T^j\beta + T^j\beta_s = dA_s^\sigma T^j\alpha + A_s^\sigma T^j\beta + B_s^\sigma c_3 T^i\alpha$. Hence, $T^j\beta_s = G_s T^j\alpha + H_s T^j\beta = \big(dA_s^\sigma - A_s + B_s^\sigma c_3 T^{i-j}\big)T^j\alpha + \big(A_s^\sigma - B_s\big)T^j\beta$, and therefore,

$$T^j G_s = T^j\big(dA_s^\sigma - A_s + B_s^\sigma c_3 T^{i-j}\big), \qquad T^j H_s = T^j\big(A_s^\sigma - B_s\big). \tag{6.8}$$

The second equality of (6.6) now gives

$$\begin{aligned}
A_s T^i\alpha + B_s T^i\beta &= T^i\alpha_s && \text{(by (6.7))}\\
&= F(\beta_s) && \text{(by (6.6))}\\
&= G_s^\sigma F(\alpha) + H_s^\sigma F(\beta) && \text{(by (6.7))}\\
&= \big(dG_s^\sigma + c_3 H_s^\sigma T^{i-j}\big)T^j\alpha + G_s^\sigma T^j\beta && \text{(by (6.4))}\\
&= (d\left[d^\sigma A_s^{\sigma^2} - A_s^\sigma + B_s^{\sigma^2} c_3^\sigma T^{i-j}\right] \\
&\quad + \left[A_s^{\sigma^2} c_3 T^{i-j} - B_s^\sigma c_3 T^{i-j}\right])T^j\alpha \\
&\quad\quad + \big(d^\sigma A_s^{\sigma^2} - A_s^\sigma + B_s^{\sigma^2} c_3^\sigma T^{i-j}\big)T^j\beta && \text{(by (6.8))}
\end{aligned}$$

This is equivalent to the following two equations:

$$T^j \cdot \Big(d^\sigma A_s^{\sigma^2} - A_s^\sigma + B_s^{\sigma^2} c_3^\sigma T^{i-j} - B_s T^{i-j}\Big) = 0, \tag{6.9}$$

$$T^i \cdot \Big(A_s^{\sigma^2} c_3 - B_s^\sigma c_3 - A_s + d\,B_s\Big) = 0. \tag{6.10}$$

For A_s, B_s, G_s and H_s, Equation (6.6) holds if and only if Equations (6.8), (6.9) and (6.10) hold. Put also

$$d^\sigma A_s^{\sigma^2} - A_s^\sigma + B_s^{\sigma^2} c_3^\sigma T^{i-j} - B_s T^{i-j} = 0, \tag{6.11}$$

$$A_s^{\sigma^2} c_3 - B_s^\sigma c_3 - A_s + d\,B_s = 0. \tag{6.12}$$

We construct rings R_s and elements A_s and B_s of $\mathcal{Z}_s(R_s)$, such that if we *choose*

$$G_s = dA_s^\sigma - A_s + B_s^\sigma c_3 T^{i-j}, \qquad H_s = A_s^\sigma - B_s,$$

then properties (1)−(5) of the proposition hold and, moreover, also Equations (6.11) and (6.12) hold.

We proceed by induction on s. Start with $s = 1$. Let $\underline{c}_3$, $\underline{d}$, be the reduction of c_3 and d in $R = \mathcal{Z}_1(R)$. Let

$$R_1 := R\big[A_1, u\big]/\big((\underline{d}A_1^{p-1} - 1)^p, u^{p-1} - \underline{d}\underline{c}_3^{-1}\big).$$

Putting α_1 and β_1 as in the proposition, with A_1 the given element of R_1 and $B_1 := \underline{d}^{-1}A_1 + u$, one checks that Equations (6.11) and (6.12) hold in $\mathcal{Z}_1(R_1) = R_1$.

Furthermore, property (2) is equivalent to requiring that the element

$$\det\begin{pmatrix} A_1 & \underline{d}A_1^p - A_1 \\ B_1 & A_1^p - B_1 \end{pmatrix} = A_1\big(A_1^p - B_1\big) - B_1\big(\underline{d}A_1^p - A_1\big) = A_1^p(A_1 - \underline{d}B_1)$$

is invertible. This holds since A_1 and $\underline{d}B_1 - A_1 = u$ are invertible.

Let S be an R–algebra as in property (5) with $s = 1$. In particular, Equations (6.9) and (6.10) have solutions $\widetilde{A}_{1+i}$ and $\widetilde{B}_{1+i}$ in $\mathcal{Z}_{1+i}(S) = \mathcal{O}_L/(T^{i+1}) \otimes_{\mathbb{Z}} S$. Note that, using Equation (6.9), Equation (6.10) becomes $T^i\big(-\underline{c}_3(\widetilde{B}_{1+i} - \underline{d}^{-1}\widetilde{A}_{1+i})^p + \underline{d}(\widetilde{B}_{1+i} - \underline{d}^{-1}\widetilde{A}_{1+i})\big) = 0$. Since $\widetilde{\alpha}_{1+i}$ and $\widetilde{\beta}_{1+i}$ generate $P \otimes_{\mathcal{O}_L\otimes\mathbb{W}(R)} \mathcal{Z}_{1+i}(S)$, a similar argument using Equation (6.8) gives that $T^i\cdot\widetilde{A}_{1+i}^p(\widetilde{A}_{1+i} - \underline{d}\widetilde{B}_{1+i})$ is T^i times a unit. Thus, we can define $f_1\colon R_1 \to S$ as the R–algebra homomorphism satisfying $T^i\widetilde{A}_{1+i} = T^i f_1(A_1)$ and $T^i\widetilde{B}_{1+i} = T^i\underline{d}^{-1}\widetilde{A}_{1+i} + T^i f_1(u)$. This concludes the base step of the induction.

Assume that the induction hypothesis holds for a given $s \in \mathbb{N}$. Let A_s' and B_s' be elements in $\mathcal{Z}_{s+1}(R_s)$ reducing to A_s and B_s respectively in $\mathcal{Z}_s(R_s)$. Let R_s' be the polynomial ring $R_s\big[\lambda, \mu\big]$. Let $\lambda_s := (0, \ldots, 0, \lambda)$ and $\mu_s := (0, \ldots, 0, \mu)$ in $\mathrm{Ker}\big(\mathcal{Z}_{s+1}(R_s') \to \mathcal{Z}_s(R_s')\big)$. Let $A_{s+1} := A_s' + \lambda_s$ and $B_{s+1} := B_s' + \mu_s$. Then, Equation (6.11) becomes

$$d^\sigma \lambda^{p^2} - \lambda^p + P_s = 0, \tag{6.13}$$

where P_s is the element of R_s' defined by $P_s = d^\sigma A_s'^{\sigma^2} - A_s'^{\sigma} + \big(B_s'^{\sigma^2} c_3^\sigma - B_s'\big)T^{i-j} + \big(\mu_s^{\sigma^2} c_3^\sigma - \mu_s\big)T^{i-j}$. Since T^{i-j} kills $\mathrm{Ker}\big(\mathcal{Z}_{s+1}(R_s') \to \mathcal{Z}_s(R_s')\big)$, we have $P_s = d^\sigma A_s'^{\sigma^2} - A_s'^{\sigma} + \big(B_s'^{\sigma^2} c_3^\sigma - B_s'\big)T^{i-j}$. For the same reason P_s is independent of the choice of B_s'. Finally, Equation (6.12) becomes

$$\mu^p c_3 - d\,\mu + Q_s = 0, \tag{6.14}$$

where Q_s is the element of R_s' defined by $Q_s = (B_s')^p c_3 - dB_s' - A_{s+1}^{\sigma^2} c_3 + A_{s+1}$. Equations (6.13), (6.14) define an ideal J_s in R_s'. Let

$$R_{s+1} := R_s'/J_s.$$

The ring R_{s+1} is an extension of R_s, finite and free as R_s-module. Define α_{s+1} and β_{s+1} as in the statement of the proposition. By construction Equation (6.6) holds and A_{s+1} and B_{s+1} reduce to A_s and B_s in $\mathcal{Z}_s(R_{s+1})$. Property (2) is equivalent to the invertibility of

$$\det\begin{pmatrix} A_{s+1} & d\,A_{s+1}^\sigma - A_{s+1} + B_{s+1}^\sigma c_3 T^{i-j} \\ B_{s+1} & A_{s+1}^\sigma - B_{s+1} \end{pmatrix}.$$

Since such element is invertible in $\mathcal{Z}_1(R_{s+1})$, we deduce from 5.5.2 that it is indeed invertible.

Let S be an R–algebra as in property (5) with $s+1$. Using the induction hypothesis on R_s, we know that there exist a unique map of R–algebras $f_s\colon R_s \to S$ such that $f_s(\alpha_s) = \widetilde{\alpha}_{s+1+i}$ and $f_s(\beta_s) = \widetilde{\beta}_{s+1+i}$ in $P \otimes_{\mathcal{O}_L\otimes\mathbb{W}(R)} \mathcal{Z}_s(S)$. Let $\widetilde{\alpha}_{s+1+i} = \widetilde{A}_{s+1+i}\alpha + \widetilde{B}_{s+1+i}\beta$. Equations (6.9) and (6.10) hold for $\widetilde{A}_{s+1+i}$ and $\widetilde{B}_{s+1+i}$. Thus, there exists a unique map of R–algebras $f_{s+1}\colon R_{s+1}\to S$ whose restriction to R_s is f_s and such that $T^i\big(f_{s+1}(A_{s+1})\big)=T^i\widetilde{A}_{s+1+i}$ and $T^i\big(f_{s+1}(B_{s+1})\big)=T^i\widetilde{B}_{s+1+i}$. By the reformulation of property (5) given at the beginning of the proof one concludes that R_{s+1}, α_{s+1} and β_{s+1} satisfy property (5). □

Until the end of this subsection we assume that the base field k over which the moduli space $\mathfrak{M}$ lives is *algebraically closed*.

Corollary 6.1.6. *Let $\underline{A}\to W_{(j,j)}$ be the universal RM abelian scheme. Let $y\in W_{(j,j)}$ be k–valued point. For every $m\in\mathbb{N}$ there exists a scheme $W^{[m]}_{(j,j)}$ finite and dominant over $W_{(j,j)}$ such that $A[p^m]\times_{W_{(j,j)}}W^{[m]}_{(j,j)}\cong A_y[p^m]\times_k W^{[m]}_{(j,j)}$.*

Proof. The case $j=0$, corresponding to the ordinary case, is easy and is left for the reader. The case $i=j=g/2$, occurring only for g even, is covered by Remark 6.1.4, where we define $W^{[m]}_{(j,j)}:=W_{(j,j)}$. We now assume $i>j>0$.

For every $n\in\mathbb{N}$, the functor, associating to a scheme T over $W_{(j,j)}$ the group of isomorphisms $\operatorname{Isom}\big(A[p^n]\times_{W_{(j,j)}}T, A_y[p^n]\times_k T\big)$ as group schemes over T endowed with an O_L-action, is represented by a scheme $\operatorname{Isom}(p^n)$, affine and of finite type over $W_{(j,j)}$ (see [Oo4, Lem. 2.4]). Let $W^{[m]}_{(j,j)}$ be the scheme theoretic image of $\operatorname{Isom}(p^{m+2})\to\operatorname{Isom}(p^m)$. It follows from Proposition 6.1.5 that for every geometric point x of $W_{(j,j)}$ one can trivialize Frobenius on the Dieudonné module of $A_x[p^{m+2}]$. Hence, one can trivialize the Dieudonné module of $A_x[p^m]$. We conclude that the reduced fiber of $W^{[m]}_{(j,j)}$ over x is non-empty. Using Dieudonné theory and properties (4) and (5) of Proposition 6.1.5 for $R=S=k$ and $s=g(m+2)$, we deduce that the reduced fibers of $W^{[m]}_{(j,j)}\to W_{(j,j)}$ are of finite cardinality. Thus, $W^{[m]}_{(j,j)}$ is quasi-finite over $W_{(j,j)}$.

