

Geometry of Hilbert Modular Varieties over Totally Ramified Primes

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1 Introduction

Let L be a totally real field with ring of integers O_L . Let $N \geq 4$ be an integer and let $\mathfrak{M}(\mu_N)$ be the fine moduli scheme over \mathbb{Z} of polarized abelian varieties with real multiplication (RM) and μ_N -level structure, satisfying the Deligne-Pappas condition. For every scheme S , we let $\mathfrak{M}(S, \mu_N) = \mathfrak{M}(\mu_N) \times_{\mathbb{Z}} S$ be the moduli scheme over S ; see [Definition 2.1](#).

Many aspects of the geometry of the modular varieties $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ are obtained via local deformation theory that factorizes according to the decomposition of p in O_L . The unramified case was considered in [\[9\]](#) (see also [\[8\]](#)). Given that, one may restrict one's attention to the case $p = \mathfrak{p}^e$ in O_L . We discuss here only the case $e = g$, that is, p is totally ramified in L .

The ramified case was first treated by Deligne and Pappas in [\[6\]](#) (the case $g = 2$ was considered in [\[2\]](#)). We recall some of their results under the assumption that p is totally ramified. Let A/k be a polarized abelian variety with RM, defined over a field k of characteristic p . Fix an isomorphism $O_L \otimes_{\mathbb{Z}} k \cong k[T]/(T^g)$. One knows that $H_{\text{dR}}^1(A)$ is a free $k[T]/(T^g)$ -module of rank 2. The elementary divisors theorem furnishes us with $k[T]/(T^g)$ -generators α and β for $H_{\text{dR}}^1(A)$ such that

$$H^1(A, O_A) = (T^i)\alpha + (T^j)\beta, \quad i \geq j, \quad i + j = g. \quad (1.1)$$

The index $j = j(A)$ gives a stratification \mathcal{S}_j of the moduli space $\mathfrak{M}(\mathbb{F}_p, \mu_N)$; the j th stratum parameterizes abelian varieties A with $j(A) \geq j$. We call this stratification the *singularity stratification*, and we call $j(A)$ the *singularity index* of A .

Using comparison of local moduli with a suitable Grassmannian variety, Deligne and Pappas gave local equations for a point in the stratum \mathcal{S}_j inside the stratum $\mathcal{S}_{j'}$ for any $j' \leq j$, implying, in particular, that \mathcal{S}_1 is the singular locus of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$. Their results imply that the \mathcal{S}_j -stratum, if nonempty, has dimension $g - 2j$. They did not prove that the \mathcal{S}_j -strata are nonempty.

To have a better understanding of the moduli space $\mathfrak{M}(\mathbb{F}_p, \mu_N)$, one would like to refine the singularity stratification. After recent works by Oort [17] and others [9, 12, 19, 20], one idea that comes to mind is to stratify the moduli space according to the isomorphism type of the p -torsion as a polarized group scheme with O_L -action. This approach is successful in the unramified case; see [8, 9]. When p is ramified, it turns out that this approach is not already desirable for $g = 2$: let k be an algebraically closed field of characteristic p and let L be a real quadratic field in which p ramifies. There are infinitely many nonisomorphic polarized group schemes with O_L -action arising as p -torsion of polarized abelian surfaces with RM by O_L (which would yield infinitely many different “strata”). This is proven in the appendix.

Still, one would like to refine the singularity stratification and study its relation to the Newton stratification and to arithmetic. To this end, we introduce another invariant. Given a polarized abelian variety A with RM, defined over a field k of characteristic p , we define its *slope* $n = n(A)$ by

$$j(A) + n(A) = a(A), \quad (1.2)$$

where $a(A)$ is the a -number of the abelian variety (for us, $a(A)$ is equal to the nullity of the Hasse-Witt matrix of A). One proves that

$$j \leq n \leq i. \quad (1.3)$$

We prove that there exists a locally closed subset of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$, denoted by $\mathfrak{W}_{(j,n)}$, that parameterizes abelian varieties with singularity index j and slope n . We prove that $\mathfrak{W}_{(j,n)}$ is a nonempty set and is a nonsingular variety of dimension

$$\dim(\mathfrak{W}_{(j,n)}) = g - (j + n). \quad (1.4)$$

The proof that $\mathfrak{W}_{(j,n)}$ is nonempty uses a construction of Moret-Bailly families and the study of the variation of (j, n) along these families, while the nonsingularity and

the dimension of $\mathfrak{W}_{(j,n)}$ follow from studying the local deformation theory via displays. The reason for calling n the slope is the following. Let $\beta_{r/g}$ denote the Newton polygon with the two slopes r/g and $(g-r)/g$, each of multiplicity g , and if g is odd, let $\beta_{1/2}$ denote the Newton polygon with unique slope $1/2$ of multiplicity $2g$. In our case, where p is totally ramified, the polygons $\beta_0, \beta_{1/g}, \dots, \beta_{1/2}$ are precisely the Newton polygons that appear on $\mathfrak{M}(\mathbb{F}_p, \mu_N)$. We also define

$$\lambda(n) = \min \left\{ \frac{n}{g}, \frac{1}{2} \right\}. \quad (1.5)$$

Then, the Newton polygon on $\mathfrak{W}_{(j,n)}$ is *constant*, equal to $\beta_{\lambda(n)}$. The proof of this result is based on the classification of the Dieudonné modules over the ring $O_L \otimes_{\mathbb{Z}} W(k)[F, V]$. The fact that the Newton polygon is constant on each $\mathfrak{W}_{(j,n)}$ allows us to confirm certain general conjectures concerning the Newton stratification on moduli spaces of abelian varieties for the moduli spaces we consider here. Such conjectures were proposed by Oort in [15] and generalized by Chai in [4]. In [16], Oort proved some of his conjectures in the “Siegel case,” that is, moduli spaces of principally polarized abelian varieties.

We prove that $\{\mathfrak{W}_{(j,n)} : 0 \leq j \leq n \leq g-j\}$ is a stratification. One consequence of the relation between n and the Newton polygon is that for $n < g/2$, both j and n go up under specialization and not only j and $j+n$. The exact determination of the boundary is somewhat involved. We prove the following result. Let

$$J = \{(j, n) : 0 \leq j \leq n \leq g-j, j, n \in \mathbb{Z}\}. \quad (1.6)$$

There exists a unique function $\Delta : 2^J \rightarrow 2^J$ determined by the following properties (we interpret elements of 2^J as subsets of J and write (j, n) for the singleton $\{(j, n)\}$):

- (i) for any integer $0 \leq j \leq g/2$, we have $\Delta(j, j) = \{(j', n') \in J : j' \leq j\}$;
- (ii) for any integer $0 \leq j \leq g/2$, we have $\Delta(j, g-j) = (j, g-j)$;
- (iii) for any integer $1 \leq j \leq g/2$, we have $\Delta(j-1, n) = \Lambda(\Delta(j, n))$, where Λ is given by an explicit recipe in [Definition 8.8](#).

We prove that

$$\overline{\mathfrak{W}_{(j,n)}} = \mathfrak{W}_{\Delta(j,n)} := \bigcup_{(j', n') \in \Delta(j,n)} \mathfrak{W}_{(j', n')}. \quad (1.7)$$

One hopes that the techniques, introduced here, of studying stratifications via p -isogenies will generalize. For example, in the Siegel case, one does not know yet the exact description of the boundary of an Ekedahl-Oort stratum (see [17]). We refer the reader to [1] for further results on the stratification defined in this paper and on the universal display of an abelian variety with RM.

2 Background and notation

2.1 Definition of the moduli problem

Throughout this paper, we fix a totally real field L of degree g over \mathbb{Q} . We denote by O_L its ring of integers, by D_L^{-1} its inverse different relative to \mathbb{Q} , by d_L its discriminant, and by $\text{Cl}(L)^+$ its strict class group. We fix a set of fractional ideals $\mathcal{R} = \{\mathfrak{I}_1, \dots, \mathfrak{I}_{h^+}\}$ of L that, endowed with their natural notion of positivity, form a complete set of representatives of $\text{Cl}(L)^+$. The moduli problem we are interested in is, roughly, that of parameterizing abelian varieties of dimension g with a given action of O_L , level structure, and polarization datum. A precise definition follows.

Definition 2.1. Let S be a scheme. Let N be a positive integer. Denote by

$$\mathfrak{M}(S, \mu_N) \longrightarrow S \quad (2.1)$$

the moduli stack over S of polarized abelian varieties with real multiplication by O_L and μ_N -level structure. It is a fibered category over the category of S -schemes. If T is a scheme over S , the objects of the stack over T are the polarized Hilbert-Blumenthal abelian schemes over T relative to O_L with μ_N -level structure, that is, quadruples $(A, \iota, \lambda, \varepsilon)$ consisting of

- (a) an abelian scheme $A \rightarrow T$ of relative dimension g ;
- (b) an O_L -action, that is, a ring homomorphism

$$\iota: O_L \hookrightarrow \text{End}_T(A); \quad (2.2)$$

- (c) a polarization

$$\lambda: (M_A, M_A^+) \xrightarrow{\sim} (\mathfrak{I}, \mathfrak{I}^+), \quad (2.3)$$

that is, an O_L -linear isomorphism on the étale site of S between the invertible O_L -module M_A of symmetric O_L -linear homomorphisms from A to its dual A^\vee and one of the fixed representatives $\mathfrak{I} \in \mathcal{R}$, identifying the positive cone of polarizations M_A^+ with \mathfrak{I}^+ ;

- (d) an O_L -linear injective homomorphism

$$\varepsilon: \mu_N \otimes_{\mathbb{Z}} D_L^{-1} \hookrightarrow A, \quad (2.4)$$

where for any scheme S over T , we define

$$(\mu_N \otimes_{\mathbb{Z}} D_L^{-1})(S) := \mu_N(S) \otimes_{\mathbb{Z}} D_L^{-1}. \quad (2.5)$$

We require that the following condition, called the *Deligne-Pappas condition*, holds:

(DP) the morphism $A \otimes_{O_L} M_A \rightarrow A^\vee$ is an isomorphism.

To ease notation, we will write \underline{A} for $(A, \iota, \lambda, \varepsilon)$.

The stack $\mathfrak{M}(S, \mu_N)$ is a disjoint union $\coprod_{\mathcal{J} \in \mathcal{R}} \mathfrak{M}(S, \mu_N, \mathcal{J})$, where $\mathfrak{M}(S, \mu_N, \mathcal{J})$ is defined as above with the proviso that the polarization module is \mathcal{J} . By works of Rapoport [18] and Deligne-Pappas [6] (building on works of others), the moduli stacks $\mathfrak{M}(S, \mu_N)$ and $\mathfrak{M}(S, \mu_N, \mathcal{J})$ are schemes for $N \geq 4$ and the morphism $\mathfrak{M}(S, \mu_N) \rightarrow S$ is flat, a locally complete intersection of relative dimension g and smooth over $S[d_L^{-1}]$. Furthermore, each geometric fiber of $\mathfrak{M}(S, \mu_N, \mathcal{J}) \rightarrow S$ is irreducible, normal, and of dimension g .

Over the complex numbers, the underlying analytic variety of $\mathfrak{M}(\mathbb{C}, \mu_N, \mathcal{J})$ is isomorphic to $\Gamma_{\mathcal{J}} \backslash \mathfrak{H}^g$, where $\Gamma_{\mathcal{J}}$ is a suitable discrete subgroup of $SL_2(L)$ acting, by a twisted diagonal action, on the g -fold product of the Poincaré upper half plane \mathfrak{H} .

2.2 (DP) versus (R)

In his paper [18], Rapoport posed the condition:

(R) $\Omega^1_{A/T}$ is a locally free $O_T \otimes_{\mathbb{Z}} O_L$ -module

instead of condition (DP). (We will refer to this condition as the *Rapoport condition*.) However, Deligne and Pappas found that condition (R) is not stable under taking limits; the problem is not with the cusps, but rather with the existence of families of polarized abelian varieties with real multiplication by O_L that generically, but not everywhere, satisfy (R). The situation is as follows.

(i) Condition (R) implies condition (DP). Indeed, it is enough to prove that an abelian variety over a field k , with real multiplication, satisfying (R), has a polarization of degree prime to ℓ for *any* prime ℓ . If k is of characteristic 0, this follows from complex uniformization. If k has positive characteristic, condition (R) and crystalline techniques allow one to lift the abelian variety with its real multiplication to characteristic 0 [18, Corollary 1.13] and hence to conclude the existence of such polarizations.