We now apply the valuative criterion of properness to prove that the morphism $W^{[m]}_{(j,j)}\to W_{(j,j)}$ is proper. Let $\mathscr{P}_y$ be the RM display associated to A_y. Let R be a complete dvr which is also a k-algebra. Let K be its fraction field. Suppose we are given a map $\phi\colon\operatorname{Spec}(R)\to W_{(j,j)}$ and a K–valued point of $W^{[m]}_{(j,j)}$ over it. It follows from Remark 6.1.3 that the Frobenius of the RM display $\mathscr{P}$ associated to the formal p–divisible group $\mathscr{G}$ over $\operatorname{Spec}(R)$ defined by ϕ, admits a $\mathcal{O}_L\otimes\mathbb{W}(R)$-basis α and β such that Frobenius is of the form given in Equation (6.4). Using Dieudonné theory and our assumption, the base change of $\mathscr{P}$ to an algebraic closure K^{alg} of K admits a $\mathcal{O}_L\otimes\mathbb{W}(K^{\mathrm{alg}})$–basis $\widetilde{\alpha}$ and $\widetilde{\beta}$ such that Frobenius satisfies $F(\widetilde{\alpha})\equiv T^j\widetilde{\alpha}+T^j\widetilde{\beta}$ and $F(\widetilde{\beta})\equiv T^i\widetilde{\alpha}$ modulo p^{m+1}. We deduce from properties (4) and (5) of Proposition 6.1.5, applied to K^{alg} and $s=g(m+2)$, that the change of basis from $\{\alpha,\beta\}$ to $\{\widetilde{\alpha},\widetilde{\beta}\}$ can be realized, at least over $\mathcal{Z}_{g(m+1)}(R')$, for some integral extension $R'\subset K^{\mathrm{alg}}$ of R. Thus, we conclude that $\mathscr{P}\otimes_{\mathcal{O}_L\otimes W(R)}\mathcal{Z}_{gm}(R')$ is equal to $\mathscr{P}_y\otimes_{\mathcal{O}_L\otimes W(k)}\mathcal{Z}_{gm}(R')$ and so $\mathscr{G}[p^m]\otimes_R R'\cong A_y[p^m]\otimes_k R'$. Note that this R' point of $W^{[m]}_{(j,j)}$ factors through K, hence through R. Thus, the morphism $W^{[m]}_{(j,j)}\to W_{(j,j)}$ is proper and quasi-finite, hence finite [EGA IV, Thm. 8.2.1]. □

Corollary 6.1.7. *The RM p-divisible group associated to the universal abelian scheme over $W_{(j,j)}$ is geometrically constant. In particular, the central leaf $\mathscr{C}_{\underline{A}_x}$ at any point x of $W_{(j,j)}$ coincides with $W_{(j,j)}$.*

Proof. Let x be a geometric point of $W_{(j,j)}$. Let $\mathcal{G}_x$ be the p-divisible group defined by x. The case $i = j = g/2$, occurring only for g even, is covered by Remark 6.1.4. The case $j = 0$ is the case of ordinary abelian varieties, where the result is well known. Assume now $i > j > 0$. Apply Proposition 6.1.5 to the $\mathcal{O}_L$-display over $R = k(x)$ associated to $\mathcal{G}_x$. The $k(x)$–algebras R_s are finite as $k(x)$–modules. Therefore, since $k(x)$ is an algebraically closed field, there exist compatible sections $R_s \to k(x)$. Note that $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}\big(k(x)\big) = \varprojlim \mathcal{Z}_s\big(k(x)\big)$. Hence, $\alpha := \varprojlim \alpha_s$ and $\beta := \varprojlim \beta_s$ are well defined and form an $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}\big(k(x)\big)$-basis of the Dieudonné module of $\mathcal{G}_x$ such that $F(\alpha) = T^j\alpha + T^j\beta$ and $F(\beta) = T^i\alpha$. Since $F \circ V = p$, we deduce that also Verschiebung V has a canonical form with respect to the basis $\{\alpha, \beta\}$ independent of x. Since the category of connected p-divisible groups and the category of displays are equivalent over perfect fields, we conclude. □

Corollary 6.1.8. *Let $0 \leq j \leq g/2$. The scheme $W_{(j,j)}$ is quasi-affine.*

Proof. If $j = g/2$, then $\dim(W_{(j,j)}) = 0$ and $W_{(j,j)}$ consists of superspecial points. The corollary is trivial in this case. Suppose $j < g/2$. By Corollary 6.1.6 there exists a finite covering $W^{[1]}_{(j,j)}$ of $W_{(j,j)}$ over which the p-torsion of the universal RM abelian scheme can be trivialized. It follows from Raynaud's trick that the pull-back of the Hodge bundle to $W^{[1]}_{(j,j)}$ is torsion [Oo3, §4]. Since the Hodge bundle is ample on $\mathfrak{M}$, it follows that $W^{[1]}_{(j,j)}$ is quasi-affine, hence so is $W_{(j,j)}$. □

Let α_p be the group scheme over k defined as the kernel of Frobenius on the additive group $\mathbb{G}_{a,k}$. We make $\mathcal{O}_L$ act on it via its quotient $\mathcal{O}_L/T = \mathbb{F}_p$.

Proposition 6.1.9. *Let $0 \leq m \leq j \leq g/2$. Let j' be either j or $g - j$. There exists a smooth connected affine scheme U_m over k, of dimension m, and a finite surjective map*

$$\psi_m \colon W^{[m]}_{(j,j')} \times_k U_m \to W_{(j-m,j')},$$

such that:

- *for every $u \in U_m(k)$ the image of $W^{[m]}_{(j,j')} \times \{u\}$ is contained in the central leaf through any point of $\psi_m\left(W^{[m]}_{(j,j')} \times \{u\}\right)$;*
- *for every $s \in W^{[m]}_{(j,j')}(k)$ the image of $\{s\} \times U_m$ is the image of $\underline{A}_s$ via iterated α_p-Hecke correspondences.*

Proof. Let $s \in W_{(j,j')}$. Define the schemes U_n for $0 \leq n \leq j$ by induction on n as follows. Let $U_0 := \mathrm{Spec}(k)$. Suppose that U_n has been defined and it is a smooth, connected affine scheme of dimension n and that every $u \in U_n$ defines an iterated α_p-quotient $\underline{A}_s \to \underline{A}_u$ of invariants $(j-n, j')$. Let U_{n+1} be the scheme over U_n whose fiber over any geometric point $u \in U_n$ is the subscheme of $\mathrm{Hom}_{\mathcal{O}_L}(\alpha_p, \underline{A}_u[p])$ of those maps for which the quotient $\underline{A}_u/\alpha_p$ has invariants $(j-(n+1), j')$. By [AG1, Prop. 6.6, Prop. 8.7] the morphism $U_{n+1} \to U_n$ is an affine bundle and the fiber over u is a non–empty open subscheme of $\mathbb{P}^1_{k(u)}$. It follows that U_{n+1} is a smooth, connected affine scheme of dimension $n+1$.

Fix m. Define the map

$$\psi_m \colon W^{[m]}_{(j,j')} \times_k U_m \longrightarrow \mathfrak{M}$$

as follows. By Corollary 6.1.6 or Remark 6.1.4 we have a canonical isomorphism $\tau_m \colon A[p^m] \times_{\mathfrak{M}} W^{[m]}_{(j,j')} \cong A_s[p^m] \times W^{[m]}_{(j,j')}$. View U_m as classifying suitable subgroup schemes of $A_s[p^m]$. Then, ψ_m is the unique map such that the pull–back of the universal RM abelian scheme via ψ_m coincides with the quotient of $A \times_{\mathfrak{M}} (W^{[m]}_{(j,j')} \times_k U_m)$ by the inverse image via τ_m of the tautological subgroup scheme of $A_s[p^m] \times_k U_m$ defined by U_m. Note that such a quotient is a RM abelian scheme by [AG1, Cor. 3.2]; in particular, the definition of ψ_m makes sense. By construction, the image of ψ_m lies in $W_{(j-m,j')}$.

To conclude it suffices to prove that ψ_m is finite and surjective. We proceed by induction on m. By Corollary 6.1.7, since ψ_0 is the identity, the proposition is true for $m=0$. Suppose that ψ_{m-1} is finite and surjective. Consider the diagram

$$\begin{array}{ccccc}
W^{[m]}_{(j,j')} \times_k U_m & \xrightarrow{\ \delta\ } & \pi_2^{-1}\left(W_{(j-m,j')}\right) \cap \pi_1^{-1}\left(W_{(j-m+1,j')}\right) & \xrightarrow{\ \pi_2\ } & W_{(j-m,j')} \\
\Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\pi_1} & & \\
W^{[m-1]}_{(j,j')} \times_k U_{m-1} & \xrightarrow{\ \psi_{m-1}\ } & W_{(j-m+1,j')}, & &
\end{array}$$

where γ is the product of the natural maps $\gamma_1 \colon W^{[m]}_{(j,j')} \to W^{[m-1]}_{(j,j')}$, $\gamma_2 \colon U_m \to U_{m-1}$ and δ is the unique morphism making the diagram commute and satisfying $\pi_2 \circ \delta = \psi_m$. By construction of U_m, for every point $s=(s_1,s_2) \in W^{[m-1]}_{(j,j')} \times_k U_{m-1}$ and any point t in the finite scheme $\gamma_1^{-1}(s_1)$ the map $\{t\} \times_k \gamma_2^{-1}(s_2) \to \pi_1^{-1}(\psi_m(s))$ is an isomorphism. Hence, δ is quasi–finite, proper (by the valuative criterion) and surjective.

By [AG1, Lem. 8.6], $\pi_1\left(\pi_2^{-1}\left(W_{(j-m,j')}\right)\right)$ has invariants $(j-m+1, j')$ and, if $j-m>0$, also $(j-m-1, j')$. Since the maps π_1 and π_2 are proper by [AG1, Lem. 8.4] and the intersection with $\pi_1^{-1}\left(W_{(j-m+1,j')}\right)$ of the fiber of π_2 over a point of $W_{(j-m,j')}$ is non–empty and finite by [AG1, Prop. 6.6], we conclude that the composite $\psi_m = \pi_2 \circ \delta$ is quasi–finite, proper and surjective as claimed. □

Corollary 6.1.10. *For every (j,n) the scheme $W_{(j,n)}$ is quasi-affine.*

Proof. By Corollary 6.1.8 the claim holds holds for the loci $W_{(n,n)}$. The locus $W_{(j,g-j)}$ is zero dimensional and, hence, quasi affine. By Proposition 6.1.9

the locus $W_{(j,n)}$ is the image via a finite map of a quasi-affine scheme. Hence, the conclusion. □

6.2. Connectedness of $\mathbb{T}_0$, $\mathbb{T}_1$ and $\mathbb{T}_2$.

Definition 6.2.1. *Let $0 \leq a \leq g$ be an integer. Let $\mathbb{T}_a$ be the closed subscheme of $\mathfrak{M}$ defined by*

$$\mathbb{T}_a := \{[\underline{A}] \in \mathfrak{M}(k) | a(\underline{A}) \geq a\} .$$

Remark 6.2.2. *By* [AG1, Lem. 4.12] *we have $\mathbb{T}_a = \amalg_{(j,n)} W_{(j,n)}$ where the union is taken over all pairs of integers (j,n) such that $0 \leq j \leq \frac{g}{2}$ and $j \leq n \leq g-j$ and $a \leq j+n$. It also follows from op. cit. that $\mathbb{T}_a$ has dimension $g-a$.*

Theorem 6.2.3. *Assume that $g > 1$. The intersection of $\mathbb{T}_1$ with any irreducible component of $\mathfrak{M}$ is connected. The same holds for $\mathbb{T}_2$ if $g > 2$.*

Proof. Suppose $g > 1$. Then, $\mathbb{T}_1$ is the complement of the ordinary locus in $\mathfrak{M}$. Hence, it is the zero locus of the Hasse invariant h. Since h is a section of the determinant of the Hodge bundle over $\mathfrak{M}$, and the Hodge bundle is ample, it follows that $\mathbb{T}_1$ is connected (cf. [Har, Cor. III.7.9]).

Assume now that $g > 2$. Let $\mathscr{C}$ be the set of connected components of the intersection of $\mathbb{T}_2$ with an irreducible component of $\mathfrak{M}$. Let $\pi_1, \pi_2 \colon \mathfrak{N} \to \mathfrak{M}$ be as in § 2.2. The Hecke correspondence $\pi_2 \circ \pi_1^{-1}$ preserves properties such as being closed, or being irreducible, or being connected, for closed subschemes not intersecting the non-singular ($j = 0$) locus of $\mathfrak{M}$, see [AG1, Prop. 8.7]. For every (j,n), it sends an irreducible component of $W_{(j,n)}$ surjectively to the union of irreducible components of loci $W_{(j',n')}$ with (j',n') in a given set $\Lambda(j,n)$ depending only on (j,n) [AG1, Prop. 8.10]. Moreover, for every $(j',n') \in \Lambda(j,n)$ we have $j' + n' \geq j + n - 1$. The Hecke correspondence has the additional property of sending each component of $\mathfrak{M}$ into a single component of $\mathfrak{M}$.

Fix a component $C \in \mathscr{C}$. By Remark 6.2.2, the irreducible components of C consist of irreducible components of strata $W^c_{(j,n)}$ with $j + n \geq 2$. We conclude that locus $\pi_2\big(\pi_1^{-1}(C)\big)$ is closed and connected, it lies in $\mathbb{T}_1$ and its irreducible components consist of union of irreducible components of loci $W^c_{(j,n)}$ for suitable pairs (j,n) with $j + n \geq 1$.

Suppose that $|\mathscr{C}| > 1$. Since $\pi_2\big(\pi_1^{-1}(\mathbb{T}_2)\big) = \mathbb{T}_1$, the irreducibility of $\mathbb{T}_1$ in each component of $\mathfrak{M}$ implies that there exist distinct connected components C_1 and C_2 such $E := \pi_2\big(\pi_1^{-1}(C_1)\big) \cap \pi_2\big(\pi_1^{-1}(C_2)\big)$ is non-empty. If two irreducible components of the loci $W_{(j,n)}$ and $W_{(j',n')}$ intersect, then $(j,n) = (j',n')$ and they must coincide, because $W_{(j,n)}$ is smooth [AG1, Cor. 7.4]. Hence, E is closed and consists of irreducible components of loci of type $W_{(j,n)}$ for suitable (j,n) with $j + n \geq 1$. By Corollary 6.1.10 the loci $W_{(j,n)}$ do not contain any complete curve. We conclude that E contains a point $[\underline{A}]$ of type (j,n) with $j + n \geq 3$ and $j \geq 1$. Note that $[\underline{A}^\vee]$ is of type (j,n) by [AG1, Lem. 8.5] and $\pi_2\big(\pi_1^{-1}(\underline{A}^\vee)\big)$

lies in $\mathbb{T}_2$. Hence, its image $\pi_2\big(\pi_1^{-1}(\underline{A}^\vee)\big)^\vee$ via the map $[\underline{A}] \mapsto [\underline{A}^\vee]$ lies in $\mathbb{T}_2$. Since $j \neq 0$ the image $\mathscr{I}$ consists of a Moret-Bailly family. In particular, it is connected. We show that $\mathscr{I}$ connects C_1 to C_2 in $\mathbb{T}_2$.