(ii) Over base schemes S in which d_L is invertible, the conditions (R) and (DP) are equivalent [6, Corollary 2.9].

(iii) Every ordinary abelian variety satisfies (R), hence (DP). The argument here is similar to the one above; the lift to characteristic 0 is provided by the Serre-Tate canonical lift.

(iv) If S is the spectrum of a field of positive characteristic p dividing d_L , but not N , there exists a pure codimension 2 subscheme of $\mathfrak{M}(S, \mu_N, \mathcal{J})$, where condition (R) does not hold. Moreover, the singular locus of $\mathfrak{M}(S, \mu_N, \mathcal{J})$ is given by

$$\mathfrak{M}(S, \mu_N, \mathcal{I})^{\text{sing}} = \mathfrak{M}(S, \mu_N, \mathcal{I}) \setminus \mathfrak{M}(S, \mu_N, \mathcal{I})^R, \quad (2.6)$$

where, by definition, $\mathfrak{M}(S, \mu_N, \mathcal{I})^R$ is the open subscheme of $\mathfrak{M}(S, \mu_N, \mathcal{I})$, where condition (R) holds [6, Proposition 4.4]. The example of $g = 2$ was worked out in detail in [2].

It is important to note that there are abelian varieties with real multiplication that do not satisfy condition (R). In fact, taking an abelian variety A with real multiplication that satisfies (R) and a general O_L -invariant subgroup H of A , the typical case is that A/H , with its canonical O_L -structure, does not satisfy (R).

3 The Deligne-Pappas condition

Let $(A, \iota)/S$ be an abelian scheme over S with real multiplication by O_L . Let $(A^\vee, \iota^\vee)/S$ be the dual abelian scheme of A with the induced O_L -action. Let $M_A := \text{Hom}_{O_L}(A, A^\vee)^{\text{sym}}$ be the O_L -module of symmetric O_L -linear homomorphisms from A to A^\vee as in Definition 2.1. As a sheaf on the étale site of S , M_A is a projective O_L -module of rank 1 generated by M_A^+ ; see [3] and [18, Proposition 1.17].

Proposition 3.1. The following are equivalent:

- (1) the natural morphism

$$\Phi : A \otimes_{O_L} M_A \longrightarrow A^\vee \quad (3.1)$$

is an isomorphism (as étale sheaves on S);

- (2) for any integer t , there exists, étale locally on S , a symmetric O_L -linear polarization

$$\lambda_t : A \longrightarrow A^\vee \quad (3.2)$$

of degree prime to t ;

- (3) as in (2) where t ranges among primes. □

Proof. (1) \Rightarrow (2). By étale localization on S , we can assume that there exists a polarization $\lambda_t \in M_A$ such that the homomorphism of O_L -modules

$$O_L/tO_L \xrightarrow{\eta} M_A/tM_A, \quad r \longmapsto r\lambda_t, \quad (3.3)$$

is an isomorphism. We have, then, the following isomorphisms:

$$A[t] \xrightarrow{\sim} A[t] \otimes_{O_L} (O_L/tO_L) \xrightarrow{1 \otimes \eta} A[t] \otimes_{O_L} (M_A/tM_A) \xrightarrow{\sim} A \otimes_{O_L} (M_A/tM_A). \quad (3.4)$$

On the other hand, by comparing degrees, one finds that the natural map

$$(A \otimes_{O_L} M_A)[t] \longrightarrow A \otimes_{O_L} (M_A/tM_A) \quad (3.5)$$

is an isomorphism. Furthermore, by assumption, Φ induces an isomorphism

$$(A \otimes_{O_L} M_A)[t] \xrightarrow{\sim} A^\vee[t]. \quad (3.6)$$

Let $A[t] \rightarrow A^\vee[t]$ be the composition of all these maps. It is an isomorphism and it coincides with λ_t restricted to $A[t]$. Hence, $\text{Ker}(\lambda_t)$ has order prime to t as wanted.

(2) \Rightarrow (3). Clear.

(3) \Rightarrow (1). On the étale site of S , M_A is a projective O_L -module of rank 1 generated by polarizations. Hence, Φ is an isogeny. Let $K := \text{Ker}(\Phi)$. It is a finite flat group scheme over S . Let ℓ be prime and choose, étale locally, a polarization λ_ℓ of degree prime to ℓ . The isogeny λ_ℓ can be factored as

$$A \longrightarrow A \otimes M_A \xrightarrow{\Phi} A^\vee, \quad (3.7)$$

where the first arrow is the isogeny $\alpha \mapsto \alpha \otimes \lambda_\ell$. We deduce that the kernel of λ_ℓ surjects onto K . Therefore, K has order prime to ℓ for any prime ℓ . \blacksquare

Corollary 3.2. Let $(A, \iota)/S$ be an abelian scheme satisfying (DP). Let $H \hookrightarrow A$ be an O_L -invariant, finite, locally free, closed subgroup scheme of A having rank p^a for some prime p and some integer a . Suppose that the following holds:

- (1) there exists $r \in O_L$ such that $H \hookrightarrow A[r]$ and $p^a \parallel \text{Norm}(r)$;
- (2) for every geometric point $s \in S$,
 - (2a) if p is prime to the characteristic of $k(s)$, the constant group scheme H_s is generated by one element as an O_L -module;
 - (2b) if p is the characteristic of $k(s)$, then the Dieudonné module associated to H_s is generated by one element as an $O_L \otimes \mathbf{W}(k(s))$ -module.

Then, the O_L -action on A descends to A/H and A/H satisfies (DP). \square

Proof. The O_L -action on A clearly descends to A/H . Let t be any prime. By [Proposition 3.1](#), it is enough to prove that étale locally A/H has a polarization of degree prime to t .

Let $\ell = t$ if $t \neq p$, and let ℓ be a prime different from p otherwise. By étale localization and [Proposition 3.1](#), we may assume that there exists a polarization $\lambda_\ell : A \rightarrow A^\vee$ of degree prime to ℓp . One may replace r by an element $r' \in O_L$ such that condition (1) still holds and r' is prime to $\ell \cdot \deg(\lambda_\ell)$. Indeed, $A[r] = A[(r)] = \bigoplus_{\mathbf{p}^i \parallel (r)} A[\mathbf{p}^i]$, and therefore the existence of r' follows from the weak approximation theorem. Without loss of generality, r is prime to $\ell \cdot \deg(\lambda_\ell)$.

The homomorphism $\lambda_\ell \circ r : A \rightarrow A^\vee$ is a polarization of degree $\text{Norm}(r)^2 \cdot \deg(\lambda_\ell)$. By hypothesis,

$$H \subset \text{Ker}(\lambda_\ell \circ r) = A[r] \oplus \text{Ker}(\lambda_\ell), \quad (3.8)$$

and H is O_L -invariant. If H is isotropic in $A[r]$ with respect to the Mumford pairing defined by $\lambda_\ell \circ r$, then $\lambda_\ell \circ r$ descends to a polarization on A/H of degree equal to $\text{Norm}(r)^2 \cdot \deg(\lambda_\ell)/\#H^2$ which is prime to ℓp , hence to t .

It remains to prove that indeed H is isotropic. We may assume that S is the spectrum of an algebraically closed field k . We distinguish two cases.

Case 1. The characteristic of k is prime to p . Then, $A[r] \otimes \mathbb{Z}_p$ is étale and isomorphic to a free $O_{L,p}/rO_{L,p}$ -module of rank 2, considered as a constant group scheme, with an alternating and perfect pairing for which the action of O_L is selfadjoint. By (2a), $H \subset A[r] \otimes \mathbb{Z}_p \subset A[r]$ is generated by one element as an O_L -module, hence isotropic.

Case 2. The characteristic of k is p . Then, the Dieudonné module of $A[r] \otimes \mathbb{Z}_p$ is isomorphic to a free $(O_L/rO_L) \otimes \mathbf{W}(k)$ -module of rank 2 [18, Lemma 1.3] with alternating and perfect pairing for which the action of O_L is selfadjoint. By (2b), the Dieudonné module of H is a $(O_L/rO_L) \otimes \mathbf{W}(k)$ -submodule generated by one element, and hence isotropic. ■

Corollary 3.3. Let $(A, \iota)/S$ be an abelian scheme satisfying (DP). Let $H \hookrightarrow A$ be an O_L -invariant, finite, locally free, closed subgroup scheme of A having rank n . Assume that the primes of O_L dividing n have residue degree 1. Then, the O_L -action descends from A to A/H and A/H satisfies (DP). □

Proof. Clearly, the action of O_L descends to the quotient A/H . Assume first that S is the spectrum of an algebraically closed field k . Note that H has an O_L -primary decomposition $H = \bigoplus_{p^i \parallel n} H[p^i]$. We may reduce to the case $H = H[p^i]$ and then, using the exact sequence

$$0 \longrightarrow H/H[p] \longrightarrow A/H[p] \longrightarrow A/H \longrightarrow 0, \quad (3.9)$$

to the case $H \subset A[p]$.

If H is trivial, then there is nothing to prove. If $H = A[p]$, then H is the direct sum of two O_L -invariant subgroup schemes, each of which satisfies condition (2) of Corollary 3.2. If H is a strict subgroup of $A[p]$, we have that H has order equal to $\text{Norm}(p)$. We may

therefore reduce to that case where H satisfies condition (2) of [Corollary 3.2](#). The corollary follows from [Corollary 3.2](#) by choosing r a local uniformizer at p which is a unit at all other primes dividing $\text{Norm}(p)$.

For a general S , by [Proposition 3.1](#), it suffices to prove that for any prime ℓ étale locally on S , there exists a polarization λ_ℓ on A/H of degree prime to ℓ . Passing to an étale covering of S , we may assume that the O_L -module $M_{A/H}$ is locally free of rank 1. Let $s \in S$ be a geometric point. Consider the reduction map

$$\gamma_s : M_{A/H} \longrightarrow M_{A_s/H_s}. \quad (3.10)$$

It is O_L -invariant and injective. Hence, there exists $m \in \mathbb{Z}$ killing the cokernel of γ_s . Let $\lambda_s \in M_{A_s/H_s}^+$. Let $\delta \in M_{A/H}$ such that $\gamma_s(\delta) = m\lambda_s$. Since δ is a polarization, $\text{Ker}(\delta)$ is a finite flat subgroup scheme. Therefore, the inclusion $\text{Ker}([m])_s \subset \text{Ker}(\delta)_s$ implies $\text{Ker}([m]) \subset \text{Ker}(\delta)$. Hence, there exists $\lambda \in M_{A/H}$ such that $\gamma_s(\lambda) = \lambda_s$. This proves that γ_s is an isomorphism. We conclude by the first part of the argument applied to A_s/H_s . ■

Remark 3.4. As the following example shows, the assumptions in [Corollary 3.3](#) are necessary. Let $g = 2$ and let p be an inert prime. Let A be an abelian surface with real multiplication by O_L satisfying (DP) with α -number equal to 1. Then, the quotient A/α_p of A by its unique O_L -invariant α_p -subgroup scheme does not satisfy (DP).

Corollary 3.5. In the notation of [Corollary 3.3](#), assume that H has order p and that $p = p^g$ is totally ramified in L . Let $\pi : A \rightarrow A/H$ be the canonical isogeny. Then, $\pi^*(M_{A/H}) = pM_A$. □

Proof. We treat only the case $S = \text{Spec}(k)$, where k is a perfect field of characteristic p , leaving the general case to the reader. Let λ be a polarization on A such that $\text{Ker}(\lambda) \supset H$. Then, the Mumford pairing on $\text{Ker}(\lambda)[p^\infty]$ induces a nondegenerate alternating O_L -pairing on the Dieudonné module of this group scheme which is isomorphic to $(O_L \otimes W(k))/p^i \oplus (O_L \otimes W(k))/p^j$ for suitable i and j . It follows that $i = j$ and, in particular, $\text{Ker}[\lambda] \supset A[p]$. On the other hand, any subgroup H of order p is isotropic with respect to any alternating pairing induced by a polarization. Hence, any polarization λ with $\text{Ker}[\lambda] \supset A[p]$ descends to A/H . Vice versa, for any $v \in M_{A/H}$, we have $\text{Ker}(\pi^*(v)) \supset A[p]$.