For $i = 1, 2$ let $[\underline{A}_i]$ be a point of C_i and let $H_i \subset A_i$ be an $\mathcal{O}_L$-invariant subgroup scheme of rank p such that $\underline{A} \cong \underline{A}_i/H_i$. Then, the moduli point corresponding to $\underline{A}^\vee/H_i^\vee \cong \underline{A}_i^\vee$ lie in $\pi_2\big(\pi_1^{-1}(\underline{A}^\vee)\big)$. Hence, $[\underline{A}_1]$ and $[\underline{A}_2]$ lie in the connected subscheme $\pi_2\big(\pi_1^{-1}(\underline{A}^\vee)\big)^\vee$ of $\mathbb{T}_2$. This contradicts the assumption that C_1 and C_2 were distinct. □

Remark 6.2.4. *The argument in the proof of Theorem* 6.2.3 *shows that if a is odd and $\mathbb{T}_a$ is connected, then $\mathbb{T}_{a+1}$ is connected. This is used in the proof in the claim that $\pi_2(\pi_1^{-1}(\mathbb{T}_{a+1})) = \mathbb{T}_a$; a claim which is false for a even, cf. Diagram B in* § 3.2. *It is an interesting question to know whether the loci $\mathbb{T}_a$ are connected for all $a \leq g-1$ or not. An affirmative answer would have strong consequences (perhaps too strong).*

6.3. Irreducibility results.

The singularity strata S_j were defined in § 3.2.

Lemma 6.3.1. *Let $g/2 \geq s \geq j \geq 0$ be integers. Let $x \in S_s$. The completed local ring $\widehat{\mathcal{O}}_{S_j,x}$ of S_j at x is a complete intersection, regular in codimension* 2. *In particular, $\widehat{\mathcal{O}}_{S_j,x}$ is a normal domain.*

Proof. One deduces as in [DP, §4.3], cf. § 4.3.2, that the completed local ring of $W^c_{(j,j)}$ at x has the presentation $k[\![a,b,c,d]\!]/(ad - bc + aT^s + dT^{g-s})$ with $a := a_jT^j + \ldots + a_{g-s-1}T^{g-s-1}$, $b := b_jT^j + \ldots + b_{s-1}T^{s-1}$, $c := c_jT^j + \ldots + c_{g-s-1}T^{g-s-1}$ and $d := d_jT^j + \ldots + d_{s-1}T^{s-1}$. Hence, $\widehat{\mathcal{O}}_{S_j,x}$ is defined by $g-2j$ equations in $2g-4j$ variables. By [DP, §4.2] the dimension of $\widehat{\mathcal{O}}_{S_j,x}$ is $g-2j$. Hence, $\widehat{\mathcal{O}}_{S_j,x}$ is a complete intersection and, in particular, Cohen-Macaulay. By loc. cit. $\widehat{\mathcal{O}}_{S_j,x}$ is smooth in codimension 2. Using Serre's criterion for normality we deduce that $\widehat{\mathcal{O}}_{S_j,x}$ is a normal domain. □

Corollary 6.3.2. *For every j, the irreducible components of $W^c_{(j,j)}$ are disjoint.*

Proof. Recall that $S_j = W^c_{(j,j)}$. The Lemma implies that for every $x \in W^c_{(j,j)}$ the ring $\widehat{\mathcal{O}}_{S_j,x}$ is a domain. In particular, $\mathcal{O}_{S_j,x}$ is a domain. Hence, there exists only one irreducible component of $W^c_{(j,j)}$ containing x. □

Proposition 6.3.3. *Let $g > 2$. Every irreducible component of $\mathfrak{M}$ contains exactly one irreducible component of the non-ordinary locus $\mathbb{T}_1 = W^c_{(0,1)}$. The same holds for the locus $W^c_{(1,1)}$.*

Proof. By Theorem 6.2.3, every irreducible component of $\mathfrak{M}$ contains exactly one connected component of $W^c_{(0,1)}$. Let $x \in W^c_{(0,1)}$. The completed local ring of $\mathfrak{M}$ at x is Cohen-Macaulay of dim g. Hence, the completed local ring of $W^c_{(0,1)}$ at x is Cohen-Macaulay of dim $g-1$ by [Eis, Prop. 18.13].

Let C be a connected component of $W^c_{(0,1)}$. Let $\{T_i\}$ be the set of irreducible components of C. Assume its cardinality is > 1. Let Z be the union of all the intersections $T_i \cap T_j$ for $i \neq j$. Then, $C \backslash Z$ is disconnected. Hence, by Hartshorne's connectedness theorem, see [Eis, Thm. 18.12], there must exist indices i and j and an irreducible component T of $T_i \cap T_j$ of codimension 1 in C and, hence, of dimension $g-2$. Since the locus $\cup_n W_{(0,n)}$ is smooth, T consists of points with singularity index > 0. Since the types (j,n) define a stratification and $W_{(j,n)}$ is pure dimensional of dimension $< g-2$ for $j > 0$ and $n > 1$, T consists of a full irreducible component of the locus $W^c_{(1,1)}$. Hence, it contains a full component of the locus $W^c_{(1,2)}$. By Lemma 6.3.4 below, the nilradical of the completed local ring $\widehat{\mathcal{O}}_{W^c_{(0,1)},x}$ at a closed point x of type $(1,2)$ is a prime ideal. This implies that the prime ideals defined by T_i and T_j in the local ring of the locus $W^c_{(0,1)}$ at a closed point of $W_{(1,2)} \cap T_i \cap T_j$ are equal. Hence, $T_i = T_j$. Contradiction. This proves the first part of the proposition.

Since $\pi_2\big(\pi_1^{-1}(W^c_{(0,1)})\big) = W^c_{(1,1)}$, the second claim follows. □

Lemma 6.3.4. *Let $g > 2$. Let x be a closed point of $W_{(1,2)}$. Let $\mathfrak{D} = W^c_{(0,1)}$ be the non-ordinary locus of $\mathfrak{M}$. Then, the nilradical of the completed local ring $\widehat{\mathcal{O}}_{\mathfrak{D},x}$ of $\mathfrak{D}$ at x is a prime ideal.*

Proof. By § 4.3.2 the completed local ring $\widehat{\mathcal{O}}_{\mathfrak{M},x}$ of $\mathfrak{M}$ at x is isomorphic to the quotient of the ring $k[\![a_0,\dots,a_{g-2},b_0,c_0,\dots,c_{g-2},d_0]\!]$ by the relations $ad-bc+aT+dT^{g-1}=0$, viz.,

$$\begin{aligned} a_0d_0 - b_0c_0 &= 0, \\ a_id_0 + a_{i-1} - b_0c_i &= 0, \qquad 1 \le i \le g-2, \\ a_{g-2} + d_0 &= 0. \end{aligned}$$

Eliminating the variables a_i, using these equations, we get

$$\widehat{\mathcal{O}}_{\mathfrak{M},x} \cong k[\![b_0,c_0,\dots,c_{g-2},d_0]\!]/ \\ \big(b_0c_0 - d_0b_0c_1 + d_0^2b_0c_2 - d_0^3b_0c_3 + \dots + (-1)^{g-2}d_0^{g-2}b_0c_{g-2} + (-1)^{g-2}d_0^g\big).$$

The equations of the non-ordinary locus can be deduced as in § 9.1.1 and coincide with equations (Eq1)–(Eq4) given there with $a_0 := b_0\big(c_1 - d_0c_2 + d_0^2c_3 + \dots - (-1)^{g-2}d_0^{g-3}c_{g-2}\big) - (-1)^{g-2}d_0^{g-1}$. If $d_0 = 0$, then a power of b_0 and c_0 is zero. The reduced ring coincides with the completion of $W^c_{(1,1)}$ at x. If $b_0 = 0$ in $\widehat{\mathcal{O}}_{\mathfrak{D},x}[d_0^{-1}]$, then $a_0 = 0$ and $c_0 = 0$. It follows that $d_0 = 0$ (contradiction). Let $h := c_1 - d_0c_2 + d_0^2c_3 + \dots - (-1)^{g-2}d_0^{g-3}c_{g-2}$. Then, $a_0 = hb_0 - (-1)^{g-2}d_0^{g-1}$. As in

§9.1.1 the lemma is reduced to proving that there exists a unique minimal prime ideal associated to the ideal I in $k[\![b_0,c_0,\ldots,c_{g-2},d_0]\!][b_0^{-1},d_0^{-1}]$ defined by

- $b_0^{p+1}+hb_0d_0^p-(-1)^{g-2}d_0^{p+g-1}=0$;
- $b_0c_0-hb_0d_0+(-1)^{g-2}d_0^g=0$.

Consider the ideal J in the ring $k[\![b_0,c_0,\ldots,c_{g-2},d_0]\!][b_0^{-1},d_0^{-1}]$ defined by

- $b_0^p+c_0d_0^{p-1}=0$ (obtained dividing by b_0 the sum of the first equation and the second equation multiplied by d_0^{p-1});
- $b_0^p\big(c_0^p-d_0^ph^p\big)+(-1)^{pg}d_0^{pg}=0$ (obtained by raising to the p–th power the second equation).

Then, the minimal primes associated to I and to J in $k[\![b_0,c_0,\ldots,c_{g-2},d_0]\!][b_0^{-1},d_0^{-1}]$ are the same. We can write the second equation as $c_0(c_0-d_0h)^p=(-1)^{pg}d_0^{pg-p+1}$. Let $f(X):=X^{p+1}-h^pX-(-1)^gd_0^{p(g-2)}=0$. Define the rings

$$R_0:=k[\![c_1\ldots,c_{g-2},d_0]\!],\quad R_1:=R_0[X]/\big(f(X)\big),\quad R_2:=R_1[b_0]/(b_0^p+Xd_0^p).$$

Since R_2 is $(d_0X,c_1,\ldots,c_{g-2},d_0,b_0)$–adically complete and separated, the homomorphism of $k[\![c_1,\ldots,c_{g-2},d_0]\!]$–algebras from $k[\![b_0,c_0,\ldots,c_{g-2},d_0]\!][b_0^{-1},d_0^{-1}]/J$ to $R_2[b_0^{-1},d_0^{-1}]$ given by $c_0\mapsto d_0X$ and $b_0\mapsto b_0$ is well defined and it is an isomorphism. It therefore suffices to prove that the nilradical of R_2 is prime.

Let P be a prime ideal of R_1 containing 0. Then,

- either X is not a p–th power in $\mathrm{Frac}(R_1/P)$ and then, since R_2 is a flat R_1–algebra, it follows that PR_2 is a prime ideal of R_2; In particular, if P is minimal in R_1 then PR_2 is minimal in R_2.
- or X is a p–th power in $\mathrm{Frac}(R_1/P)$ and then $Xd_0^p=t^p$ for some $t\in\mathrm{Frac}(R_1/P)$. In this case let P_2 be a minimal prime ideal of R_2 containing P. By the going down theorem we must have $P_2\cap R_1=P$. Hence, P_2 defines a prime ideal in $(R_2/P)\otimes_{R_1}\mathrm{Frac}(R_1/P)\cong\mathrm{Frac}(R_1/P)[b_0]/(b_0+t)^p$. Hence, P_2 must be the kernel of $R_2\to R_2/P\to\mathrm{Frac}(R_1/P)$ the latter map being $b_0\mapsto -t$. Hence, P_2 is unique.

In any case the map $\mathrm{Spec}(R_2)\to\mathrm{Spec}(R_1)$ defines a one to one correspondence between the irreducible components of $\mathrm{Spec}(R_2)$ and those of $\mathrm{Spec}(R_1)$. Therefore, it suffices to prove that the nilradical of R_1 is prime. We show that in fact R_1 is a domain.

Assume that the polynomial $X^{p+1}-c_1^pX-(-1)^{pg}d_0^{p(g-2)}$ factors as the product of the monic polynomials $f_1(X)=X^{n_1}+\ldots+\alpha_1X+\alpha_0$ and $f_2(X)=X^{n_2}+\ldots+\beta_1X+\beta_0$ over $k[\![c_1,d_0]\!]$. Then, we have $\alpha_0\beta_0=-(-1)^{pg}d_0^{p(g-2)}$. Without loss of generality we may assume that $\alpha_0=u_0d_0^m$ for some integer $p(g-2)\geq m>0$ and some $u_0\in R_0$ not divisible by d_0. Since $f(X)\equiv X^{p+1}-c_1^pX=(X-c_1)^pX$ in the polynomial ring over $R_0/(d_0,c_2,\ldots,c_{g-2})\cong k[\![c_1]\!]$ (which is factorial), we must have $\beta_0 = \pm c_1^n + v_0 d_0$ for some integer $n > 0$ and $v_0 \in R_0$. In particular, $\beta_0 \equiv 0$ mod (d_0, c_1). Since $\alpha_0\beta_0 = u_0 d_0^m(\pm c_1^n + v_0 d_0)$, then $\pm u_0 c_1^n d_0^m = -(-1)^{pg} d_0^{p(g-2)} - u_0 v_0 d_0^{m+1}$. Since u_0 and c_1 are not divisible by d_0, we must have $m = p(g-2)$. Hence, $\beta_0 u_0 = -(-1)^{pg}$ i. e., β_0 is a unit (contradiction). This implies that the polynomial $f(X)$ is irreducible over R_0. Since R_0 is local and regular, it is also factorial and, in particular, normal. It follows from [Eis, Cor. 4.12] that R_1 is an integral domain. □

The following lemma shows that the situation is different if we start with a closed point x of $W_{(1,1)}$.

Lemma 6.3.5. *Let x be a closed point of $\mathcal{D} = W^c_{(0,1)}$ of type $(1,1)$. The completed local ring $\widehat{\mathcal{O}}_{\mathfrak{D},x}$ of $\mathfrak{D}$ at x has exactly two minimal associated prime ideals. Each of them has height 1 in $\widehat{\mathcal{O}}_{\mathfrak{M},x}$.*

Proof. As in the proof of Lemma 6.3.4 the completed local ring $\widehat{\mathcal{O}}_{\mathfrak{M},x}$ of $\mathfrak{M}$ at x is isomorphic to

$$k[\![b_0, c_0, \ldots, c_{g-2}, d_0]\!]/ \big(b_0c_0 - d_0b_0c_1 + d_0^2b_0c_2 - d_0^3b_0c_3 + \ldots + (-1)^{g-2}d_0^{g-2}b_0c_{g-2} + (-1)^{g-2}d_0^g\big).$$

The equations of the non-ordinary locus can be deduced as in § 9.2 and coincide with equations (Eq1)–(Eq4) given there. The reduced subscheme defined by $d_0 = 0$ coincides with the $W^c_{(1,1)}$ locus. Inverting d_0, we get that the non-ordinary locus in $k[\![b_0, c_0, \ldots, c_{g-2}, d_0]\!][d_0^{-1}]$ is defined by the ideal I:

- $b_0c_0 - d_0b_0c_1 + d_0^2b_0c_2 - d_0^3b_0c_3 + \ldots + (-1)^{g-2}d_0^{g-2}b_0c_{g-2} + (-1)^g d_0^g = 0;$
- $-b_0^{p^2} + d_0^{p^2} - c_0^p d_0^{p^2-p} = 0.$

Let $h := c_1 - d_0c_2 + d_0^2c_3 + \ldots - (-1)^{g-2}d_0^{g-3}c_{g-2}$. Consider the ideal J in the ring $k[\![b_0, c_0, \ldots, c_{g-2}, d_0]\!][d_0^{-1}]$ defined by

- $-b_0^p + d_0^p - c_0 d_0^{p-1} = 0;$
- $b_0^p\big(c_0^p - d_0^p h^p\big) + (-1)^{pg} d_0^{pg} = 0.$

Then, the minimal primes associated to I and to J in $k[\![b_0, c_0, \ldots, c_{g-2}, d_0]\!][d_0^{-1}]$ are the same.

We can write the second equation as $(c_0 - d_0)(c_0 - d_0h)^p = (-1)^{pg} d_0^{pg-p+1}$. Let $f(X) := X^{p+1} - X^p - h^pX + h^p - (-1)^g d_0^{p(g-2)} = 0$. Define the rings

$$R_0 := k[\![c_1 \ldots, c_{g-2}, d_0]\!], \quad R_1 := R_0[X]/\big(f(X)\big), \quad R_2 := R_1[b_0]/(b_0^p - d_0^p + Xd_0^p).$$

Since R_2 is $(d_0X, c_1, \ldots, c_{g-2}, d_0, b_0)$–adically complete and separated, the map of $k[\![c_1, \ldots, c_{g-2}, d_0]\!]$–algebras from $k[\![b_0, c_0, \ldots, c_{g-2}, d_0]\!][d_0^{-1}]/J$ to $R_2[d_0^{-1}]$ given by $c_0 \mapsto d_0X$ and $b_0 \mapsto b_0$ is well defined. It is easily checked that it is an isomorphism. As in the proof of Lemma 6.3.4 one concludes that the map $\mathrm{Spec}(R_2) \to$

$\mathrm{Spec}(R_1)$ defines a one to one correspondence between the irreducible components of $\mathrm{Spec}(R_2)$ and those of $\mathrm{Spec}(R_1)$. It thus suffices to prove that R_1 has 2 minimal prime ideals.

By Hensel's lemma, $f(X)$ admits a unique root $x \in R_0$ which is congruent to 1 modulo the maximal ideal of R_0. Write $f(X) = (X - x)q(X)$ with $q(X)$ prime to $X - x$. Let $R := R_0[X]/\big(q(X)\big)$. We claim that R is a domain. Since $k[\![c_1 \ldots, c_{g-2}, d_0]\!]$ is local and regular, it is also factorial and, in particular, normal. Therefore, by [Eis, Cor. 4.12], R is a domain if and only if the polynomial $q(X)$ is irreducible. It suffices to check the irreducibility of the reduction $s(X)$ of $q(X)$ modulo $(c_1, \ldots, c_{g-2})$.

Let V be a normal, local, noetherian extension of $k[\![d_0]\!]$ such that $s(X)$ admits a root $z \in V$. Let y be the image of x in V. Since $(X - y)s(X) = X^{p+1} - X^p - (-1)^g d_0^{p(g-2)}$ with y a unit, the element z is not a unit and $z = 1+(-1)^g d_0^{p(g-2)} z^{-p}$. Hence, $z = (z')^p$ where z' satisfies $(z')^{p+1} - (z')^p - (-1)^g d_0^{(g-2)} = 0$. Applying inductively the same trick we find that there exists a positive integer r prime to p and an element w in the maximal ideal of V such that $w^{p+1} - w^p - (-1)^g d_0^r = 0$. Hence, $p\mathrm{val}_V(w) = \mathrm{val}_V(w^{p+1} - w^p) = r\mathrm{val}_V(d_0)$. Hence, $\mathrm{val}_V(d_0)$ is a multiple of p. Hence, the degree of $k[\![d_0]\!] \subset V$ is $\geq p$ and it must then be equal to p, proving that $s(X)$ is irreducible as claimed. It follows that

$$R_1 \cong R_0 \times R$$

is the product of two integral domains of dimension $g - 1$ which are flat R_0–algebras. Since minimal associated primes behave nicely under localization [Eis, Thm. 3.10(d)], the zero ideal in $R_2[d_0^{-1}] \cong k[\![b_0, c_0, \ldots, c_{g-2}, d_0]\!][d_0^{-1}]/J$ is contained in exactly two minimal prime ideals, each of codimension 1. □

7. Intersection theory on a singular surface

We survey here intersection theory on complete surfaces with isolated normal singularities. The main references for this theory are [Arc, RT1, RT2]; see also [Mum, II (b)].

By a *singular surface* we mean in this section an irreducible projective normal algebraic surface over an algebraically closed field.

In [Mum, RT1, RT2] the fundamentals of intersection theory on singular surfaces are presented only over the complex numbers. The reason for that is that resolution of singularities for surfaces in characteristic p was not known at those times. In fact, even the situation over the complex numbers was not yet a common knowledge as one gathers from the assumptions made in [RT1, §1] and the addendum [RT2]. Since then a very strong result about resolution of singularities in arbitrary characteristic was obtained by Lipman [Lip2], building on the works of Zariski and Abhyankar [Lip2, Introd.]. Indeed, [Lip2, I §2] proves that resolution of singularities for surfaces can be achieved in arbitrary characteristic by a succession of normalizations and blow-ups. In particular, the results of [Lip2] (see also §26 of loc. cit.) show that the set-up [RT1, §1] can be achieved in arbitrary characteristic. The thesis of Archibald [Arc] contains a thorough discussion and development of intersection theory on singular surfaces (of not necessarily locally principal divisors) and comparison with other available intersection theories such as, for example, Snapper-Kleiman's [Kle].

7.1. Definition of the intersection number.

Given a singular surface V, one can find a resolution of singularities,

$$\pi : V^* \longrightarrow V,$$

such that V^* is non-singular, π is an isomorphism over the set $V^\circ := V \setminus V^{\text{sing}}$, $\pi^{-1}(V^{\text{sing}}) := \Upsilon$ (the "fundamental manifold") is of pure dimension 1, each irreducible component of it is non-singular, every two irreducible components have at most simple intersections, no three components have a common point. In fact, V^* can be obtained by a succession of blow-ups and normalizations. Moreover, any two such resolutions are dominated by a third one. Cf. [RT2, §1]

Let $C \subset V$ be an irreducible curve. Define $\widetilde{C}$, the *strict transform*, as the closure in V^* of $\pi^{-1}(C \cap V^\circ)$. One says that $C_1 \equiv_{\mathbb{Q}} C_2$ on V, and calls this relation *algebraic equivalence with division*, if for some $m > 0$ and some $\pi : V^* \longrightarrow V$, $m(\widetilde{C}_1 - \widetilde{C}_2)$ is algebraically equivalent to a divisor supported on Υ. This notion is independent of V^* and defines an equivalence relation. Given a resolution of singularities $\pi : V^* \longrightarrow V$ as above, let

$$\mu_1, \dots, \mu_s$$

be the irreducible components of Υ. Let

$$\mathbf{d} = (\mu_i \cdot \mu_j)_{i,j=1,\dots,s},$$

be the intersection matrix. It is an invertible, symmetric, negative definite matrix with no negative elements except on the diagonal. It follows that

$$\mathbf{k} = -\mathbf{d}^{-1}$$

is a symmetric, positive definite matrix with no negative elements.

Let C, D be two curves on V. One can find V^* as above such that in addition: $\widetilde{C}, \widetilde{D}$ have no common point on Υ, neither passes through a point of $\mu_i \cdot \mu_j$ and they intersect each μ_i simply. The contribution to the intersection multiplicity coming from V^{sing} is then

$$\sum_{i,j} \mathbf{k}_{ij} [\widetilde{C} \cdot \mu_i][\widetilde{D} \cdot \mu_j] = (\dots, \widetilde{C} \cdot \mu_i, \dots) \mathbf{k} \, {}^t(\dots, \widetilde{D} \cdot \mu_i, \dots).$$

It will be convenient to denote the vector $(\dots, \widetilde{C} \cdot \mu_i, \dots)$ by C^Υ. The total intersection number is

$$C \cdot D = \widetilde{C} \cdot \widetilde{D} + C^\Upsilon \, \mathbf{k} \, {}^t D^\Upsilon \tag{7.1}$$

One can prove [RT1] that this defines a symmetric bilinear pairing on divisor classes modulo $\equiv_{\mathbb{Q}}$.

7.2. Pull-back and intersection.

Let $\mu_1, \dots, \mu_s$ be the irreducible components of Υ. We want to define for an irreducible curve C in V a divisor C^* in V^*, such that

$$C^* = \widetilde{C} + \sum_{i=1}^{s} \gamma_i \mu_i, \tag{7.2}$$

and such that

$$C^* \cdot \mu_j = 0, \quad \forall j. \tag{7.3}$$

Since $C^* \cdot \mu_j = \widetilde{C} \cdot \mu_j + \sum_{i=1}^{s} \gamma_i \mu_i \mu_j$, we see that we need to solve the equation $\mathbf{d}\,{}^t(\gamma_1, \dots, \gamma_s) = -\,{}^tC^{\Upsilon}$. This has a unique solution given by

$${}^t(\gamma_1, \dots, \gamma_s) = {}^t(\gamma_1(C), \dots, \gamma_s(C)) = \mathbf{k}\,{}^tC^{\Upsilon}. \tag{7.4}$$

The definition of C^* extends by linearity to any divisor.

Proposition 7.2.1. *The following identities hold.*

1. *Let C be a divisor on V, then $C^* \cdot \mu_j = 0$ for all $j = 1, \dots, s$.*

2. *Let C, D be divisors on V, then $C^* \cdot D^* = C \cdot D$.*

3. *Let C be a divisor on V and D a divisor on V^*, then $C^* \cdot D = C \cdot \pi_* D$.*

Proof. The first part follows from the definition and the calculation above. For part (2), on the one hand, we have

$$C \cdot D = \widetilde{C} \cdot \widetilde{D} + C^{\Upsilon}\,\mathbf{k}\,{}^tD^{\Upsilon},$$

and on the other hand

$$\begin{aligned}
C^* \cdot D^* &= \left(\widetilde{C} + \sum_i \gamma_i(C)\mu_i\right) \cdot \left(\widetilde{D} + \sum_i \gamma_i(D)\mu_i\right) \\
&= \left(\sum_i \gamma_i(C)\mu_i\right) \cdot D^* + C^* \cdot \left(\sum_i \gamma_i(D)\mu_i\right) + \widetilde{C} \cdot \widetilde{D} \\
&\qquad - \left(\sum_i \gamma_i(C)\mu_i\right) \cdot \left(\sum_i \gamma_i(D)\mu_i\right) \\
&= \widetilde{C} \cdot \widetilde{D} - \left(\sum_i \gamma_i(C)\mu_i\right) \cdot \left(\sum_i \gamma_i(D)\mu_i\right) \\
&= \widetilde{C} \cdot \widetilde{D} - \sum_{i,j} \gamma_i(C)\gamma_j(D)\mu_i \cdot \mu_j \\
&= \widetilde{C} \cdot \widetilde{D} - (\gamma_1(C), \cdots, \gamma_s(C))\, \mathbf{d}\, {}^t(\gamma_1(D), \cdots, \gamma_s(D)) \\
&= \widetilde{C} \cdot \widetilde{D} - (C^\Upsilon\, \mathbf{k})\mathbf{d}(\mathbf{k}\, {}^t D^\Upsilon) \\
&= \widetilde{C} \cdot \widetilde{D} + C^\Upsilon\, \mathbf{k}\, {}^t D^\Upsilon.
\end{aligned}$$

For part (3), we calculate that

$$\begin{aligned}
C^* \cdot D &= C^* \cdot (\pi_* D)^* - C^* \cdot (D - (\pi_* D)^*) \\
&= C^* \cdot (\pi_* D)^* \\
&= C \cdot \pi_* D.
\end{aligned}$$

□

7.3. Adjunction.

Let $K[V^*]$ be the canonical divisor of V^* and let

$$K = \pi_* K[V^*].$$

We note that K is the unique extension of the canonical divisor on V° and hence is independent of the choice of V^*. We call it the *canonical divisor* of V. One may ask if K satisfies the adjunction formula. The answer is NO as we show by a simple example:

Suppose that $\Upsilon = \mu$ is irreducible and $\mu^2 = -n$. This happens for example in the case of the blow-up at the origin of the cone over the curve $x^n + y^n = z^n$. Then $\mu \cdot (\mu + K[V^*]) = 2g(\mu) - 2$ and therefore $\mu \cdot K[V^*] = 2g(\mu) - 2 + n$. Let C be a nonsingular curve passing simply through the point $\pi(\mu)$ then $C^* = \widetilde{C} + \frac{1}{n}\mu$.