Since A satisfies (DP), there exists $\gamma \in M_A$ of degree prime to p . In particular, $O_{L_p} \cdot \gamma = M_A \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let $\lambda \in M_A$. Write $\lambda = r \cdot \gamma$ with $r \in O_{L_p}$. Then, the Dieudonné module of the p -part of $\text{Ker}(\lambda)$ is isomorphic to $(O_L \otimes W(k)/(r))^2$. Hence, the inclusion $\text{Ker}[\lambda] \supset A[p]$ is equivalent to $\text{val}_p(r) \geq 1$, which, in turn, is equivalent to $\lambda \in pM_A$. ■

4 Displays with real multiplication

4.1 Displays

Let k be a perfect field of characteristic $p > 0$. Let A_0 be an abelian variety over k of dimension g of p -rank equal to 0. One can associate to A_0 its covariant Dieudonné module. It is a module over the Witt vectors $\mathbf{W}(k)$ of k , of rank $2g$, endowed with a σ -linear morphism F and a σ^{-1} -linear morphism V . The assumption on the p -rank implies that V is topologically nilpotent. The Dieudonné module coincides with the dual of $H_{\text{crys}}^1(A_0/\mathbf{W}(k))$. In [21], the notion of a display over a ring R is introduced. It is also proven there that the notion of a Dieudonné module over $\mathbf{W}(k)$, such that V is topologically nilpotent, is equivalent to the notion of a display over k . Furthermore, it is shown that the deformation theory of A_0 is equivalent to the deformation theory of the associated display. What is of interest to us is that the language of displays allows one to describe explicitly the equicharacteristic deformation theory of A_0 (possibly with extra structure, e.g., O_L -action), obtaining the deformation of the Frobenius morphism.

4.2 O_L -displays

The reader is referred to [21] for the definition and theory of displays. We define here the notion of display with real multiplication by O_L and reformulate it in a language that emphasizes the O_L -linear structure of this setting.

Definition 4.1 (cf. [21, Definition 1]). Let R be a ring. Let $\mathbf{W}(R)$ be the Witt vectors over R and let σ be the Frobenius morphism on $\mathbf{W}(R)$. A *display with real multiplication* by O_L over R , or an O_L -display over R , is a quadruple (P, Q, V^{-1}, F) , where

- (1) P is a projective $O_L \otimes \mathbf{W}(R)$ -module of rank 2;
- (2) $Q \subset P$ is a finitely generated $O_L \otimes \mathbf{W}(R)$ -submodule of P such that $I_R P \subset Q \subset P$ and P/Q is a direct summand of the $\mathbf{W}(R)$ -module $P/I_R P$;
- (3) $F : P \rightarrow P$ is linear with respect to O_L and σ -linear with respect to $\mathbf{W}(R)$;
- (4) $V^{-1} : Q \rightarrow P$ is linear with respect to O_L and σ -linear with respect to $\mathbf{W}(R)$, and $V^{-1}(Q)$ generates P as a $\mathbf{W}(R)$ -module.

We require that for any $w \in \mathbf{W}(R)$ and any $y \in P$, we have

$$V^{-1}(w^V \cdot y) = w \cdot F(y). \quad (4.1)$$

One imposes a further nilpotence condition as in [21, Definition 11].

Remark 4.2. There exists an $O_L \otimes \mathbf{W}(R)$ -submodule L and a $\mathbf{W}(R)$ -submodule T of P satisfying $P = L \oplus T$ and $Q = L \oplus I_R T$. This decomposition is not canonical, though.

Remark 4.3. Let k be a perfect field of characteristic $p > 0$. Let (A, ι) be an abelian variety over k with p -rank equal to 0 and with real multiplication by O_L . Let (\mathbb{D}, F, V) be the co-variant Dieudonné module of A . It is a projective $O_L \otimes_{\mathbb{Z}} \mathbf{W}(k)$ -module of rank 2 [18], and the Frobenius and Verschiebung morphisms of \mathbb{D} are O_L -linear. Let Q be the image of V . The exact sequence of $O_L \otimes_{\mathbb{Z}} \mathbf{W}(k)$ -modules

$$0 \longrightarrow Q \longrightarrow P \longrightarrow P/Q \longrightarrow 0 \quad (4.2)$$

lifts the $O_L \otimes_{\mathbb{Z}} k$ -exact sequence

$$0 \longrightarrow \mathrm{Lie}(A^\vee)^* \longrightarrow H_{1, \mathrm{dR}}(A/k) \longrightarrow \mathrm{Lie}(A) \longrightarrow 0. \quad (4.3)$$

The module L is chosen as an $O_L \otimes_{\mathbb{Z}} \mathbf{W}(k)$ -lift of $\mathrm{Lie}(A^\vee)^*$, while the module T is chosen as a $\mathbf{W}(k)$ -lift of $\mathrm{Lie}(A)$ which mod p splits the exact sequence. The reader may check that, when $\mathrm{Lie}(A^\vee)^*$ is not a free $O_L \otimes_{\mathbb{Z}} k$ -module, one cannot choose T to be O_L -invariant.

Definition 4.4 (cf. [21, Definition 18]). Let $(P_1, Q_1, V_1^{-1}, F_1)$ and $(P_2, Q_2, V_2^{-1}, F_2)$ be two displays (resp., O_L -displays) over R . A *bilinear form of displays* (resp., O_L -displays) is a map

$$\langle \cdot, \cdot \rangle : P_1 \times P_2 \longrightarrow \mathbf{W}(R), \quad (4.4)$$

(resp., $\mathrm{Hom}(O_L, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbf{W}(R)$) such that

- (a) $\langle \cdot, \cdot \rangle$ is $\mathbf{W}(R)$ -bilinear (resp., $O_L \otimes \mathbf{W}(R)$ -bilinear);
- (b) $\langle V_1^{-1}(x), V_2^{-1}(y) \rangle = \langle x, y \rangle$ for any $x \in Q_1$ and $y \in Q_2$.

Proposition 4.5. Let $(P_1, Q_1, V_1^{-1}, F_1)$ and $(P_2, Q_2, V_2^{-1}, F_2)$ be two O_L -displays over R . The trace map $\mathrm{Tr} : L \rightarrow \mathbb{Q}$ defines a one-to-one correspondence between the following:

- (i) the set $\{\Phi : P_1 \times P_2 \rightarrow \mathbf{W}(R)\}$ of bilinear forms of displays such that $\Phi(rx, y) = \Phi(x, ry)$ for any $r \in O_L$, $x \in P_1$, and $y \in P_2$;
- (ii) the set $\{\langle \cdot, \cdot \rangle : P_1 \times P_2 \rightarrow \mathrm{Hom}(O_L, \mathbb{Z}) \otimes \mathbf{W}(R)\}$ of bilinear forms of O_L -displays.

□

Proof. See [6, Section 2.11]. ■

Definition 4.6. Let (P, Q, V^{-1}, F) be an O_L -display over R . An O_L -*polarization* is an alternating bilinear form of O_L -displays. We say that it is *principal* if its image is equal to $\mathrm{Hom}(O_L, \mathbf{W}(R))$.

Theorem 4.7 (cf. [21]). Let \underline{A}_0 be a polarized abelian variety with real multiplication by O_L over a field k of characteristic p of p -rank equal to 0. Let $(P_0, Q_0, V_0^{-1}, F_0)$ be the associated polarized O_L -display. Let R be a complete, Noetherian, local ring with residue field k . There is an equivalence of categories between the category of polarized O_L -displays over R deforming $(P_0, Q_0, V_0^{-1}, F_0)$ and the category of polarized abelian schemes over R with real multiplication by O_L deforming \underline{A}_0 . \square

4.3 Solving Frobenius equations

In this section, we discuss the solvability of equations of the form $x^{\sigma^n} = bx$ for certain p -adic rings. This is a step in providing a normal form for O_L -displays; see [Proposition 4.10](#).

Let R be a Henselian ring of positive characteristic p with a separably closed residue field k . Let $h(T)$ be an Eisenstein polynomial in $\mathbf{W}(\mathbb{F}_{p^a})[T]$. Define the ring

$$B := \mathbf{W}(R)[T]/(h(T)) \quad (4.5)$$

and the *Frobenius* ring automorphism $\sigma : B \rightarrow B$ which is the identity on T and is given on $\mathbf{W}(R)$ by $(x_0, x_1, \dots) \mapsto (x_0^{p^a}, x_1^{p^a}, \dots)$. For any integer n , define the group homomorphism $\phi_n : B^* \rightarrow B^*$ by

$$\phi_n(\lambda) = \frac{\lambda^{\sigma^n}}{\lambda}. \quad (4.6)$$

Proposition 4.8. For any nonzero integer n , the homomorphism $\phi_n : B^* \rightarrow B^*$ is surjective. \square

Proof. We can assume, without loss of generality, that n is a positive integer. Define ring automorphisms

$$\mathbf{W}(R)[T] \xrightarrow{\sigma} \mathbf{W}(R)[T], \quad \mathbf{W}(R)[[T]] \xrightarrow{\sigma} \mathbf{W}(R)[[T]] \quad (4.7)$$

to be Frobenius on $\mathbf{W}(R)$ and to satisfy $\sigma(T) = T$. The ring $\mathbf{W}(R)$ is p -adically complete and separated [21]. In particular, the ring B is p -adically complete and separated. Since $h(T)$ is Eisenstein, the σ -equivariant ring homomorphism $\mathbf{W}(R)[T] \rightarrow B$ extends to a surjective σ -equivariant ring homomorphism

$$\mathbf{W}(R)[[T]] \longrightarrow B, \quad (4.8)$$

which is surjective also on units. It suffices to prove that for all $m \in \mathbf{W}(\mathbf{R})[[T]]^*$, there exists $\lambda \in \mathbf{W}(\mathbf{R})[[T]]^*$ such that

$$\lambda^{\sigma^n} = m\lambda. \quad (4.9)$$

Write

$$\lambda = \lambda_0 + \lambda_1 T + \lambda_2 T^2 + \cdots, \quad m = m_0 + m_1 T + m_2 T^2 + \cdots \quad (4.10)$$

with $\lambda_i, m_i \in \mathbf{W}(\mathbf{R})$ for all $i \in \mathbb{N}$ and $m_0 \in \mathbf{W}(\mathbf{R})^*$. We need to solve the system of equations

$$\begin{aligned} \lambda_0^{\sigma^n} &= m_0 \lambda_0, \\ \lambda_1^{\sigma^n} &= m_1 \lambda_0 + m_0 \lambda_1, \\ &\vdots \\ \lambda_t^{\sigma^n} &= \sum_{i=0}^t m_i \lambda_{t-i}, \end{aligned} \quad (4.11)$$

Proceeding by induction on t , it is enough to prove the following claim.

Claim 4.9. Let $A \in \mathbf{W}(\mathbf{R})^*$ and $D \in \mathbf{W}(\mathbf{R})$, and let n be a positive integer. The equation

$$x^{\sigma^n} = Ax + D \quad (4.12)$$

admits a solution in $\mathbf{W}(\mathbf{R})$. Moreover, if $D = 0$ modulo the maximal ideal of $\mathbf{W}(\mathbf{R})$, the solution can be chosen in $\mathbf{W}(\mathbf{R})^*$. \square

Proof. Let S_t and P_t with $t \in \mathbb{N}$, respectively, be the addition and multiplication polynomials for the Witt ring. Recall that

- (i) $S_t(\alpha_0, \dots, \alpha_t; \beta_0, \dots, \beta_t) = \alpha_t + \beta_t + \text{polynomial in } \alpha_0, \dots, \alpha_{t-1}, \beta_0, \dots, \beta_{t-1};$
- (ii) $P_t(\alpha_0, \dots, \alpha_t; \beta_0, \dots, \beta_t) = \alpha_t \beta_0^{p^t} + \beta_t \alpha_0^{p^t} + \text{polynomial in } \alpha_0, \dots, \alpha_{t-1}, \beta_0, \dots, \beta_{t-1}.$

Write

$$x = (x_0, x_1, x_2, \dots), \quad A = (A_0, A_1, A_2, \dots), \quad D = (D_0, D_1, D_2, \dots). \quad (4.13)$$

We need to solve the equation

$$\begin{aligned} x^{\sigma^n} &= (x_0^{p^{an}}, \dots, x_t^{p^{an}}, \dots) \\ &= (S_0(P_0(A_0, x_0); D_0), \dots, \\ &\quad S_t(P_0(A_0; x_0), \dots, P_t(A_0, \dots, A_t; x_0, \dots, x_t); D_0, \dots, D_t), \dots). \end{aligned} \quad (4.14)$$

For $t = 0$, we have

$$x_0^{p^{an}} - A_0 x_0 - D_0 = 0, \quad A_0, D_0 \in R. \quad (4.15)$$

The left-hand side is a separable polynomial because $A_0 \in R^*$. Since the residue field of R is separably closed, by Hensel's lemma, we conclude that such a solution x_0 exists. If $D_0 = 0$ modulo the maximal ideal of R , then x_0 can be chosen in R^* .