We find that

$$\begin{aligned} C \centerdot (C+K) &= C^* \centerdot (C^* + K[V^*]) \\ &= \left(\widetilde{C} + \frac{1}{n}\mu\right) \centerdot \left(\widetilde{C} + \frac{1}{n}\mu + K[V^*]\right) \\ &= \widetilde{C}^2 + \frac{1}{n} + \widetilde{C} \centerdot K[V^*] + \frac{1}{n}\mu \centerdot K[V^*] \\ &= \widetilde{C}^2 + \widetilde{C} \centerdot K[V^*] + \frac{1}{n}\left(\mu^2 + \mu \centerdot K[V^*] + n + 1\right) \\ &= 2g(\widetilde{C}) - 2 + \frac{2g(\mu)+n-1}{n} \\ &= 2g(C) - 2 + \frac{2g(\mu)+n-1}{n}. \end{aligned}$$

Since the term $(2g(\mu)+n-1)/n$ is not zero in general, we see that adjunction does not hold in the same way.

Proposition 7.3.1. *Define a vector* κ^Υ *as*

$$\begin{aligned} \kappa^\Upsilon &= -\mathbf{k}\,{}^t(2g(\mu_1) - 2 - \mu_1^2, \dots, 2g(\mu_s) - 2 - \mu_s^2) \\ &= -\mathbf{k}\,{}^t(\mu_1 \centerdot K[V^*], \dots, \mu_s \centerdot K[V^*]) \\ &= -\mathbf{k}\,{}^t K[V^*]^\Upsilon. \end{aligned}$$

Then

$$K[V^*] = K^* + \sum_i \kappa_i \mu_i, \tag{7.5}$$

and

$$C \centerdot (C+K) = 2g(C) - 2 + C^\Upsilon\, \mathbf{k}\,{}^t(C^\Upsilon + K[V^*]^\Upsilon). \tag{7.6}$$

Proof. Write $K[V^*] = K^* + \sum_i \kappa_i \mu_i$, where the κ_i need to be calculated. We have

$$\begin{aligned} 2g(\mu_i) - 2 - \mu_i^2 &= K[V^*] \centerdot \mu_i \\ &= K^* \centerdot \mu_i + \sum_j \kappa_j \mu_j \centerdot \mu_i \\ &= \sum_j \kappa_j \mu_j \centerdot \mu_i. \end{aligned}$$

We conclude that ${}^t(2g(\mu_1) - 2 - \mu_1^2, \dots, 2g(\mu_s) - 2 - \mu_s^2) = \mathbf{d}\,{}^t(\kappa_1, \dots, \kappa_s)$.

Write $C^* = \widetilde{C} + \sum_i \gamma_i(C)\mu_i$ and use

$$C \centerdot (C+K) = C^* \centerdot (C^* + K^*) = \widetilde{C} \centerdot (C^* + K^*).$$

We get,

$$\begin{aligned} C \cdot (C+K) &= \widetilde{C} \cdot \left(\widetilde{C} + \sum_i \gamma_i(C)\mu_i + K[V^*] - \sum_i \kappa_i \mu_i \right) \\ &= \widetilde{C}^2 + \sum_i \gamma_i(C) \widetilde{C} \cdot \mu_i + \widetilde{C} \cdot K[V^*] - \sum_i \kappa_i \widetilde{C} \cdot \mu_i \\ &= \widetilde{C}^2 + C^{\Upsilon} \, \mathbf{k} \, {}^t C^{\Upsilon} + \widetilde{C} \cdot K[V^*] - C^{\Upsilon} \, {}^t \kappa^{\Upsilon} \\ &= 2g(C) - 2 + C^{\Upsilon} \, \mathbf{k} \, {}^t C^{\Upsilon} - C^{\Upsilon} \, {}^t \kappa^{\Upsilon} \\ &= 2g(C) - 2 + C^{\Upsilon} \, \mathbf{k} \, {}^t (C^{\Upsilon} + K[V^*]^{\Upsilon}). \end{aligned}$$

□

Remark 7.3.2. *Observe that if C passes through none of the singular points then adjunction holds in the usual sense.*

8. Hilbert modular surfaces

Let L be a real quadratic field. We let $\mathfrak{M} = \mathfrak{M}(\mu_N)$ be the moduli space with μ_N level structure, where $N \geq 4$, $(N,p) = 1$.

8.1. The inert case.

8.1.1. Calculation of some intersection numbers. Assume $p > 2$ in this section. To conform with the notation in § 7 we let V be the Satake compactification of $\mathfrak{M}$, V^* be a smooth toroidal compactification of V, $\pi : V^* \longrightarrow V$ be the projection, V° be the complement in $\mathfrak{M}$ of the singular locus of V. We also let $D_i = W_{(\{i+1\})}$. Let $C(N)$ be the degree of $\mathfrak{M}$ over the coarse moduli space of abelian surfaces with RM and no level structure.

Let $\eta = \frac{1}{2}\zeta_L(-1)C(N)$. We know [BG] that each D_i is a disjoint union of η non-singular rational curves, that D_1 and D_2 intersect transversely, the set of intersection points is the set of superspecial points, and that

$$D_1 \cdot D_2 = \eta(p^2+1). \tag{8.1}$$

Let h be the total Hasse invariant [Go1, Thm. 2.1]. It is a section of $\mathbb{L}_1^{p-1} \otimes \mathbb{L}_2^{p-1}$. Over V° the Kodaira-Spencer isomorphism gives that $\det \Omega^1_{V/k} \cong \mathbb{L}_1^2 \otimes \mathbb{L}_2^2$, thus

$$K \sim \frac{2}{p-1}(h) = \frac{2}{p-1}(D_1 + D_2), \tag{8.2}$$

hence this also holds over V (since V is normal and $V - V^\circ$ is of codimension 2).

Note also that over V° we have $\mathbb{L}_i^p \mathbb{L}_{i+1}^{-1} \cong \mathcal{O}_{V^\circ}(D_i)$, as follows from the properties of the partial Hasse invariants [Go1]. Since D_i is closed in V° we conclude that $\mathbb{L}_i^p \mathbb{L}_{i+1}^{-1}$ extends to V and therefore we may define unique classes $\ell_i \in$

$CH(V) \otimes \mathbb{Q}$ so that

$$c_1(\mathbb{L}_i^p \mathbb{L}_{i+1}^{-1}) = p\ell_i - \ell_{i+1}, \quad i = 1, 2.$$

Now,

$$\begin{aligned} D_1 \centerdot K &= \textstyle\sum_{C \in D_1} C \centerdot K \\ &= -2\eta - \textstyle\sum_{C \in D_1} C^2 && \text{(adjunction, each } C \cong \mathbb{P}^1) \\ &= -2\eta - D_1^2 && (D_1 \text{ is a disjoint union of its components).} \end{aligned}$$

On the other hand,

$$\begin{aligned} D_1 \centerdot K &= \tfrac{2}{p-1} D_1 \centerdot (D_1 + D_2) && \text{(Equation (8.2))} \\ &= \tfrac{2}{p-1} D_1^2 + \tfrac{2}{p-1}\eta(p^2+1) && \text{(Equation (8.1)).} \end{aligned}$$

This yields

$$D_1^2 = -2p\eta, \qquad D_2^2 = -2p\eta.$$

Solving for ℓ_1, ℓ_2, one finds

$$\ell_1^2 = 0, \qquad \ell_2^2 = 0, \qquad \ell_1\ell_2 = \eta. \tag{8.3}$$

8.1.2. On ampleness. The sections of the line bundle $\mathbb{L}_1^{a_1}\mathbb{L}_2^{a_2}$ are Hilbert modular forms of weight (a_1, a_2). This motivates our interest in its ampleness.

Theorem 8.1.1. *The class $a_1\ell_1 + a_2\ell_2$ is ample if and only if $pa_1 > a_2 > \frac{1}{p}a_1$.*

Proof. We prove the claim by using the Nakai-Moishezon criterion [Kle, III.1, Thm. 1], cf. [Har, App. A, Thm. 5.1]. Though, strictly speaking, this criterion uses Snapper-Kleinman's intersection theory, we can use the Reeve-Tyrrell intersection theory, since the theories agrees when both are defined [Arc, Thm. 2.5.15]. We first make some preliminary calculations.

Let C be a component of D_1. We have $C^2 = -2 - C \centerdot K$ by adjunction. On the other hand, $C \centerdot K = \frac{2}{p-1} C \centerdot (D_1 + D_2) = \frac{2}{p-1}(C^2 + p^2 + 1)$, where we have used that D_1 is a disjoint union of its components, one of which is C, and that $C \centerdot D_2$ is the set of superspecial points on C, which has cardinality p^2+1 [BG, Thm. 6.1]. Therefore, $C^2 = -2 - \frac{2}{p-1}(C^2 + p^2 + 1)$, which gives

$$C^2 = -2p. \tag{8.4}$$

We conclude that $C \centerdot D_1 = C^2 = -2p$ and $C \centerdot D_2 = p^2 + 1$. Using that $D_1 = p\ell_1 - \ell_2$, $D_2 = p\ell_2 - \ell_1$, we solve for ℓ_1, ℓ_2 and get

$$C \centerdot \ell_1 = -1, \qquad C \centerdot \ell_2 = p. \tag{8.5}$$

We conclude that if $C \centerdot (a_1\ell_1 + a_2\ell_2) > 0$ then $pa_2 > a_1$. By symmetry, if C is a component of D_2 such that $C \centerdot (a_1\ell_1 + a_2\ell_2) > 0$ then $pa_1 > a_2$.

Applying the Nakai-Moishezon criterion to the class $a_1\ell_1 + a_2\ell_2$, we conclude that if $a_1\ell_1 + a_2\ell_2$ is ample then $pa_1 > a_2 > \frac{1}{p}a_1$. We now claim that the converse also holds. It is enough to prove that for every irreducible curve C we have $C \cdot (a_1\ell_1 + a_2\ell_2) > 0$. If C is contained in $D_1 \cup D_2$ then this follows from our calculations above. Else, write $a_1\ell_1 + a_2\ell_2 = b_1D_1 + b_2D_2$. One checks that b_1, b_2 are both positive. Since C is generically ordinary, it intersect the non-ordinary locus $D_1 \cup D_2$ by the "Raynaud trick" [Oo3, §4], hence has positive intersection with $b_1D_1 + b_2D_2$. □

8.2. The split case.

To conform with the notation of § 7, we let V be the Satake compactification of $\mathfrak{M}$, the moduli space with μ_N-level structure, V^* be a smooth toroidal compactification of $\mathfrak{M}$, $\pi : V^* \to V$ be the projection and V° be the complement in $\mathfrak{M}$ of the singular locus of V.

One knows that the non-ordinary locus consists of two divisors $D_1 = W_{(\{1\},\emptyset)}$ and $D_2 = W_{(\emptyset,\{1\})}$ that intersect transversely; the intersection being the set of superspecial points. We also know that each D_i is a disjoint union of non-singular curves. See [BG, Thm. 6.1]. However, we have very little information on the components of the D_i. They are not Moret-Bailly families and one can show that they are not Shimura curves. Here by a "Shimura curve" we mean the following. Let $B/\mathbb{Q}$ be a quaternion algebra split at infinity. Fix a maximal order $\mathcal{O}_B$ of B and a positive involution $*$ of B fixing $\mathcal{O}_B$. There is a moduli space for special polarized abelian surfaces with multiplication by $\mathcal{O}_B$ (such that $*$ is the Rosati involution) [Dri, §4 Dfn. and Prop. 4.4]. It is easy to see that every abelian surface A with multiplication by $\mathcal{O}_B$ over a field k is either simple or isogenous to E^2 where E is an elliptic curve. In particular, if $\mathrm{char}(k) = p > 0$ then A is either ordinary or supersingular.

Assume now that $\mathcal{O}_L \subset \mathcal{O}_B$ and that $*$ preserves $\mathcal{O}_L$, then we get a forgetful morphism to the Hilbert moduli space $\mathfrak{M}$. We call the images of such curves, and their images under Hecke correspondences, Shimura curves.