Assume that we found a solution $\{x_0, \dots, x_{t-1}\}$ for the first t equations. The $(t+1)$ th equation is

$$\begin{aligned} 0 &= x_t^{p^{an}} - S_t(P_0(A_0; x_0), \dots, P_t(A_0, \dots, A_t; x_0, \dots, x_t); D_0, \dots, D_t) \\ &= x_t^{p^{an}} - D_t - P_t(A_0, \dots, A_t; x_0, \dots, x_t) \\ &\quad + \text{pol}\{D_0, \dots, D_{t-1}, A_0, \dots, A_{t-1}, x_0, \dots, x_{t-1}\} \\ &= x_t^{p^{an}} - D_t - x_t A_0^{p^{at}} + A_t x_0^{p^{at}} + \text{pol}\{D_0, \dots, D_{t-1}, A_0, \dots, A_{t-1}, x_0, \dots, x_{t-1}\}. \end{aligned} \quad (4.16)$$

This amounts to solving an equation of the form

$$x_t^{p^{an}} - A_0^{p^{at}} x_t - C = 0 \quad (4.17)$$

with $C \in R$. Since $A_0 \in R^*$ and R is Henselian with separably closed residue field, we conclude that the equation admits a zero in R . ■

This concludes the proof of [Proposition 4.8](#). ■

4.4 Normal form for O_L -displays

Let k be an algebraically closed field of positive characteristic p . Suppose that $p = p^g$ is totally ramified in L . In particular,

$$O_L \otimes \mathbf{W}(k) \xrightarrow{\sim} \mathbf{W}(k)[T]/(h(T)), \quad (4.18)$$

where $h(T)$ is an Eisenstein polynomial of degree g .

Proposition 4.10 (normal form). Let $(P_0, Q_0, V_0^{-1}, F_0)$ be an O_L -display over k . Let $\bar{P}_0 := P_0 \otimes (\mathbf{W}(k)/p\mathbf{W}(k))$, $\bar{Q}_0 := \text{Image}(Q_0 \subset P_0 \rightarrow \bar{P}_0)$, and $\bar{F}_0 : \bar{P}_0 \rightarrow \bar{P}_0$ be the reduction of F_0 . Let $\langle \cdot, \cdot \rangle_0 : P_0 \times P_0 \rightarrow \text{Hom}(O_L, \mathbf{W}(k))$ be a principal O_L -polarization. There exist α_0 and β_0 in P_0 such that

- (a) $P_0 = (O_L \otimes W(k))\alpha_0 \oplus (O_L \otimes W(k))\beta_0$;
 (b) the Hodge filtration $\bar{Q}_0 = \text{Ker}(\bar{F}_0) \subset \bar{P}_0$ is defined by

$$\bar{Q}_0 = (\bar{T}^i)\bar{\alpha}_0 \oplus (\bar{T}^j)\bar{\beta}_0 \subset \bar{P}_0 = \frac{O_L}{pO_L}\bar{\alpha}_0 \oplus \frac{O_L}{pO_L}\bar{\beta}_0, \quad (4.19)$$

where $i + j = g$ and $0 \leq j \leq i \leq g$;

- (c) there exist
 (c1) a nonnegative integer $m \geq j$,
 (c2) a unit $c_3 \in (O_L \otimes W(k))^*$ such that

$$F_0(\alpha_0) = T^m\alpha_0 + T^j\beta_0, \quad F_0(\beta_0) = c_3T^i\alpha_0. \quad (4.20)$$

□

Proof. It follows from the properties of $\langle \cdot, \cdot \rangle_0$ that \bar{Q}_0 is a maximal totally isotropic $O_L \otimes k$ -submodule of \bar{P}_0 . Moreover, $P_0 \otimes k$ is a free $O_L \otimes k$ -module of rank 2. Note that $O_L \otimes k$ is an Artinian local ring. Hence, we may find α_0 and β_0 in P_0 such that (a) and (b) hold. Note that $T^i \cdot T^j$ is equal to p up to a unit in $O_L \otimes W(k)$. From (b), we deduce that F_0 is of the form

$$F_0(\alpha_0) = c_1T^j\alpha_0 + c_2T^j\beta_0, \quad F_0(\beta_0) = c_3T^i\alpha_0 + c_4T^i\beta_0, \quad (4.21)$$

where c_1, c_2, c_3 , and c_4 are in $O_L \otimes W(k)$. Since

$$\langle F_0(\alpha_0), F_0(\beta_0) \rangle_0 = p \langle \alpha_0, \beta_0 \rangle_0^\sigma \quad (4.22)$$

and $\langle \alpha_0, \beta_0 \rangle_0$ generates $\text{Hom}(O_L, W(k))$ by assumption, we conclude that

$$c_1c_4 - c_2c_3 \in (O_L \otimes W(k))^*. \quad (4.23)$$

Case 1 (c_2 and c_3 are units). Substituting α_0 with $\alpha_0 - c_3^{-1}c_4\beta_0$, we can assume that $c_4 = 0$. Note that the new c_2 and c_3 are units. Let $c_1 = \epsilon T^y$ with ϵ a unit and $y \geq 0$. Note that $O_L \otimes W(k) = W(k)[T]/(h(T))$, where $h(T)$ is an Eisenstein polynomial. Moreover, the automorphism of $O_L \otimes W(k)$ that is trivial on O_L and is given by Frobenius on $W(k)$ corresponds to the automorphism of $W(k)[T]/(h(T))$ that is trivial on T and is equal to Frobenius on $W(k)$. We may now apply [Proposition 4.8](#) to $W(k)[T]/(h(T))$ and deduce that there exists $\lambda \in (O_L \otimes W(k))^*$ such that $\epsilon^{-1} = \lambda^\sigma \lambda^{-1}$. Substituting α_0 with $\lambda\alpha_0$, we can assume that $c_1 = T^y$. Let $m = y + j$. Substituting β_0 with $c_2\beta_0$, we can assume that $c_2 = 1$ as wanted.

Case 2 (c_2c_3 is not a unit). Then, c_1 and c_4 are units. Consider the change of variables

$$\alpha' = a\alpha_0 + c\beta_0, \quad \beta' = bT^{i-j}\alpha_0 + d\beta_0, \quad (4.24)$$

subordinate to the conditions: a, b, c , and d are in $O_L \otimes \mathbf{W}(k)$ and $ad - bcT^{i-j}$ is a unit. Then, (a) and (b) still hold. The new matrix of Frobenius is

$$\frac{1}{ad - bcT^{i-j}} \begin{pmatrix} d & -bT^{i-j} \\ -c & a \end{pmatrix} \begin{pmatrix} c_1T^j & c_3T^i \\ c_2T^j & c_4T^i \end{pmatrix} \begin{pmatrix} a^\sigma & b^\sigma T^{i-j} \\ c^\sigma & d^\sigma \end{pmatrix}. \quad (4.25)$$

Hence, the coefficient c'_4 of β' in $F_0(\beta')$ is

$$\frac{-c_1cb^\sigma + c_2ab^\sigma - c_3cd^\sigma + c_4ad^\sigma}{ad - bcT^{i-j}}. \quad (4.26)$$

The conditions

- (i) $f_1(b, c, d) = c_4d^\sigma + c_2b^\sigma$ is a unit,
- (ii) $f_2(b, c, d) = c_1cb^\sigma + c_3cd^\sigma$ is a unit,
- (iii) $f_3(b, c, d) = (c_1cb^\sigma + c_3cd^\sigma)d - bc(c_4d^\sigma + c_2b^\sigma)T^{i-j}$ is a unit,

are equivalent to the fact that the product f of the functions f_i does not vanish identically modulo \mathfrak{p} on $\mathbb{A}^3(k)$. Since k is infinite, this is equivalent to require that f modulo \mathfrak{p} , which is a polynomial, is not zero. This follows since c_1 and c_4 are units. If we put

$$a = (c_1cb^\sigma + c_3cd^\sigma)(c_4d^\sigma + c_2b^\sigma)^{-1}, \quad (4.27)$$

we get that $c'_4 = 0$ and therefore c'_2 and c'_3 must be units. We conclude as in [Case 1](#). ■

4.4.1 A formula for Verschiebung. Define the Verschiebung map $V_0 : P_0 \rightarrow P_0$ as $V_0 := (V_0^{-1})^{-1}$. Since k is perfect, V_0 exists. We find that

$${}^V\langle y, F_0x \rangle = {}^V\langle V_0^{-1}V_0y, V_0^{-1}px \rangle = \langle V_0y, px \rangle = {}^{VF}\langle V_0y, x \rangle, \quad (4.28)$$

which gives

$$\langle V_0(y), x \rangle = \langle y, F_0(x) \rangle^{\sigma^{-1}}, \quad \forall x, y \in P. \quad (4.29)$$

Suppose that, with respect to an $O_L \otimes \mathbf{W}(k)$ -basis $\{\alpha_0, \beta_0\}$ of P_0 , Frobenius has the 2×2 matrix with entries in $O_L \otimes \mathbf{W}(k)$

$$F_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (4.30)$$

Then, the matrix of V_0 , with respect to the same basis, is

$$V_0 = \frac{\langle \alpha_0, \beta_0 \rangle^{\sigma^{-1}}}{\langle \alpha_0, \beta_0 \rangle} \cdot \begin{pmatrix} D^{\sigma^{-1}} & -B^{\sigma^{-1}} \\ -C^{\sigma^{-1}} & A^{\sigma^{-1}} \end{pmatrix}. \quad (4.31)$$

Definition 4.11. In the notation of [Proposition 4.10](#), let

$$n = \begin{cases} m & \text{if } m \leq i, \\ i & \text{otherwise.} \end{cases} \quad (4.32)$$

Note that $i \geq n \geq j$.

Lemma 4.12. The notation is as in [Proposition 4.10](#) and [Definition 4.11](#).

(1) Let ℓ be a positive integer. The rank of \bar{F}_0^ℓ on \bar{Q}_0 is

$$\text{rank}_{\bar{Q}_0}(\bar{F}_0^\ell) = g - j - \min(i, \ell n). \quad (4.33)$$

In particular,

$$\text{rank}_{\bar{Q}_0}(\bar{F}_0) = g - (j + n). \quad (4.34)$$

(2) The following conditions are equivalent:

(a) $n = i$;

(b) \bar{F}_0^2 is zero on \bar{P}_0 . □

Proof. Claim (1) follows from an easy calculation using [Proposition 4.10](#). The equivalence of (a) and (b) follows from (1). ■

Note that the O_L -display $(P_0, Q_0, F_0, V_0^{-1})$ is superspecial if and only if either (a) or (b) holds.

5 Key definitions j , n , \mathcal{S}_j , and $\mathfrak{W}_{(j,n)}$

Let p be totally ramified in O_L , $p = p^g$. Choose an Eisenstein polynomial $h(T)$ over \mathbb{Z}_p and an isomorphism $O_L \otimes \mathbf{W}(k) \xrightarrow{\sim} \mathbf{W}(k)[T]/(h(T))$. Let $(A, \iota)/k$ be an abelian variety with real multiplication by O_L over a perfect field k of characteristic p . Let $(P_0, Q_0, V_0^{-1}, F_0)$ be the display associated to (A, ι) as in [Section 4.1](#). Let $j = j(A, \iota)$ and $n = n(A, \iota)$ be the invariants associated to $(P_0, Q_0, V_0^{-1}, F_0)$ as in [Proposition 4.10](#). We call j the *singularity index* of (A, ι) and n the *slope* of (A, ι) .

Note that T^{g-j} is the minimal power of T annihilating $\text{Lie}(A)$ and $j+n$ is equal to the α -number of A by [Lemma 4.12](#). In particular, j and n are, indeed, invariants of (A, ι) ; they satisfy the following numerical restrictions:

$$0 \leq j \leq n \leq g-j. \quad (5.1)$$

For later use, we define

$$\lambda(n) := \min \left\{ \frac{n}{g}, \frac{1}{2} \right\}, \quad J := \{(j, n) \mid 0 \leq j \leq n \leq g-j, j, n \in \mathbb{Z}\} \quad (5.2)$$

(a set of cardinality $([g/2] + 1)(g - [g/2] + 1)$).