In the following, we obtain some information on the field of definition and genus of the components of the divisors D_1, D_2.

8.2.1. Fields of definition We examine the field of definition of the superspecial points and the non-ordinary locus, under some restriction on N and p. The following lemma holds for any totally real field L of degree $g > 1$ and for any prime p.

Lemma 8.2.1. *Let $N \geq 3$ be an integer such that $N|(p-1)$ or $N|(p+1)$. Every superspecial point on the moduli space $\mathfrak{M}$ of RM abelian varieties with μ_N-level structure can be defined over $\mathbb{F}_{p^2}$.*

Proof. We use Honda-Tate theory for which [Wat] is a good reference. Consider the Weil numbers $\pm p$ over $\mathbb{F}_{p^2}$. There exist elliptic curves $E_\pm$ over $\mathbb{F}_{p^2}$ with that Weil number. The endomorphism ring of $E_\pm$ after tensoring with $\mathbb{Q}$ is "the" quaternion algebra $B_{p,\infty}$ over $\mathbb{Q}$ ramified at p and ∞. However, one easily sees that if $f \in \mathrm{End}_{\overline{\mathbb{F}}_p}(E_\pm)$ and $mf \in \mathrm{End}_{\mathbb{F}_{p^2}}(E_\pm)$, for some non-zero integer m, then $f \in \mathrm{End}_{\mathbb{F}_{p^2}}(E_\pm)$. It follows that $\mathrm{End}_{\mathbb{F}_{p^2}}(E)$ is a maximal order in $B_{p,\infty}$.

The Frobenius endomorphism $\pi := \mathrm{Fr}_{p^2} : E \longrightarrow E$ is equal to $\pm p$. It follows that $E_\pm[N] \subseteq E_\pm(\mathbb{F}_{p^2})$ iff $N|(\pi - 1)$ in $\mathrm{End}(E_\pm)$. But $\pi = \pm p$ as an endomorphism and we conclude that $E_\pm[N] \subseteq E_\pm(\mathbb{F}_{p^2})$ iff $N|(\pm p - 1)$ as integers.

Note that $\mathrm{End}(E_\pm^g) = M_g(\mathrm{End}(E_\pm))$ is defined over $\mathbb{F}_{p^2}$. It follows that any $\mathcal{O}_L$ structure on $E_\pm^g$ is defined over $\mathbb{F}_{p^2}$. Note also that $E_\pm^g$ has an obvious polarization defined over $\mathbb{F}_{p^2}$ induced from the canonical identification of E with its dual, and hence (using that polarization to identify the polarizations with the symmetric positive elements of $\mathrm{End}(E_\pm^g)$) every polarization of $\mathrm{End}(E_\pm^g)$ is defined over $\mathbb{F}_{p^2}$.

To conclude the proof, we notice that by a theorem of P. Deligne [Shi, Thm. 3.5] every superspecial abelian variety of dimension $g > 1$ is isomorphic over $\overline{\mathbb{F}}_p$ to E^g and, under our assumptions, $\mu_N \cong \mathbb{Z}/N\mathbb{Z}$ as group schemes over $\mathbb{F}_{p^2}$. □

Corollary 8.2.2. *Every component of D_i is defined over $\mathbb{F}_{p^2}$.*

Proof. It is enough to show that if C is a component of D_i then $\sigma(C) = C$ if $\sigma \in \mathrm{Gal}(\overline{\mathbb{F}}_{p^2}/\mathbb{F}_{p^2})$.

We first note that D_i is defined over $\mathbb{F}_p$. Let $x \in C$ be a superspecial point (such exists, because $D_i \setminus W_{(1,1)}$ is quasi-affine by applying [Oo3, Thm. 6.5], but see also below). It is a $\mathbb{F}_{p^2}$ rational point of V and hence $\sigma(C)$ is also a component of D_i passing through x. However, there is a unique such component passing through x. We conclude that $\sigma(C) = C$. □

8.2.2. Calculation of intersection numbers We shall make the following assumption regarding continuity of intersection numbers (cf. Equation (8.3), Remark 8.3.3).

Assumption: $\quad \ell_1^2 = 0, \qquad \ell_2^2 = 0, \qquad \ell_1\ell_2 = \eta.$

It follows that

$$D_1^2 = 0, \qquad D_2^2 = 0, \qquad D_1 \cdot D_2 = (p-1)^2\eta. \tag{8.6}$$

Therefore,

$$
\begin{aligned}
0 &= D_1^2 \\
&= \sum_{C\in D_1} C^2 \\
&= \sum_{C\in D_1} (2g(C) - 2 - C \centerdot K) \\
&= \sum_{C\in D_1} (2g(C) - 2) - D_1 \centerdot K \\
&= \sum_{C\in D_1} (2g(C) - 2) - (p-1)\ell_1 \centerdot 2(\ell_1 + \ell_2) \\
&= \sum_{C\in D_1} (2g(C) - 2) - 2(p-1)\eta.
\end{aligned} \tag{8.7}
$$

That is,

$$(p-1)\eta = \sum_{C\in D_1} (g(C) - 1). \tag{8.8}$$

This already shows that on average the genus of components of C should be greater than 1.

We can do slightly better. Assume that $N \geq 3$ and either $N|(p-1)$ or $N|(p+1)$. Let $\{C_1, \ldots, C_\ell\}$ be the irreducible components of D_1. Let r_i be the number of superspecial points on C_i. Let g_i be the genus of C_i, and $G = \sum_{i=1}^{\ell} g_i$. Then $R := \sum_{i=1}^{\ell} r_i = (p-1)^2\eta$ and, together with Equation (8.8), we get,

$$R = (p-1)\sum_{i=1}^{\ell}(g_i - 1) = (p-1)(G - \ell). \tag{8.9}$$

We have the estimate $r_i > 0$ (because $D_i \setminus W_{(1,1)}$ is quasi-affine), but since $r_i = \deg\left(\mathbb{L}_2^{p-1}|_{C_i}\right)$ (existence of partial Hasse invariants and simplicity of their zeros [Go1, Thm. 2.1]) we actually have $r_i \geq p-1$. Summing over the components, we get

$$R \geq (p-1)\ell. \tag{8.10}$$

We obtain the following:

Proposition 8.2.3. *Assume that $N \geq 3$ and $N|(p-1)$ or $N|(p+1)$. Then the average genus g of the non-ordinary locus satisfies the inequality $g = G/\ell \geq 2$.*

Proposition 8.2.4. *The line bundle $\mathbb{L}_1^{n_1}\mathbb{L}_2^{n_2}$ is ample if and only if both n_1 and n_2 are positive.*

The proof is along the same lines as the proof of Theorem 8.1.1.

8.3. The ramified case.

Again, to conform with the notation of § 7, we let V be the Satake compactification of $\mathfrak{M}$, the moduli space with μ_N-level structure, V^* be a smooth toroidal compactification of $\mathfrak{N}$ (sic!), $\pi : V^* \to V$ the projection. Let $V^\circ = V \setminus V^{\text{sing}}$. For every $S \subset W_{(1,1)}$, let $\mu_S = \pi^{-1}(S)$.

8.3.1. The local structure of the moduli space. First we compute the local deformation theory at a point of $\mathfrak{M}$. It follows from Example 4.3.1 that the moduli space is regular at points of type $(0,n)$, $0 \leq n \leq 2$. By loc. cit., at a point of type $(0,2)$, the universal deformation ring is $k[\![c_0, c_1]\!]$. Recall Remark 6.1.4. We may take $m = \infty$ and $c_3 = 1$ as in (6.3) so that the universal Frobenius is

$$F = \begin{pmatrix} 0 & T^2 \\ 1 & -c_0^\sigma - c_1^\sigma T \end{pmatrix}.$$

A deformation has type $(0,1)$ if and only if it is not ordinary. This is equivalent to $TF^2 \equiv 0 \pmod{T^2}$. Equivalently,

$$\begin{pmatrix} 0 & 0 \\ 1 & -c_0^\sigma \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & -c_0^{\sigma^2} \end{pmatrix} = 0 \mod T.$$

This gives the condition $c_0 = 0$. We conclude that in the local deformation space the condition for deforming into $W_{(0,1)}$ is given by $c_0 = 0$ and it defines a smooth formal curve.

By Example 4.3.2, at a point of type $(1,1)$ the universal deformation ring R is defined by

$$k[\![a_0, b_0, c_0, d_0]\!]/(a_0 + d_0, a_0 d_0 - b_0 c_0) \cong k[\![a_0, b_0, c_0]\!]/(a_0^2 + b_0 c_0).$$

Hence, $\operatorname{Spec}(R)$ is a cone. By (6.3) we may take $m = \infty$ and $c_3 = 1$ so that the universal Frobenius is given by

$$F = \begin{pmatrix} -b_0^\sigma & T + a_0^\sigma \\ T - a_0^\sigma & -c_0^\sigma \end{pmatrix}.$$

In order to have deformation of $(0,1)$ we must have $TF^2 = 0 \pmod{T^2}$, which is equivalent to

$$F^2 = \begin{pmatrix} -b_0^\sigma & a_0^\sigma \\ -a_0^\sigma & -c_0^\sigma \end{pmatrix}\begin{pmatrix} -b_0^{\sigma^2} & a_0^{\sigma^2} \\ -a_0^{\sigma^2} & -c_0^{\sigma^2} \end{pmatrix} = 0.$$

This gives the system of equations modulo p:

$$b_0^{p+1} - a_0^{p+1} = 0, \quad a_0 b_0^p + c_0 a_0^p = 0, \quad b_0 a_0^p + a_0 c_0^p = 0, \quad -a_0^{p+1} + c_0^{p+1} = 0.$$

If $b_0 = 0$ it follows that a_0 and $c_0 = 0$ are nilpotent. The associated reduced scheme is the point we started with. Inverting b_0, the second equation can be eliminated using $b_0(a_0 b_0^p + c_0 a_0^p) = a_0 b^{p+1} - a_0^2 a_0^p = a_0(b_0^{p+1} - a_0^{p+1})$. If $a_0 = 0$, the associated reduced scheme is the point we started with. Inverting a_0 we deduce from $a_0(b_0 a_0^p + a_0 c_0^p) = b_0(a_0^{p+1} - c_0^{p+1})$ and from the other relations that $b_0 a_0^p +$

$a_0 c_0^p = 0$. Hence, on the complement of the point we are reduced to the equations

$$a_0^2 + b_0 c_0 = 0, \quad b_0^{p+1} - a_0^{p+1} = 0, \quad -a_0^{p+1} + c_0^{p+1} = 0. \tag{8.11}$$

We conclude that the non-ordinary locus consists of $p+1$ branches given by $b_0 = \zeta a_0$ and $c_0 = \zeta^{-1} a_0$ for ζ a $p+1$-st root of unity.

Finally, we compute the structure of $\pi_1 \colon \mathfrak{N} \to \mathfrak{M}$. The morphism π_1 is proper [AG1, Lem. 8.4]. Outside $\pi_1^{-1}\big(W_{(1,1)}\big)$ it is one-to-one [AG1, Prop. 6.5] and so is an isomorphism. Since $\mathfrak{M}\backslash W_{(1,1)}$ is smooth, we conclude that $\pi_1^{-1}\big(\mathfrak{M}\backslash W_{(1,1)}\big)$ is smooth. Let $s \in W_{(1,1)}$. Let $R := k[\![a_0, b_0, c_0]\!]/(a_0^2 + b_0 c_0)$ be the completed local ring of $\mathfrak{M}$ at s. Let $A \to \operatorname{Spec}(R)$ be the universal abelian scheme over R. Using the theory of local models §4.3.2, we can find a $R \otimes_k k[T]/(T^2)$-basis α, β of $H^1_{\mathrm{dR}}(A/R)$ such that the relative cotangent space $H^0(A, \Omega_{A/R})$ in $H^1_{\mathrm{dR}}(A/R)$ is generated as $R \otimes_{\mathbb{Z}} k[T]/(T^2)$-module by $(T + a_0)\alpha + b_0\beta$ and $c_0\alpha + (T - a_0)\beta$. The scheme $\mathfrak{N} \times_{\mathfrak{M}} \operatorname{Spec}(R)$ can be interpreted as representing the Grassmannian of $R \otimes_k k[T]/(T^2)$ rank 1 submodules of $H^0(A, \Omega_{A/R})$, free as R-modules and killed by T. Any such module is generated by an element $TX\alpha + TZ\beta$ which is zero in $H^1_{\mathrm{dR}}(A/R)/H^0(A, \Omega_{A/R})$. Hence,

$$\mathfrak{N} \times_{\mathfrak{M}} \operatorname{Spec}(R) \cong \mathbf{Proj}\; R[X, Z]/(a_0 X + c_0 Z, -b_0 X + a_0 Z), \tag{8.12}$$

Proposition 8.3.1. *The following hold:*

1. *the singular points of $\mathfrak{M}$ are the cusps and the points contained in $W_{(1,1)}$;*
2. *the variety $\mathfrak{N}$ is smooth over k;*
3. *$\pi \colon \mathfrak{N} \to \mathfrak{M}$ is the blow-up along $W_{(1,1)}$;*
4. *for every $s \in W_{(1,1)}$, the scheme μ_s is a non-singular rational curve with self intersection -2.*

Proof. The first assertion is a summary of part of the discussion above. Next, it follows from (8.12) that $\mathfrak{N}$ is a smooth variety.