The singularity index, being a measure of the degeneracy of a morphism of vector bundles, defines closed sets \mathcal{S}_j of the moduli space $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ whose geometric points consists of the geometric points x of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ for which $j(\underline{A}_x) \geq j$. Deligne and Pappas [\[6\]](#) proved the following facts:

- (1) \mathcal{S}_1 is not empty and coincides with $\mathfrak{M}(\mathbb{F}_p, \mu_N)^{\text{sing}}$;
- (2) \mathcal{S}_j is of pure dimension $g - 2j$ if it is nonempty;
- (3) $\mathcal{S}_j \setminus \mathcal{S}_{j+1}$ is nonsingular.

Since n measures the degeneracy of a morphism between vector bundles on \mathcal{S}_j , we may define locally closed subsets $\mathfrak{W}_{(j,n)}$ of \mathcal{S}_j , indeed of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$, as follows. The geometric points of $\mathfrak{W}_{(j,n)}$ consist of the geometric points x of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ for which $j(\underline{A}_x) = j$ and $n(\underline{A}_x) = n$. If T is a subset of J , we define a constructible subset \mathfrak{W}_T by

$$\mathfrak{W}_T := \bigcup_{(j,n) \in T} \mathfrak{W}_{(j,n)}. \quad (5.3)$$

5.1 Examples

Consider the case $g = 1$. Then, $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ is the moduli space of elliptic curves with μ_N -level structure. Note that $j = 0$, that is, $\mathcal{S}_0 = \mathfrak{M}(\mathbb{F}_p, \mu_N)$. Moreover, $\mathfrak{W}_{(0,0)}$ coincides with the ordinary locus and $\mathfrak{W}_{(0,1)}$ coincides with the supersingular locus.

Consider the case $g = 2$. This has been extensively studied in [\[2\]](#). We rephrase the results using our notation. The only possible j 's are $j = 0$ and $j = 1$. The locus \mathcal{S}_0 coincides with the Rapoport locus of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$. The locus \mathcal{S}_1 coincides with the non-smooth locus of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ and consists of the moduli points associated to supersingular Hilbert-Blumenthal abelian surfaces not satisfying (R). For the loci $\mathfrak{W}_{(j,n)}$, we have the following possibilities:

- (i) $\mathfrak{W}_{(0,0)}$ is the ordinary locus of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$. It is open dense in $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ and smooth of dimension 2;

- (ii) $\mathfrak{W}_{(0,1)}$ is affine and smooth of dimension 1. It is parameterized by finitely many nontrivial open subsets of $\mathbb{P}_{\mathbb{F}_2}^1$;
- (iii) $\mathfrak{W}_{(0,2)}$ consists of the moduli points associated to superspecial Hilbert-Blumenthal abelian surfaces satisfying (R);
- (iv) $\mathfrak{W}_{(1,1)}$ consists of the moduli points associated to superspecial Hilbert-Blumenthal abelian surfaces not satisfying (R).

Note that $\mathcal{S}_1 = \mathfrak{W}_{(1,1)}$ and $\mathcal{S}_0 \setminus \mathcal{S}_1 = \mathfrak{W}_{(0,0)} \cup \mathfrak{W}_{(0,1)} \cup \mathfrak{W}_{(0,2)}$. It is proven in [2] that the Zariski closure of $\mathfrak{W}_{(0,1)}$ coincides with $\mathfrak{W}_{(0,1)} \cup \mathfrak{W}_{(0,2)} \cup \mathfrak{W}_{(1,1)}$. Furthermore, [2] gives a formula for the number of components of each stratum.

6 Moret-Bailly families

Throughout this section, we assume that $p = p^g$ is totally ramified in O_L .

In this section, we construct, in characteristic p , families of abelian varieties with real multiplication over \mathbb{P}^1 . The method is that of dividing a single abelian variety by a variable subgroup parameterized by \mathbb{P}^1 , a method introduced by Moret-Bailly [13]. These families are used in the proof that the collection $\{\mathfrak{W}_{(j,n)}\}$ forms a stratification in Section 8, and in the proof that $\mathfrak{W}_{(j,n)}$ is nonempty in Section 10.

Let k be an algebraically closed field of positive characteristic p . Let (P, Q, F, V^{-1}) be a principally polarized O_L -display over k . We use \bar{X} to denote the reduction of X (a module, a morphism, etc.) modulo p . Let α and β be elements of P such that F and $V = (V^{-1})^{-1}$ are in normal form as in Proposition 4.10. In particular,

$$\bar{Q} = \text{Ker}(\bar{F}) = (\bar{T}^i) \bar{\alpha} \oplus (\bar{T}^j) \bar{\beta}. \quad (6.1)$$

Definition 6.1. Let

$$P_\gamma := \frac{1}{p} \mathbf{W}(k) \gamma + P, \quad \text{where } Q \ni \gamma := w(a_1) T^{g-1} \alpha + w(a_2) T^{g-1} \beta, \quad (6.2)$$

($w : k \rightarrow W(k)$ is the Teichmüller lift) with $(a_1 : a_2) \in \mathbb{P}_k^1$ and $a_1 = 0$ if $j = 0$. Define

$$Q_\gamma := Q + F^{-1}(\mathbf{W}(k) \gamma). \quad (6.3)$$

Lemma 6.2. The pair (P_γ, Q_γ) is endowed with a unique structure of display with O_L -action such that

$$(P, Q, F, V^{-1}) \hookrightarrow (P_\gamma, Q_\gamma, F_\gamma, V_\gamma^{-1}) \quad (6.4)$$

is a morphism of displays. □

Proof. Note that F and V^{-1} extend uniquely to automorphisms of $P \otimes \mathbb{Q} = Q \otimes \mathbb{Q}$. Since $T\gamma \in pP$, it follows that P_γ is O_L -invariant. Since $F^{-1} = Vp^{-1}$, it follows that, also, Q_γ is O_L -invariant. It suffices to verify that $Q_\gamma \subset P_\gamma$ and that $V^{-1}(Q_\gamma)$ is contained in P_γ and generates it; the existence of F follows automatically. Note that $V(P_\gamma) = Q_\gamma$. The inclusion $Q_\gamma \subset P_\gamma$ is equivalent to $V(\gamma) \in pP_\gamma$ which, in turn, is equivalent to $\bar{V}(\bar{\gamma}) = 0$. The last equality follows from [Proposition 4.10](#) and [Section 4.4.1](#). ■

6.1 The computation of \bar{Q}_γ

We note that \bar{Q}_γ is the kernel of the map \bar{F}_γ , where $\bar{F}_\gamma : \bar{P}_\gamma \rightarrow \bar{P}_\gamma$ is the induced Frobenius on the reduction modulo p of P_γ . Thus,

$$\bar{Q}_\gamma \cong F^{-1}(W(k)\gamma + pP)/(W(k)\gamma + pP). \quad (6.5)$$

In the notation of [Proposition 4.10](#), we find that

$$\begin{aligned} & F^{-1}(W(k)\gamma + pP) \\ &= \{d_1\alpha + d_2\beta \mid d_1, d_2 \in L \otimes W(k), F(d_1\alpha + d_2\beta) \in W(k)\gamma + pP\} \\ &= \{d_1\alpha + d_2\beta \mid d_1, d_2 \in L \otimes W(k), (d_1^\sigma T^m + d_2^\sigma c_3 T^i)\alpha + d_1^\sigma T^j \beta \in W(k)\gamma + pP\} \\ &= \{T^{g-j-1} \delta_1^{\sigma^{-1}} \alpha + T^{g-i-1} \delta_2^{\sigma^{-1}} \beta \mid \\ &\quad \delta_1, \delta_2 \in O_L \otimes W(k), (T^{m-j} \delta_1 + \delta_2 c_3) a_2 - \delta_1 a_1 \equiv 0 \pmod{T}\}. \end{aligned} \quad (6.6)$$

6.2 The O_L -structure of \bar{Q}_γ

For fixed a_1 and a_2 , we compute, using the isomorphism $\bar{Q}_\gamma \cong F^{-1}(W(k)\gamma + pP)/(W(k)\gamma + pP)$, the minimal nonnegative integer r such that

$$\bar{T}^r \bar{Q}_\gamma = 0. \quad (6.7)$$

We distinguish cases.

Case 6.3 ($j > 0$). The equality $\bar{T}^s \bar{Q}_\gamma = 0$ is equivalent to

$$\bar{T}^s (\bar{T}^{g-j-1} \delta_1^{\sigma^{-1}} \bar{\alpha} + \bar{T}^{g-i-1} \delta_2^{\sigma^{-1}} \bar{\beta}) \in k\bar{\gamma} \quad (6.8)$$

for all $\delta_1, \delta_2 \in k[T]/(T^g)$ satisfying

$$(\bar{T}^{m-j} \delta_1 + \delta_2 \bar{c}_3) \bar{T}^{g-1} a_2 - \delta_1 \bar{T}^{g-1} a_1 = 0. \quad (6.9)$$

Note that $s = i + 1$ satisfies (6.8). On the other hand, $\delta_1 = \delta_2 = \bar{T}$ always satisfies (6.9) and gives

$$\bar{T}^s (\bar{T}^{g-j} \bar{\alpha} + \bar{T}^{g-i} \bar{\beta}) \in k\bar{\gamma}, \quad (6.10)$$

which implies that $s \geq i - 1$. Hence, we conclude that

$$i - 1 \leq r \leq i + 1. \quad (6.11)$$

We proceed to examine when does $s = i$ or $i - 1$ satisfy (6.8) for all δ_1 and δ_2 as in (6.9).

(i) The case $s = i$. We rewrite (6.8) as

$$\bar{T}^{g-1+(i-j)} \delta_1^{\sigma^{-1}} \bar{\alpha} + \bar{T}^{g-1} \delta_2^{\sigma^{-1}} \bar{\beta} \in k\bar{\gamma} \quad (6.12)$$

for all δ_1 and δ_2 as in (6.9). Note that the linear dependence condition (6.12) is equivalent to

$$a_1 \delta_2^{\sigma^{-1}} - a_2 \bar{T}^{i-j} \delta_1^{\sigma^{-1}} = 0 \bmod \bar{T}. \quad (6.13)$$

Hence, we need to examine when does the following implication hold:

$$(\bar{T}^{m-j} a_2 - a_1) \delta_1 + a_2 \bar{c}_3 \delta_2 = 0 \bmod \bar{T} \implies a_1^\sigma \delta_2 - a_2^\sigma \bar{T}^{i-j} \delta_1 = 0 \bmod \bar{T} \quad (6.14)$$

for all δ_1 and δ_2 as in (6.9). Treating δ_1 and δ_2 as free variables in k , the implication is equivalent to

$$(\bar{T}^{m-j} a_2 - a_1) a_1^\sigma + \bar{T}^{i-j} \bar{c}_3 a_2^{\sigma+1} = 0 \bmod \bar{T}. \quad (6.15)$$

(a) Suppose that $i > j$ and $n > j$. Then, $m > j$ and (6.15) holds if and only if $a_1 = 0$.

(b) Suppose that $i > j$ and $m = n = j$. Then, (6.15) holds if and only if $a_1(a_1 - a_2) = 0$.

(c) Suppose that $i = j$. Then, (6.15) holds if and only if the couple $(a_1 : a_2)$ is among the $p + 1$ distinct solutions to the equation $\bar{c}_3 a_2^{p+1} + \bar{T}^{m-j} a_1^p a_2 - a_1^{p+1} = 0$.

(ii) The case $s = i - 1$. We rewrite (6.8) as

$$\bar{T}^{g-1+i-(j+1)} \delta_1^{\sigma^{-1}} \bar{\alpha} + \bar{T}^{g-2} \delta_2^{\sigma^{-1}} \bar{\beta} \in k\bar{\gamma} \quad (6.16)$$

for all δ_1 and δ_2 as in (6.9).

- (a) Suppose that $i \geq j + 2$ and $n > j$. First, take $\delta_1 = \delta_2 = \bar{T}$ to deduce from (6.16) that $a_1 = 0$, and hence $a_2 \neq 0$. Note that $m > j$. Therefore, we may rewrite (6.9) as

$$\delta_2 \bar{c}_3 a_2 = 0 \bmod \bar{T} \quad (6.17)$$

or simply as $\delta_2 = 0 \bmod \bar{T}$. But, if $a_1 = 0$ and $\delta_2 = 0 \bmod \bar{T}$, then (6.16) holds. To sum up, this case is possible if and only if $a_1 = 0 \bmod \bar{T}$.

- (b) Suppose that $i \geq j + 2$ and $n = j$. Then, also $m = j$. As before, taking $\delta_1 = \delta_2 = \bar{T}$, we find that $a_1 = 0$. Hence, (6.9) is equivalent to

$$(\delta_1 + \delta_2 \bar{c}_3) a_2 = 0 \bmod \bar{T}, \quad (6.18)$$

which can be solved with $\delta_2 = 1$ and $\delta_1 = -\bar{c}_3$. Assigning these to (6.16), we get that $\bar{T}^{g-2} \bar{\beta} \in k\bar{\gamma}$, which is a contradiction. Hence, if $i \geq j + 2$ and $n = j$, then $s = i - 1$ is not possible.

- (c) Suppose that $i = j + 1$. Again, $\delta_1 = \delta_2 = \bar{T}$ gives $a_1 = 0$. Now, δ_1 and δ_2 need to satisfy

$$\bar{T}^{m-j} \delta_1 + \delta_2 \bar{c}_3 = 0 \bmod \bar{T}. \quad (6.19)$$

Whether $m = j$ or $m > j$, there exists a solution with $\delta_1 = 1$. But, then (6.16) is just

$$\bar{T}^{g-1} \bar{\alpha} + \bar{T}^{g-2} \delta_2^{\sigma^{-1}} \bar{\beta} \in k\bar{\gamma}, \quad (6.20)$$

which is impossible.

- (d) Suppose that $i = j$. Here, already $\delta_1 = \epsilon_1 T$ and $\delta_2 = \epsilon_2 T$, where ϵ_i are appropriate units, give a contradiction.

Case 6.4 ($j = 0$). In this case, $a_1 = 0$ by definition, and we may assume, without loss of generality, that $a_2 = 1$ and $\gamma = T^{g-1} \beta$. It follows that

$$\begin{aligned} & F^{-1}(\mathbf{W}(k)\gamma + pP) \\ &= \{T^{g-1} \delta_1^{\sigma^{-1}} \alpha + T^{-1} \delta_2^{\sigma^{-1}} \beta \mid \delta_1, \delta_2 \in O_L \otimes \mathbf{W}(k), T^m \delta_1 + \delta_2 c_3 \equiv 0 \bmod T\}. \end{aligned} \quad (6.21)$$

Table 6.1

$r = i - 1$	$i \geq j + 2$	$n > j$	$a_1 = 0$
$r = i$	$i = j + 1$	$n > j$	$a_1 = 0$
	$i > j$	$n = j$	$a_1(a_1 - a_2) = 0$
	$i = j$	$n = j$	the $p + 1$ distinct solutions of $\bar{c}_3 a_2^{p+1} + \bar{T}^{m-j} a_1^p a_2 - a_1^{p+1} = 0 \bmod \bar{T}$
$r = i + 1$	otherwise		

(a) Suppose that $n > 0$. Then, $m > 0$ and, therefore, δ_2 cannot be a unit. Thus,

$$F^{-1}(\mathbf{W}(k)\gamma + pP) = \{T^{g-1}\delta_1^{\sigma^{-1}}\alpha + \delta_2^{\sigma^{-1}}\beta \mid \delta_1, \delta_2 \in O_L \otimes \mathbf{W}(k)\}. \quad (6.22)$$

Hence, $r = \max(g - 1, 1)$.

(b) Suppose that $n = 0$. Then, $m = 0$. Taking $\delta_2 = 1$ in (6.21), one concludes that $r = g$.

We summarize our results in Table 6.1.

6.3 The O_L -structure of $\bar{P}_\gamma/\bar{Q}_\gamma$

It follows from our definitions that $\bar{F}(\bar{P}/\bar{Q})$ is isomorphic, as an $O_L \otimes \mathbf{W}(k)$ -module, to $(T^m)/(T^i)\bar{\alpha}$, where F is given in a normal form as in Proposition 4.10, that is, $F(\alpha) = T^m\alpha + T^j\beta$ and $F(\beta) = c_3T^i\alpha$ with $n = \min(i, m)$ and $c_3 \in (O_L \otimes \mathbf{W}(k))^*$. Therefore,

$$\bar{T}^s \bar{F} \left(\frac{\bar{P}}{\bar{Q}} \right) \begin{cases} = 0 & \text{if } s \geq i - n, \\ \neq 0 & \text{if } s < i - n. \end{cases} \quad (6.23)$$

Note that knowing i , the calculation of the minimal s annihilating $\bar{F}(\bar{P}/\bar{Q})$ gives n . We are interested in finding n for the modules P_γ , equivalently, for $\bar{P}_\gamma/\bar{Q}_\gamma$. Since i , and hence j , for these modules is given in Section 6.2, we focus on the calculation of the minimal s such that $\bar{T}^s \bar{F}_\gamma(\bar{P}_\gamma/\bar{Q}_\gamma) = 0$.

We have $P_\gamma = P + \mathbf{W}(k)p^{-1}\gamma$ and

$$\bar{P} \ni \overline{T p^{-1} \gamma} = \bar{u}(a_1 \bar{\alpha} + a_2 \bar{\beta}) \quad \text{for some } \bar{u} \in (k[\bar{T}]/(\bar{T}^g))^*. \quad (6.24)$$

Hence,

$$\bar{T}^s \bar{F}_\gamma(\bar{P}_\gamma/\bar{Q}_\gamma) = 0 \quad \text{if } s \geq i - n + 1. \quad (6.25)$$

Note also that if $i - n - 2 \geq 0$ (which implies $n < i$ and, therefore, $m = n$), then

$$T^{i-n-2}F(\alpha) = T^{i-2}\alpha + T^{g-n-2}\beta. \quad (6.26)$$

Applying F to both sides and reducing modulo pP , we find that

$$\bar{T}^{i-n-2}\bar{F}^2(\bar{\alpha}) = \bar{T}^{i+n-2}\bar{\alpha} + \bar{T}^{g-2}\bar{\beta} \quad (6.27)$$

is not proportional to $\bar{\gamma}$. Therefore, $T^{i-n-2}F(\alpha)$ does not belong to Q_γ and, hence, T^{i-n-2} does not kill $\bar{F}(\bar{P}_\gamma/\bar{Q}_\gamma)$. Thus,

$$\bar{T}^s\bar{F}_\gamma(\bar{P}_\gamma/\bar{Q}_\gamma) \neq 0 \quad \text{if } 0 \leq s \leq i - n - 2. \quad (6.28)$$

We proceed to examine, for $s = i - n - 1$ and $s = i - n$, under which conditions we have $\bar{T}^s\bar{F}_\gamma(\bar{P}_\gamma/\bar{Q}_\gamma) = 0$.

6.3.1 The case $\bar{T}^{i-n}\bar{F}_\gamma(\bar{P}_\gamma/\bar{Q}_\gamma) = 0$. This happens if and only if $\bar{T}^{i-n}\bar{F}_\gamma^2(\bar{P}_\gamma) = 0$; equivalently,

$$T^{i-n}F_\gamma^2(p^{-1}\gamma) \in pP_\gamma. \quad (6.29)$$

Fix a_1 and a_2 . Then, the last condition can be written explicitly: the element

$$\begin{aligned} T^{i-n}F_\gamma^2\left(\frac{\gamma}{p}\right) &= uF\left(w(a_1)^\sigma(T^{i+m-n-1}\alpha + T^{g-n-1}\beta) + w(a_2)^\sigma c_3 T^{2i-n-1}\alpha\right) \\ &= uT^{i-n}\left(w(a_1)^{\sigma^2}T^{2m-1} + w(a_2)^{\sigma^2}c_3^\sigma T^{i+m-1} + w(a_1)^{\sigma^2}c_3 T^{g-1}\right)\alpha \\ &\quad + u\left(w(a_1)^{\sigma^2}T^{g+m-n-1} + w(a_2)^{\sigma^2}c_3^\sigma T^{g+i-n-1}\right)\beta, \end{aligned} \quad (6.30)$$

with $u \in (O_L \otimes \mathbf{W}(k))^*$, is proportional to γ modulo pP . Equivalently, the following equality should hold modulo T :

$$(a_1^{\sigma^2}\bar{T}^{2m-j-n} + a_2^{\sigma^2}\bar{c}_3^\sigma\bar{T}^{i-j+m-n} + a_1^{\sigma^2}\bar{c}_3\bar{T}^{i-n})a_2 - (a_1^{\sigma^2}\bar{T}^{m-n} + a_2^{\sigma^2}\bar{c}_3^\sigma\bar{T}^{i-n})a_1 \equiv 0 \quad (6.31)$$

- (i) if $j < n < i$, then $m = n < i$ and $i - j > 0$. Hence, the above equality is equivalent to $a_1 = 0$;

Table 6.2

$s = i - n - 1$	$j < n < i$	$a_1 = 0$
$s = i - n$	$j = n < i$	$a_1(a_1 - a_2) = 0$
	$j \leq n = i$	the $p^2 + 1$ distinct solutions mod \bar{T} of (6.32)
$s = i - n + 1$	otherwise	

- (ii) if $j = n < i$, the equality is equivalent to $a_1(a_1 - a_2) = 0$;
- (iii) if $n = i$, then (6.31) is a nonzero homogeneous polynomial equation of degree $p^2 + 1$ in the variables a_1 and a_2 . There are, therefore, $p^2 + 1$ distinct points $(a_1 : a_2)$ forming the solutions of the equation

$$(a_1^{p^2} \bar{T}^{2m-g} + a_2^{p^2} \bar{c}_3^p \bar{T}^{m-j} + a_1^{p^2} \bar{c}_3) a_2 - (a_1^{p^2} \bar{T}^{m-n} + a_2^{p^2} \bar{c}_3^p) a_1 \equiv 0. \quad (6.32)$$

6.3.2 The case $\bar{T}^{i-n-1} \bar{F}_\gamma(\bar{P}_\gamma/\bar{Q}_\gamma) = 0$. Note that we must have $i - n - 1 \geq 0$ and, hence, $m = n$. The condition is equivalent to require that $\bar{T}^{i-n-1} \bar{F}_\gamma^2(\bar{P}_\gamma) = 0$. Since $F^2(\bar{\beta}) = 0$, necessary and sufficient conditions are that

- (i) $\bar{T}^{i-n-1} \bar{F}^2(\bar{\alpha}) = \bar{T}^{i+n-1} \bar{\alpha} + \bar{T}^{g-1} \bar{\beta} \in k\bar{\gamma}$;
- (ii) $F(a_1^\sigma (\bar{T}^{i+m-n-2} \bar{\alpha} + \bar{T}^{g-n-2} \bar{\beta}) + a_2^\sigma c_3 \bar{T}^{2i-n-2} \bar{\alpha}) \in k\bar{\gamma}$ is proportional to γ modulo pP .

Hence, one of the following must hold:

- (i) $j < n$. Then, $m = n \leq i - 1$ and the first equation gives $a_1 = 0$. Conversely, if $a_1 = 0$, both equations hold;
- (ii) $j = n$. Then, the first equation implies that $a_1 = a_2$. The second equation implies that $a_1 = 0$, a contradiction. Hence, this case never holds.

We summarize our results in Table 6.2 that gives the minimal s so that $\bar{T}^s \bar{F}_\gamma(\bar{P}_\gamma/\bar{Q}_\gamma) = 0$.

6.4 Notation

Let k be a field. We denote by $\widetilde{\mathfrak{M}}(k, \mu_N)$ (resp., $\widetilde{\mathfrak{M}}(k, \mu_N, \mathcal{I})$) the coarse moduli space of abelian varieties (A, ι) with real multiplication by O_L and μ_N -level structure (resp., such that (M_A, M_A^+) is isomorphic étale locally to $(\mathcal{I}, \mathcal{I}^+)$). The natural morphism $\mathfrak{M}(k, \mu_N) \rightarrow \widetilde{\mathfrak{M}}(k, \mu_N)$ is finite. It takes S_j and $\mathfrak{W}_{(j,n)}$ to their counterpart in $\widetilde{\mathfrak{M}}(k, \mu_N)$.

Proposition 6.5. Let $(A, \iota) \rightarrow \text{Spec}(k)$ be an abelian variety with O_L -multiplication over an algebraically closed field of characteristic p satisfying (DP) and with polarization module isomorphic to \mathcal{I} . Assume that $g > 1$. Suppose that the moduli point $[A] \in \widetilde{\mathfrak{M}}(k, \mu_N)$

satisfies

$$[A] \in \mathfrak{W}_{(0,n)} \quad \text{with } n > 0. \quad (6.33)$$

There exists a unique O_L -invariant subgroup scheme $H \subset A[T]$ of order p . It is isomorphic to α_p . Moreover, A/H satisfies

- (1) the O_L -action on A descends to A/H and the abelian variety A/H satisfies (DP) with polarization module isomorphic to $p\mathfrak{I}$;
- (2) the moduli point $[A/H]$ of $(A/H, \iota)$ satisfies

$$[A/H] \in \begin{cases} \mathfrak{W}_{(1,n)} & \text{if } n < g, \\ \mathfrak{W}_{(1,n-1)} & \text{if } n = g. \end{cases} \quad (6.34) \quad \square$$

If $[A] \in \mathfrak{W}_{(0,0)}$, there exist two O_L -invariant subgroup schemes $H \subset A[T]$ of order p : one isomorphic to $\mathbb{Z}/p\mathbb{Z}$ and one to μ_p . Furthermore, $[A/H]$ lies in $\mathfrak{W}_{(0,0)}$.