Let $\widetilde{V^o}$ be the blow-up of $\mathfrak{M}$ at $W_{(1,1)}$. Since $W_{(1,1)}$ is reduced, we also get that the inverse image of $W_{(1,1)}$ is a disjoint union of curves and, hence, is a divisor. By the universal property of blow-up we get a birational map $\rho \colon \mathfrak{N} \to \widetilde{V^o}$ compatible with the projections onto $\mathfrak{M}$. It is an isomorphism over $\mathfrak{M}\backslash W_{(1,1)}$. The completed local ring of $\mathfrak{M}$ at a point of $W_{(1,1)}$ is isomorphic to $R = k[\![a_0, b_0, c_0]\!]/(a_0^2 + b_0 c_0)$. Since the blow-up is defined in terms of **Proj** of the ideal defining $W_{(1,1)}$ and $W_{(1,1)}$ is reduced, the fibre product $\widetilde{V^o} \times_{\mathfrak{M}} \operatorname{Spec}(R)$ coincides with the blow-up of $\operatorname{Spec}(R)$ at its closed point. In particular, the inverse image of the closed point of R in $\widetilde{V^o} \times_{\mathfrak{M}} \operatorname{Spec}(R)$ is isomorphic to $\mathbb{P}^1_k$ and has self intersection -2. Using (8.12) one easily checks that the base change of ρ to the product of the completed local rings at the points of $W_{(1,1)}$ is an isomorphism. Hence, ρ is an isomorphism. □

8.3.2. Calculation of some intersection numbers. Assume that $p > 2$ in this section. Let D be the reduced divisor that is equal to the non-ordinary locus of V. Let h be the total Hasse invariant, $h \in \Gamma(V^\circ, \det \mathbb{E}^{p-1})$; it admits a square root $\sqrt{h} \in \Gamma(V^\circ, \det \mathbb{E}^{(p-1)/2})$ - see [AG2]. We have $(\sqrt{h}) = D$. It follows from the Kodaira-Spencer isomorphism that (initially on V°, but then on V)

$$K \sim \frac{4}{p-1} D. \tag{8.13}$$

We know [BG, Thm. 5.3] that the number of components of D is $\eta = \frac{1}{2}\zeta_L(-1)C(N)$, where $C(N)$ is the degree of the level structure, and that the number of points of $W_{(1,1)}$ is also η. We also note that Proposition 8.3.1 implies that the variety V^* is suitable for calculating the intersections of divisors support on D. The following calculations are done using the results and notations of § 7. On the one hand,

$$\begin{aligned}
D^2 &= (D^*)^2 \\
&= \left(\sum_{C \in D} \widetilde{C} + \frac{p+1}{2} \mu_{W_{(1,1)}} \right)^2 \\
&= \left(\sum_{C \in D} \widetilde{C} \right)^2 + (p+1) \left(\sum_{C \in D} \widetilde{C} \right) \cdot \mu_{W_{(1,1)}} + \frac{(p+1)^2}{4} \mu^2_{W_{(1,1)}} \\
&= \sum_{C \in D} \widetilde{C}^2 + (p+1) \sum_{u \in W_{(1,1)}} \sum_{C \in D} \widetilde{C} \cdot \mu_u + \frac{(p+1)^2}{4} \sum_{u \in W_{(1,1)}} \mu_u^2 \\
&= \sum_{C \in D} \widetilde{C}^2 + (p+1)^2 \eta + \frac{(p+1)^2}{4} (-2)\eta \\
&= \sum_{C \in D} \widetilde{C}^2 + \frac{(p+1)^2}{2} \eta.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\textstyle\sum_{C \in D} \widetilde{C}^2 &= \textstyle\sum_{C \in D} (-2 - \widetilde{C} \cdot K[V^*]) && \text{(adjunction on } V^*) \\
&= \textstyle\sum_{C \in D} (-2 - \widetilde{C} \cdot K^*) && \text{(Prop. 7.3.1 + Prop. 8.3.1)} \\
&= -2\eta - \textstyle\sum_{C \in D} C^* \cdot K^* && \text{(Prop. 7.2.1)} \\
&= -2\eta - \textstyle\sum_{C \in D} C \cdot K && \text{(Prop. 7.2.1)} \\
&= -2\eta - \textstyle\sum_{C \in D} C \cdot \frac{4}{p-1} D && \text{(Equation (8.13))} \\
&= -2\eta - \tfrac{4}{p-1} D^2.
\end{aligned}$$

We conclude that $D^2 = -2\eta - \frac{4}{p-1}D^2 + \frac{(p+1)^2}{2}\eta$, which gives:

Proposition 8.3.2. *The self intersection of D is given by*

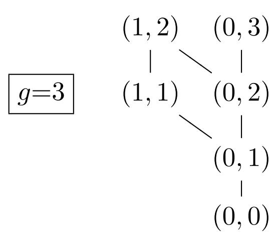
$$D^2 = \frac{(p-1)^2}{2}\eta.$$

Remark 8.3.3. *Note that if we could argue by 'continuity of intersection numbers', we could write $D = \frac{p-1}{2}(\ell_1 + \ell_2)$, whence $D^2 = \frac{(p-1)^2}{2}\ell_1 \cdot \ell_2 = \frac{(p-1)^2}{2}\eta$.*

9. Hilbert modular threefolds

Let L be a totally real cubic field. In this section we study the local structure of the moduli variety $\mathfrak{M}$. Given the results for $g = 2$ and the unramified case, we may restrict our attention to the case when $p = \mathfrak{p}^3$ is maximally ramified. Assume that henceforth.

We recall from § 3.2 the strata and their hierarchy in terms of "being in the closure" as encoded in the following diagram



To begin with, it follows from Example 4.3.1 that the locus $W^c_{(j,n)}$ for $j = 0$ and $n = 0, \dots, 3$, or for $j = 1$ and $n = 1, 2$ (performing a similar computation), is formally smooth at points of type (j, n') with $n' \geq n$. Thus, we are interested in the structure of the strata $W^c_{(0,1)}$ at a point of type $(1, 1)$ and $(1, 2)$, and $W^c_{(0,2)}$ at a point of type $(1, 2)$.

9.1. Points of type $(1, 2)$.

In this case $j = 1$, $i = 2$, and, since the point is superspecial, we may assume $m = \infty$ and $c_3 = 1$ in Equation (6.3). The universal deformation space is of the form (cf. Example 4.3.3):

$$\begin{aligned} &k[\![a_0, a_1, b_0, c_0, c_1, d_0]\!]/(a_0d_0 - b_0c_0, a_0 + a_1d_0 - b_0c_1, a_1 + d_0) \\ &\qquad\qquad \cong k[\![a_0, b_0, c_0, c_1, d_0]\!]/(a_0d_0 - b_0c_0, a_0 - d_0^2 - b_0c_1). \end{aligned}$$

The results of §5.6 imply that the universal "mod p" Frobenius is given over this ring by

$$F = \begin{pmatrix} -b_0^\sigma & T^2 + a_0^\sigma - d_0^\sigma T \\ T + d_0^\sigma & -c_0^\sigma - c_1^\sigma T \end{pmatrix}. \tag{9.1}$$

9.1.1. The non-ordinary locus $W^c_{(0,1)}$. By Corollary 5.6.3, the condition that the deformation is non-ordinary is equivalent to the condition

$$\begin{pmatrix} -b_0^\sigma & a_0^\sigma \\ d_0^\sigma & -c_0^\sigma \end{pmatrix} \begin{pmatrix} -b_0^{\sigma^2} & a_0^{\sigma^2} \\ d_0^{\sigma^2} & -c_0^{\sigma^2} \end{pmatrix} \equiv 0 \pmod{T}.$$

This gives the following system of equations:

$$\text{(Eq1)} \qquad b_0^{p+1} + a_0 d_0^p = 0$$
$$\text{(Eq2)} \qquad b_0 a_0^p + a_0 c_0^p = 0$$
$$\text{(Eq3)} \qquad d_0 b_0^p + c_0 d_0^p = 0$$
$$\text{(Eq4)} \qquad d_0 a_0^p + c_0^{p+1} = 0$$
$$\text{(Eq5)} \qquad a_0 d_0 - b_0 c_0 = 0$$
$$\text{(Eq6)} \qquad a_0 - d_0^2 - b_0 c_1 = 0.$$

We note that if any of the variables a_0, b_0, c_0, or d_0 is zero then so is a power of all the others. In this case, the associated reduced subscheme defines a smooth 1-dimensional deformation which coincides with the $j = 1$ locus, generically having invariants $(1, 1)$. Else, to find the components of the non-ordinary locus, we may invert a_0, b_0, c_0, and d_0. Using (Eq5) one checks that

$$b_0 \cdot \text{(Eq4)} = d_0 \cdot \text{(Eq2)}, \quad b_0 \cdot \text{(Eq3)} = d_0 \cdot \text{(Eq1)}, \quad b_0^p \cdot \text{(Eq2)} = a_0^p \cdot \text{(Eq1)}.$$

Thus, we may consider only the three equations (Eq1), (Eq5), (Eq6). Substituting using $a_0 = b_0 c_1 + d_0^2$ we reduce to the equations

$$b_0^{p+1} + b_0 c_1 d_0^p + d_0^{p+2} = 0, \quad b_0 c_1 d_0 + d_0^3 - b_0 c_0 = 0$$

in the ring $k[\![b_0, c_0, c_1, d_0]\!][b_0^{-1}, c_0^{-1}, c_1^{-1}, d_0^{-1}]$. Multiply the second equation by d_0^{p-1} and subtract from the first equation to reduce to the equations

$$b_0^p + c_0 d_0^{p-1} = 0, \quad d_0^3 - b_0 c_0 + b_0 c_1 d_0 = 0.$$

In order to compute the components of the non–ordinary locus through the given point, one proceeds as in the proof of 6.3.4 and computes the minimal prime ideals of $k[\![b_0, c_0, c_1, d_0]\!][b_0^{-1}, c_0^{-1}, c_1^{-1}, d_0^{-1}]$ associated to the ideal defined by the equations

$$b_0^p + c_0 d_0^{p-1} = 0, \quad d_0^{2p+1} + c_0^{p+1} - c_0 c_1^p d_0^p = 0.$$

As in loc. cit., one concludes that those prime ideals are in one to one correspondence with the minimal prime ideals associated to the ideal (0) in the ring $R_1 := k[\![c_1, d_0]\!][c_0]/(c_0^{p+1} - c_0 c_1^p d_0^p + d_0^{2p+1})$ not containing d_0. Since the polynomial $c_0^{p+1} + d_0^{2p+1}$ in the variable c_0 is irreducible over $k[\![d_0]\!]$, one concludes that R_1 is a domain.

We conclude that the non-ordinary locus is locally irreducible at points of type $(1,2)$. One can also calculate that the tangent space at a point of type $(1,2)$ to the deformation space into non-ordinary abelian varieties (given by (Eq1)-(Eq6)) is three dimensional and conclude that every point of type $(1,2)$ is a singular point of $W^c_{(0,1)}$.

9.1.2. The locus $W^c_{(0,2)}$. We next consider the problem of deforming a point of type $(1,2)$ into the $(0,2)$ locus. The condition that the a-number is at least 2 is equivalent to the condition $TF^2 \equiv 0 \pmod{T^3}$, where F is given by

$$F = \begin{pmatrix} -b_0^\sigma & T^2 + a_0^\sigma - d_0^\sigma T \\ T + d_0^\sigma & -c_0^\sigma - c_1^\sigma T \end{pmatrix}.$$

This is equivalent to the following matrix being congruent to 0 modulo T^2:

$$\begin{pmatrix} -b_0^\sigma & a_0^\sigma - d_0^\sigma T \\ T + d_0^\sigma & -c_0^\sigma - c_1^\sigma T \end{pmatrix} \begin{pmatrix} -b_0^{\sigma^2} & a_0^{\sigma^2} - d_0^{\sigma^2} T \\ T + d_0^{\sigma^2} & -c_0^{\sigma^2} - c_1^{\sigma^2} T \end{pmatrix}.$$

This provides the following equations:

$$\text{(Eq1)} \qquad a_0 d_0 - b_0 c_0 = 0$$
$$\text{(Eq2)} \qquad a_0 - d_0^2 - b_0 c_1 = 0$$
$$\text{(Eq3)} \qquad b_0^{p+1} + a_0 d_0^p = 0$$
$$\text{(Eq4)} \qquad a_0 - d_0^{p+1} = 0$$
$$\text{(Eq5)} \qquad d_0 b_0^p + c_0 d_0^p = 0$$
$$\text{(Eq6)} \qquad b_0^p + c_0 + c_1 d_0^p = 0$$
$$\text{(Eq7)} \qquad b_0 a_0^p + a_0 c_0^p = 0$$
$$\text{(Eq8)} \qquad b_0 d_0^p - a_0 c_1^p + d_0 c_0^p = 0$$
$$\text{(Eq9)} \qquad d_0 a_0^p + c_0^{p+1} = 0$$
$$\text{(Eq10)} \qquad a_0^p - d_0^{p+1} + c_1 c_0^p + c_0 c_1^p = 0.$$

We now substitute using (Eq4) $a_0 = d_0^{p+1}$ and obtain the following equations in the variables b_0, c_0, c_1, and d_0:

$$
\begin{array}{ll}
\text{(Eq1)} & d_0^{p+2} - b_0c_0 = 0 \\
\text{(Eq2)} & d_0^{p+1} - d_0^2 - b_0c_1 = 0 \\
\text{(Eq3)} & b_0^{p+1} + d_0^{2p+1} = 0 \\
\text{(Eq5)} & d_0b_0^p + c_0d_0^p = 0 \\
\text{(Eq6)} & b_0^p + c_0 + c_1d_0^p = 0 \\
\text{(Eq7)} & b_0d_0^{p^2+p} + d_0^{p+1}c_0^p = 0 \\
\text{(Eq8)} & b_0d_0^p - d_0^{p+1}c_1^p + d_0c_0^p = 0 \\
\text{(Eq9)} & d_0^{p^2+p+1} + c_0^{p+1} = 0 \\
\text{(Eq10)} & d_0^{p^2+p} - d_0^{p+1} + c_1c_0^p + c_0c_1^p = 0.
\end{array}
$$

We distinguish two cases:

Case 1: $d_0 = 0$.
This implies that a power of b_0 and of c_0 is zero. The associated reduced subscheme is the smooth curve given by c_1, which is the $(1, 1)$ curve already noticed above.