Proof. Use Sections 6.2 and 6.3, Corollaries 3.3 and 3.5. For the last assertion, note that $\mathfrak{W}_{(0,0)}$ is the ordinary locus of $\widetilde{\mathfrak{M}}_{(k,\mu_N)}$. ■

Proposition 6.6. Let $(A, \iota) \rightarrow \text{Spec}(k)$ be an abelian variety with O_L -multiplication over an algebraically closed field of characteristic p satisfying (DP) and with polarization module isomorphic to \mathfrak{I} . Suppose that the moduli point $[A] \in \widetilde{\mathfrak{M}}(k, \mu_N)$ satisfies

$$[A] \in \mathfrak{W}_{(j,n)} \quad \text{with } j > 0. \quad (6.35)$$

There exists a Moret-Bailly family of abelian varieties $\mathcal{A} \rightarrow \mathbb{P}_k^1$. Denote by $\mathcal{A}_{(a:b)}$ the fiber over $(a:b) \in \mathbb{P}_k^1$. The family $\mathcal{A} \rightarrow \mathbb{P}_k^1$ satisfies the following:

- (1) $\mathcal{A} \rightarrow \mathbb{P}_k^1$ is an abelian scheme with O_L -multiplication satisfying (DP) and has polarization module isomorphic to $p\mathfrak{I}$;
- (2) moreover,
 - (2a) if $j < n < i$, then

$$[\mathcal{A}_{(a:b)}] \in \begin{cases} \mathfrak{W}_{(j+1,n)} & \text{if } (a:b) = (0:1), \\ \mathfrak{W}_{(j-1,n)} & \text{for all other points of } \mathbb{P}_k^1; \end{cases} \quad (6.36)$$

- (2b) if $n = j < i$, then

$$[\mathcal{A}_{(a:b)}] \in \begin{cases} \mathfrak{W}_{(j,n)} & \text{for } (0:1), (1:1), \\ \mathfrak{W}_{(j-1,n)} & \text{for all other points of } \mathbb{P}_k^1; \end{cases} \quad (6.37)$$

(2c) if $n = i = j$, then

$$[\mathcal{A}_{(a:b)}] \in \begin{cases} \mathfrak{W}_{(j,n)} & \text{for exactly } (p+1)\text{-points } (a:b) \in \mathbb{P}_k^1, \\ \mathfrak{W}_{(j-1,n+1)} & \text{for exactly } (p^2-p)\text{-points } (a',b') \in \mathbb{P}_k^1, \\ \mathfrak{W}_{(j-1,n)} & \text{for all other points of } \mathbb{P}_k^1; \end{cases} \quad (6.38)$$

(2d) if $j < i = n$, let $j' := j + 1$ if $i \geq j + 2$ and $j' = j$ otherwise. Then,

$$[\mathcal{A}_{(a:b)}] \in \begin{cases} \mathfrak{W}_{(j',n+(j-j'))} & \text{if } (a:b) = (0:1), \\ \mathfrak{W}_{(j-1,n+1)} & \text{for the } p^2\text{-solutions } \neq (0:1) \text{ of (6.32),} \\ \mathfrak{W}_{(j-1,n)} & \text{for all other points of } \mathbb{P}_k^1. \end{cases} \quad (6.39)$$

□

Proof. Use Sections 6.2 and 6.3 and Corollaries 3.3 and 3.5. ■

7 Deformations and the local structure of $\mathfrak{W}_{(j,n)}$

In this section, we construct equicharacteristic deformations of abelian varieties with real multiplication, using the theory of O_L -displays as in Definition 4.1. In particular, we construct a model for the completion of the local ring of \mathcal{S}_j at a point x such that the corresponding abelian variety has singularity index j . The method of Deligne-Pappas, following Grothendieck, is to study the deformation theory via the variation of the Hodge filtration. Thus, the completed local ring $O_{\mathfrak{M}(k, \mu_N), x}$ of $\mathfrak{M}(k, \mu_N)$ at x (resp., $O_{\mathcal{S}_j, x}$ of \mathcal{S}_j at x) is identified with the completion of (a closed subset of) the Grassmannian of certain O_L -modules in a rank 2 free $O_L \otimes k(x)$ -module (coming from the variation of $H^0(\Omega^1) \subset H_{\text{dR}}^1$). This already gives that the local ring $O_{\mathcal{S}_j, x}$ is smooth of dimension $g - 2j$. However, the inherent in this method is that one does not get an explicit expression for the variation of Frobenius, and, therefore, more delicate questions, like the local structure of $\mathfrak{W}_{(j,n)}$, must be approached using other methods.

We refer to Proposition 4.10 for notation. In particular, our starting point is an O_L -display $(P_0, Q_0, V_0^{-1}, F_0)$, over an algebraically closed characteristic p -field k , given in normal form: it has basis $\{\alpha_0, \beta_0\}$ and F_0 is described with respect to the basis by the matrix

$$\begin{pmatrix} T^m & c_3 T^i \\ T^j & 0 \end{pmatrix}. \quad (7.1)$$

Proposition 7.1. Define $R := k[[f_j, \dots, f_{i-1}]]$. There exists a polarized O_L -display $(P, Q, V^{-1}, F, \langle \cdot, \cdot \rangle)$ over R with an $O_L \otimes \mathbf{W}(R)$ -basis $\{\alpha, \beta\}$ of P such that the following properties hold:

(i) there is an isomorphism of polarized O_L -displays

$$(P_k, Q_k, V_k^{-1}, F_k) \xrightarrow{\sim} (P_0, Q_0, V_0^{-1}, F_0); \quad (7.2)$$

(ii) we have $Q = (O_L \otimes \mathbf{W}(R))T^i\alpha \oplus (O_L \otimes \mathbf{W}(R))T^j\beta + I_R P$;

(iii) the Frobenius is determined by the following identities:

$$F(\alpha) := F_0(\alpha_0) + w(f_j)T^j\alpha + \dots + w(f_{i-1})T^{i-1}\alpha, \quad F(\beta) := F_0(\beta_0). \quad (7.3)$$

In the above formulas, we view $P_0 = (O_L \otimes \mathbf{W}(k))\alpha_0 \oplus (O_L \otimes \mathbf{W}(k))\beta_0$ as a subset of P via the identification $(\alpha_0, \beta_0) \mapsto (\alpha, \beta)$ and the inclusion $\mathbf{W}(k) \subset \mathbf{W}(R)$. (We recall that $w(x)$ denotes the Teichmüller lift of x .) \square

The proof is given in the following subsections.

7.1.1 The definition of (P, Q, F, V^{-1}) . Define P and Q by

$$\begin{aligned} P &= (O_L \otimes \mathbf{W}(R))\alpha \oplus (O_L \otimes \mathbf{W}(R))\beta, \\ Q &= (O_L \otimes \mathbf{W}(R))\alpha' \oplus (O_L \otimes \mathbf{W}(R))\beta' + I_R P, \end{aligned} \quad (7.4)$$

where $\alpha' := T^i\alpha$ and $\beta' := T^j\beta$. Define $F : P \rightarrow P$ as the unique O_L -linear and σ -linear map such that (7.3) holds.

We, further, define

$$\begin{aligned} T &= \mathbf{W}(R)\alpha \oplus \dots \oplus \mathbf{W}(R)T^{i-1}\alpha \oplus \mathbf{W}(R)\beta \oplus \dots \oplus \mathbf{W}(R)T^{j-1}\beta, \\ L &= \mathbf{W}(R)\alpha' \oplus \dots \oplus \mathbf{W}(R)T^{j-1}\alpha' \oplus \mathbf{W}(R)\beta' \oplus \dots \oplus \mathbf{W}(R)T^{i-1}\beta'. \end{aligned} \quad (7.5)$$

One verifies from the definitions that the kernel of the reduction \bar{F} of F to \bar{P} is $\bar{Q} = \text{Image}(Q \rightarrow P/I_R P)$.

We now proceed to the definition of V^{-1} : since $P \hookrightarrow P \otimes \mathbb{Q}$, the map V^{-1} exists on $P \otimes \mathbb{Q}$ and is given by the formula $V^{-1} = p^{-1}F$. Clearly, it is O_L -linear and is a σ -linear homomorphism satisfying $V^{-1}(w^V \cdot y) = w \cdot F(y)$ for any $w \in \mathbf{W}(R)$ and any $y \in Q$. Furthermore, $V_k^{-1} = V_0^{-1}$. It remains to verify that $V^{-1}(Q) \subset P$ and generates it.

Note that $P = L \oplus T$ and $Q = L \oplus I_R T$. Since $F(\alpha'), F(\beta') \equiv 0$ modulo p , the following identities show that $V^{-1}(Q) \subset P$:

$$V^{-1}(w^V \cdot y) = wF(y) \quad \text{if } y \in T, \quad V^{-1}(x) = p^{-1}F(x) \quad \text{if } x \in L. \quad (7.6)$$

Finally, since $V_k^{-1} = V_0^{-1}$, by Nakayama's lemma, $V^{-1}(Q)$ generates P .

7.1.2 The pairing on P . Let

$$\langle \cdot, \cdot \rangle : P \times P \longrightarrow \text{Hom}(O_L, \mathbb{Z}) \otimes W(R) \quad (7.7)$$

be the unique $O_L \otimes W(R)$ -bilinear alternating pairing such that

$$\langle \alpha, \beta \rangle = \langle \alpha_0, \beta_0 \rangle_0. \quad (7.8)$$

This pairing extends to $P \otimes \mathbb{Q}$. Let $a, b, c, d \in O_L \otimes W(R)$. An easy calculation gives $\langle V^{-1}(a\alpha + c\beta), V^{-1}(b\alpha + d\beta) \rangle = \langle a\alpha + c\beta, b\alpha + d\beta \rangle$.

Proposition 7.2. Let k be a separably closed field of characteristic p . Let $\underline{A}_0 := (A_0, \iota_0, \lambda_0, \varepsilon_0)/k$ be a polarized abelian variety with real multiplication by O_L and μ_N -level structure, where $N \geq 4$. Suppose that the associated moduli point $[\underline{A}_0] \in \mathfrak{M}(k, \mu_N)$ belongs to $\mathfrak{W}_{(j,n)}$. Let $R := k[[f_j, \dots, f_{i-1}]]$. There exists a deformation

$$\underline{A} \longrightarrow \text{Spec}(R) \quad (7.9)$$

of $\underline{A}_0 \rightarrow \text{Spec}(k)$ such that

- (a) $\underline{A} \rightarrow \text{Spec}(R)$ is a polarized abelian scheme with real multiplication by O_L and level μ_N satisfying (DP);
- (b) the induced map $\psi : \text{Spec}(R) \rightarrow \mathfrak{M}(\mathbb{F}_p, \mu_N)$ factors via \mathcal{S}_j and is an isomorphism onto the completed local ring of \mathcal{S}_j at $[\underline{A}_0]$;
- (c) letting (P, Q, V^{-1}, F) be the polarized O_L -display of \underline{A} , there exist an integer $m \geq n$, an element $c_3 \in (O_L \otimes W(k))^*$, and an $O_L \otimes W(R)$ -basis $\{\alpha, \beta\}$ of P such that the Frobenius map is defined by

$$\alpha \longmapsto T^m \alpha + w(f_j)T^j \alpha + \dots + w(f_{i-1})T^{i-1} \alpha + T^j \beta, \quad \beta \longmapsto c_3 T^i \alpha. \quad (7.10)$$

□

To prove the proposition we argue as follows.

We take the deformation \underline{A} of \underline{A}_0 corresponding to the display constructed in [Proposition 7.1](#). Parts (a) and (c) and the fact that the map ψ factors via \mathcal{S}_j follow from [Proposition 7.1](#) and [Theorem 4.7](#). It remains to prove that ψ is an isomorphism onto the completed local ring Γ of \mathcal{S}_j at $[\underline{A}_0]$. By Deligne-Pappas [6], the ring Γ is a power series ring

in $(g-2j)$ -variables over k . Since $N \geq 4$, the scheme $\mathfrak{M}(k, \mu_N)$ is a fine moduli scheme and $\text{Spec}(\Gamma)$ is naturally identified with a closed subscheme of the universal equicharacteristic deformation space of \underline{A}_0 . Since R is a formal power series ring in $(g-2j)$ -variables over k , it is enough to prove that the map ψ is injective on tangent spaces. Using the moduli property of Γ , this follows from [Section 7.2.1](#).