Case 2: we invert d_0.

We now multiply each equation by a suitable power of d_0 so that to substitute expressions of the form $c_0d_0^p$ by $-b_0^pd_0$ (using (Eq5)). We remark that the elimination of c_0 was justified by (Eq6). We arrive at the following system of equations in b_0, c_1, and d_0:

$$
\begin{array}{ll}
\text{(Eq1')} & d_0^{2p+1} + b_0^{p+1} = 0 \\
\text{(Eq2')} & d_0^{p+1} - d_0^2 - b_0c_1 = 0 \\
\text{(Eq6')} & b_0^pd_0^{p-1} - b_0^p + c_1d_0^{2p-1} = 0 \\
\text{(Eq7')} & b_0d_0^{2p^2-p-1} - b_0^{p^2} = 0 \\
\text{(Eq8')} & -b_0d_0^{p^2-1} + d_0^{p^2}c_1^p + b_0^{p^2} = 0 \\
\text{(Eq9')} & d_0^{2p^2+p} - b_0^{p^2+p} = 0 \\
\text{(Eq10')} & d_0^{2p^2} - d_0^{p^2+1} - b_0^{p^2}c_1 - d_0^{p^2-2p+1}b_0^pc_1^p = 0.
\end{array}
$$

Note that (Eq1') implies that $b_0 \neq 0$ and implies (Eq7') and (Eq9'). We may therefore consider only the system

$$
\begin{aligned}
&\text{(Eq1')} && d_0^{2p+1} + b_0^{p+1} = 0 \\
&\text{(Eq2')} && d_0^{p+1} - d_0^2 - b_0 c_1 = 0 \\
&\text{(Eq6')} && b_0^p d_0^{p-1} - b_0^p + c_1 d_0^{2p-1} = 0 \\
&\text{(Eq8')} && -b_0 d_0^{p^2-1} + d_0^{p^2} c_1^p + b_0^{p^2} = 0 \\
&\text{(Eq10')} && d_0^{2p^2} - d_0^{p^2+1} - b_0^{p^2} c_1 - d_0^{p^2-2p+1} b_0^p c_1^p = 0.
\end{aligned}
$$

We now show that (Eq1') and (Eq2') imply (Eq6') and (Eq8'), (Eq10'). Indeed, multiplying (Eq6') by b_0 we get

$$b_0(\mathrm{Eq6}') = d_0^{2p+1} + (-d_0^2 + d_0^{p+1}) d_0^{2p-1} - d_0^{2p+1} d_0^{p-1} = 0.$$

Multiplying (Eq8') by b_0^p, we get

$$b_0^p(\mathrm{Eq8}') = d_0^{2p+1} d_0^{p^2-1} + (-d_0^2 + d_0^{p+1})^p d_0^{p^2} + (-d_0^{2p+1})^p = 0.$$

Finally,

$$(\mathrm{Eq10}') = d_0^{2p^2} - d_0^{p^2+1} - (-d_0^{2p+1})^{p-1}(-d_0^2 + d_0^{p+1}) - d_0^{p^2-2p+1}(-d_0^2 + d_0^{p+1})^p = 0.$$

Hence, we are left with the system of equations

$$
\begin{aligned}
&\text{(Eq1')} && d_0^{2p+1} + b_0^{p+1} = 0 \\
&\text{(Eq2')} && d_0^{p+1} - d_0^2 - b_0 c_1 = 0.
\end{aligned}
$$

Recall that these equations are taken in a ring where d_0 is invertible, viz. in the ring $k[\![b_0, c_1, d_0]\!][d_0^{-1}]$. If I is the ideal generated by the equations (Eq1'), (Eq2') then the ring $k[\![b_0, c_1, d_0]\!][d_0^{-1}]/I$ is equal to the ring

$$k[\![b_0, c_1, d_0]\!][b_0^{-1}, d_0^{-1}]/(d_0^{2p+1} + b_0^{p+1}, d_0^{p+1} - d_0^2 - b_0 c_1).$$

Hence, we can eliminate c_1, putting $c_1 = d_0^2(d_0^{p-1} - 1)b_0^{-1}$ (note that $c_1^{p+1} = -d_0(d_0^{p-1} - 1)^{p+1}$, justifying the substitution) and conclude that the $(0,2)$-locus is given locally at a point $(1,2)$ by the irreducible equation

$$d_0^{2p+1} + b_0^{p+1} = 0$$

in the ring $k[\![b_0, d_0]\!]$ and hence is irreducible there.

9.2. Points of type $(1,1)$.

In this case $j = n = 1$ and $i = 1$. Hence, we may assume that $c_3 = 1$ in (6.3). The universal deformation space of $[\underline{A}_0]$ is defined by the ring

$$R := k[\![a_0, a_1, b_0, c_0, c_1, d_0]\!]/(a_0 d_0 - b_0 c_0, a_1 d_0 + a_0 - b_0 c_1, a_1 + d_0).$$

The matrix M of Frobenius F of the universal display is defined by

$$\begin{pmatrix} T - b_0^\sigma + d_0^\sigma & T^2 + a^\sigma - c^\sigma \\ T + d_0^\sigma & -c^\sigma \end{pmatrix}$$

with $a := a_0 + a_1 T$ and $c := c_0 + c_1 T$. The deformations in the non-ordinary locus, i. e., inside $W^c_{(0,1)}$, are defined by the condition that $T^2 F^2 = 0$ modulo T. This is equivalent to require that $M \cdot M^\sigma = 0 \bmod T$, i. e., to the vanishing of

$$\begin{pmatrix} -b_0^p + d_0^p & a_0^p - c_0^p \\ d_0^p & -c_0^p \end{pmatrix} \begin{pmatrix} -b_0^{p^2} + d_0^{p^2} & a_0^{p^2} - c_0^{p^2} \\ d_0^{p^2} & -c_0^{p^2}, \end{pmatrix}$$

which is equal to

$$\begin{pmatrix} b_0^{p^2+p} - b_0^p d_0^{p^2} - b_0^{p^2} d_0^p + d_0^{p^2+p} + a_0^p d_0^{p^2} - c_0^p d_0^{p^2} & -a_0^{p^2} b_0^p + a_0^{p^2} d_0^p + b_0^p c_0^{p^2} - c_0^{p^2} d_0^p - a_0^p c_0^{p^2} + c_0^{p^2+p} \\ -b_0^{p^2} d_0^p + d_0^{p^2+p} - c_0^p d_0^{p^2} & a_0^{p^2} d_0^p - c_0^{p^2} d_0^p + c_0^{p^2+p} \end{pmatrix}.$$

Hence, we get the following seven equations in the variables a_0, a_1, b_0, c_0, c_1 and d_0:

$$\text{(Eq1)} \qquad b_0^{p^2+p} - b_0^p d_0^{p^2} - b_0^{p^2} d_0^p + d_0^{p^2+p} + a_0^p d_0^{p^2} - c_0^p d_0^{p^2} = 0$$

$$\text{(Eq2)} \qquad -b_0^{p^2} d_0^p + d_0^{p^2+p} - c_0^p d_0^{p^2} = 0$$

$$\text{(Eq3)} \qquad -a_0^{p^2} b_0^p + a_0^{p^2} d_0^p + b_0^p c_0^{p^2} - c_0^{p^2} d_0^p - a_0^p c_0^{p^2} + c_0^{p^2+p} = 0$$

$$\text{(Eq4)} \qquad a_0^{p^2} d_0^p - c_0^{p^2} d_0^p + c_0^{p^2+p} = 0$$

$$\text{(Eq5)} \qquad a_0 d_0 - b_0 c_0 = 0$$

$$\text{(Eq6)} \qquad a_1 d_0 + a_0 - b_0 c_1 = 0$$

$$\text{(Eq7)} \qquad a_1 + d_0 = 0\ .$$

Case 1: Assume $d_0 = 0$. Then, a power of b_0 is 0 from (Eq1), a power of c_0 is 0 from (Eq4), $a_0 = 0$ from (Eq6) and $a_1 = 0$ from (Eq7). The only free variable left is c_1. Hence, the reduced subscheme defined by $d_0 = 0$ is 1-dimensional and coincides with universal deformation space inside the locus $W^c_{(1,1)}$, as already known.

Case 2: Let us invert $d_0 = 0$. Then

- $d_0^p(\text{Eq1}) = (-b_0^p + d_0^p)(\text{Eq2}) + d_0^{p^2}(\text{Eq5})^p$;
- $c_0^{p^2}(\text{Eq2}) = -d_0^{p^2}(\text{Eq4}) + d_0^p(\text{Eq5})^{p^2}$;
- $d_0^{p^2+p}(\text{Eq3} - \text{Eq4}) = -b_0^p d^p (\text{Eq5})^{p^2} - d_0^{p^2} c_0^{p^2} (\text{Eq5})^p + b_0^p c_0^{p^2} (\text{Eq2})$.

Hence, using $a_1 = -d_0$, $a_0 = d_0^2 + b_0 c_1$, (Eq5) and $d_0^{-p}(\text{Eq2})$, the system of equations (Eq1)–(Eq7) becomes equivalent to the system of equations

- $d_0^3 + b_0 c_1 d_0 - b_0 c_0 = 0$;
- $-b_0^p + d_0^p - c_0 d_0^{p-1} = 0$;

in $k[\![b_0, c_0, c_1, d_0]\!][d_0^{-1}]$.

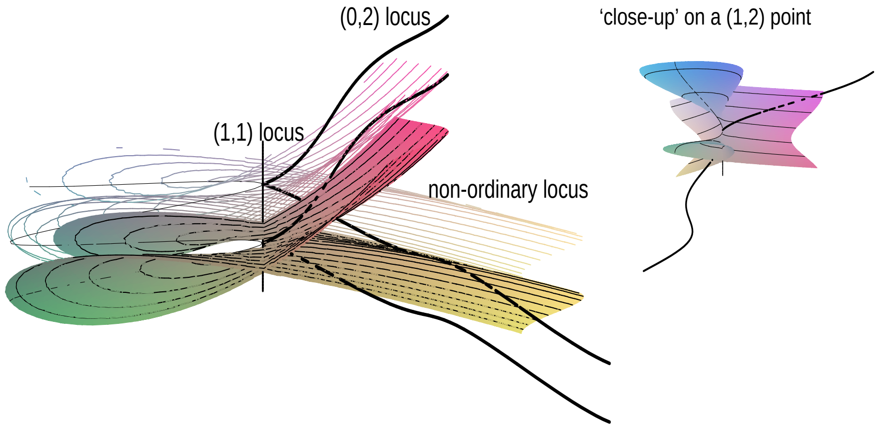


Figure 9.1: Hilbert modular threefold - maximally ramified case.

It follows from 6.3.5 that the nilradical of the ideal defined by these equations has exactly two minimal prime ideals. Hence, the locus $W^c_{(0,1)}$ is not analytically irreducible at the points of $W_{(1,1)}$. Studying the tangent space it is easily seen that $W^c_{(1,1)}$ is singular in $W^c_{(0,1)}$.

9.3. Summary.

We now come to some conclusions concerning the global structure of moduli space $\mathfrak{M}$ for L cubic totally real field and p maximally ramified in L.

Let $\mathfrak{B}$ be any component of $\mathfrak{M}$. By Proposition 6.3.3 the non-ordinary locus is irreducible. The locus $W^c_{(1,1)} = W_{(1,1)} \cup W_{(1,2)}$ is irreducible and non-singular, by loc. cit. and (3.1). The locus $W^c_{(0,2)} = W_{(0,2)} \cup W_{(1,2)} \cup W_{(0,3)}$ is a union of Moret-Bailly families, each component is singular only at the unique point (cf. [AG1, Prop. 6.6]) of $W_{(1,2)}$ lying on it. The components of the locus $W^c_{(0,2)}$ are disjoint, because intersection points can only be of type $(1,2)$, and by § 9.1.2 the locus is locally irreducible there. One can prove that the $W^c_{(0,1)}$ locus, and the $W^c_{(1,1)}$ locus are irreducible in each component of the moduli space in a different way. In fact, a similar use of the correspondences $\pi_1\pi_2^{-1}$, $\pi_2\pi_1^{-1}$, shows that one is irreducible if an only if the other is. We know by Theorem 6.2.3 that $\mathbb{T}_2 = W^c_{(1,1)} \cup W^c_{(0,2)}$ is connected, we know that each component of $W^c_{(0,2)}$ meets $W^c_{(1,1)}$ at a unique point, and we know that the locus $S_1 = W_{(1,1)} \cup W_{(1,2)}$ is non-singular. The implies that there is a unique component of $W_{(1,1)}$ in every component of $\mathfrak{M}$.

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