7.2.1 The map ψ is injective on tangent spaces. This amounts to proving that for any tangent vector

$$t : R = k[[f_j, \dots, f_{i-1}]] \longrightarrow k[\varepsilon]/(\varepsilon^2), \quad (7.11)$$

the polarized O_L -display $(P_t, Q_t, V_t^{-1}, F_t, \langle \cdot, \cdot \rangle_t)$, obtained by base change via t from $(P, Q, V^{-1}, F, \langle \cdot, \cdot \rangle)$, is *trivial* if and only if $t(f_\ell) = 0$ for all $\ell = j, \dots, i-1$.

By [Proposition 7.1](#), we have $Q = (O_L \otimes W(R))T^i\alpha \oplus (O_L \otimes W(R))T^j\beta + I_R P$ and that V^{-1} is given, with respect to the bases $\{T^i\alpha, T^j\beta\}$ of Q and $\{\alpha, \beta\}$ of P , by the matrix

$$\begin{pmatrix} p^{-1}T^{m+i} + p^{-1}T^iZ & c_3p^{-1}T^g \\ p^{-1}T^g & 0 \end{pmatrix}, \quad (7.12)$$

where $Z := w(f_j)T^j + \dots + w(f_{i-1})T^{i-1}$ and the expression $p^{-1}T^g$ is a unit of $O_L \otimes \mathbb{Z}_p$ (similarly, for $p^{-1}T^{m+i}$ and $p^{-1}T^iZ$) and, in particular, makes sense mod p . Assume that

$$(P_t, Q_t, V_t^{-1}, F_t, \langle \cdot, \cdot \rangle_t) \cong (P_0, Q_0, V_0^{-1}, F_0, \langle \cdot, \cdot \rangle_0) \times_k k[\varepsilon]/(\varepsilon^2). \quad (7.13)$$

Then, there exists an $O_L \otimes W(k[\varepsilon])$ -linear isomorphism

$$\Phi : P_t \xrightarrow{\sim} P_t, \quad (7.14)$$

restricting to the identity on P_0 such that

- (1) $\Phi(Q_t) = Q_t$;
- (2) $\langle \Phi(\alpha), \Phi(\beta) \rangle_t = \langle \alpha, \beta \rangle_t$;
- (3) the following diagram commutes:

$$\begin{array}{ccc} Q_t & \xrightarrow{\Phi} & Q_t \\ V_0^{-1} \downarrow & & \downarrow V_t^{-1} \\ P_t & \xrightarrow{\Phi} & P_t. \end{array} \quad (7.15)$$

We write

$$\Phi(\alpha) = A\alpha + C\beta, \quad \Phi(\beta) = B\alpha + D\beta, \quad (7.16)$$

where $A, B, C, D \in \mathcal{O}_L \otimes \mathbf{W}(k[\varepsilon])$ and $A-1, D-1, C$, and B are equivalent to 0 mod ε that is, via the map $\mathcal{O}_L \otimes \mathbf{W}(k[\varepsilon]) \rightarrow \mathcal{O}_L \otimes \mathbf{W}(k)$. By construction, $\langle \Phi(\alpha), \Phi(\beta) \rangle_t = \langle \alpha, \beta \rangle_0$, which implies that

$$AD - BC = 1. \quad (7.17)$$

By assumption, the matrix of V_0^{-1} , with respect to the bases $\{T^i\alpha, T^j\beta\}$ and $\{\alpha, \beta\}$, is

$$\begin{pmatrix} p^{-1}T^{m+i} & c_3p^{-1}T^g \\ p^{-1}T^g & 0 \end{pmatrix}. \quad (7.18)$$

We proceed to find out the implications of condition (3).

We first note that condition (1) implies that $\Phi(T^j\beta) = T^jB\alpha + T^jD\beta \in Q_t$, and hence $T^jB = T^i\chi$ for a suitable $\chi \equiv 0 \pmod{\varepsilon}$. Then, one computes

$$\begin{aligned} V_t^{-1}(\Phi(T^j\beta)) &= V_t^{-1}(T^jB\alpha + T^jD\beta) \\ &= ((p^{-1}T^{m+i}) + t_*(p^{-1}ZT^i))\chi^\sigma\alpha \\ &\quad + (p^{-1}T^g)\chi^\sigma\beta + c_3(p^{-1}T^g)D^\sigma\alpha \\ &= c_3(p^{-1}T^g)\alpha; \end{aligned} \quad (7.19)$$

the last equality follows from σ being zero on the kernel of $\mathbf{W}(k[\varepsilon]) \rightarrow \mathbf{W}(k)$. On the other hand,

$$\Phi(V_0^{-1}(T^j\beta)) = \Phi(c_3(p^{-1}T^g)\alpha) = c_3(p^{-1}T^g)A\alpha + c_3(p^{-1}T^g)C\beta. \quad (7.20)$$

Since c_3 and $(p^{-1}T^g)$ are units, we deduce that $A = 1$ and $C = 0$. From the determinant condition, we also get that $D = 1$. Finally, we compute that

$$\begin{aligned} V_t^{-1}(\Phi(T^i\alpha)) &= V_t^{-1}(T^i\alpha) \\ &= ((p^{-1}T^{m+i}) + t_*(p^{-1}ZT^i))\alpha + (p^{-1}T^g)\beta, \\ \Phi(V_0^{-1}(T^i\alpha)) &= \Phi((p^{-1}T^{m+i})\alpha + (p^{-1}T^g)\beta) \\ &= ((p^{-1}T^{m+i}) + (p^{-1}T^g)B)\alpha + (p^{-1}T^g)\beta. \end{aligned} \quad (7.21)$$

Comparing the two expressions, we find that $(p^{-1}T^g)B = t_*(p^{-1}ZT^i)$, whence

$$B = t_*(w(f_j)) + t_*(w(f_{j+1}))T + \cdots + t_*(w(f_{i-1}))T^{i-j-1}. \quad (7.22)$$

Since $B = T^{i-j}x$, we conclude that $t(f_\ell) = 0$ for all $\ell = j, \dots, i-1$.

Remark 7.3. The matrix of the operator \bar{F} on \bar{P}/\bar{Q} , with respect to the basis $\{\bar{\alpha}, \dots, \bar{T}^{i-1}\bar{\alpha}, \bar{\beta}, \dots, \bar{T}^{j-1}\bar{\beta}\}$, is

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ f_j & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ f_{j+1} & f_j & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{n-1} & f_{n-2} & f_{n-3} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ b + f_n & f_{n-1} & f_{n-2} & \cdots & & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ f_{n+1} & b + f_n & f_{n-1} & \cdots & & & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{i-2} & & & \cdots & b + f_n & f_{n-1} & & \cdots & 0 & 0 & \cdots & 0 \\ f_{i-1} & f_{i-2} & & \cdots & f_{n+1} & b + f_n & f_{n-1} & \cdots & f_j & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (7.23)$$

where $b = 1$ if $m < i$ and $b = 0$ if $m \geq i$.

Corollary 7.4. Let x be a geometric point of $\mathfrak{W}_{(j,n)}$. Then, using the above notation, for any $j \leq n' \leq n$, the equations defining $\mathfrak{W}_{(j,n')}$ at the formal neighborhood of x in \mathcal{S}_j are

$$f_j = \cdots = f_{n'-1} = 0. \quad (7.24)$$

In particular, $\mathfrak{W}_{(j,n)}$ is a locally irreducible, nonsingular subscheme of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$. \square

8 $\Gamma_0(p)$ -correspondences

Assume that $p = p^g$ is totally ramified in L . In this section, we study the Hecke correspondence arising from the $\Gamma_0(p)$ -level structure. We prove in [Proposition 8.10](#) that the image

of $\mathfrak{W}_{(j,n)}$ via such correspondence is the union of certain sets $\mathfrak{W}_{(j',n')}$. We use it to prove in [Theorem 8.14](#) that the $\mathfrak{W}_{(j,n)}$'s define a *stratification* of $\mathfrak{M}(\mathbb{F}_p, \mu_N)$, that is, the closure of $\mathfrak{W}_{(j,n)}$ is the union of some of the sets $\mathfrak{W}_{(j',n')}$.

Let A be an abelian variety with a polarization module M_A and a polarization datum identifying (M_A, M_A^+) with an element \mathcal{I} of $\mathcal{R} = \{\mathcal{I}_1, \dots, \mathcal{I}_{h+j}\}$. Let H be an O_L -invariant subgroup scheme of A of order p . By [Corollary 3.5](#), the polarization module of A/H is pM_A , and hence identified with $p\mathcal{I}$. Thus, we fix, for every $\mathcal{I} \in \mathcal{R}$, an isomorphism $p\mathcal{I} \cong \mathcal{I}'$, where $\mathcal{I}' \in \mathcal{R}$ is uniquely determined. Then, for every A with polarization datum, also A/H is equipped with a polarization datum.

Definition 8.1. Let $\mathfrak{N} := \mathfrak{M}(\mathbb{F}_p, \mu_N, \Gamma_0(p))$ be the moduli space over \mathbb{F}_p of polarized abelian schemes with real multiplication by O_L , with μ_N -level structure and O_L -invariant finite flat subgroup scheme of degree p . Let $\mathfrak{M} := \mathfrak{M}(\mathbb{F}_p, \mu_N)$ and let

$$\begin{array}{ccc} & \mathfrak{N} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathfrak{M} & & \mathfrak{M} \end{array} \quad (8.1)$$

be the two projections defined by

$$\pi_1([\underline{A}, H]) = \underline{A}, \quad \pi_2([\underline{A}, H]) = \underline{A}/H. \quad (8.2)$$

Definition 8.2. Define the involution $^\vee : \mathfrak{M} \rightarrow \mathfrak{M}$ by mapping an abelian scheme \underline{A} with real multiplication by O_L onto its dual \underline{A}^\vee . Define an involution ρ on \mathfrak{N} by

$$\rho([\underline{A}, H]) = [(\underline{A}/H)^\vee, H^\vee]. \quad (8.3)$$

Remark 8.3. The involution ρ fits in the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{N} & \xrightarrow{\rho} & \mathfrak{N} \\ \pi_2 \downarrow & & \downarrow \pi_1 \\ \mathfrak{M} & \xrightarrow{^\vee} & \mathfrak{M}. \end{array} \quad (8.4)$$

Similarly, with the role of π_1 and π_2 interchanged.

Lemma 8.4. The morphisms $\pi_i : \mathfrak{N} \rightarrow \mathfrak{M}$ are proper. □

Proof. We first prove that π_1 is proper applying the valuative criterion. Let R be a dvr with fraction field K , let $[\underline{A}]$ be in $\mathfrak{M}(R)$, and let $[\underline{A}_K, H_K]$ be in $\mathfrak{N}(K)$. The Zariski closure

H of H_K in A defines an O_L -invariant subgroup scheme, finite and flat of order p over $\mathrm{Spec}(R)$. Then, $[\underline{A}, H]$ is in $\mathfrak{M}(R)$ and it is the unique extension of $[\underline{A}_K, H_K]$ over $\mathrm{Spec}(R)$. It follows that π_1 is proper. Thus, $\pi_1 \circ \rho = {}^\vee \circ \pi_2$ is also proper. Hence, so is $\pi_2 = {}^\vee \circ {}^\vee \circ \pi_2$. ■

Lemma 8.5. The involution ${}^\vee$ on \mathfrak{M} sends $\mathfrak{W}_{(j,n)}$ to $\mathfrak{W}_{(j,n)}$. □

Proof. Let A be an abelian scheme with real multiplication by O_L over a scheme of characteristic p . By [Proposition 3.1](#), there exists a prime-to- p O_L -polarization on A that induces an isomorphism of the Hodge filtration of A and its dual A^\vee . In particular, the singularity index and the slope of A coincide with those of A^\vee , that is, $j(A) = j(A^\vee)$ and $n(A) = n(A^\vee)$. ■

Lemma 8.6 (inversion lemma). Let $x = [\underline{A}_x] \in \mathfrak{M}$ be a geometric point. Then,

$$\begin{aligned} \pi_2^{-1}([\underline{A}_x]) &= \rho(\pi_1^{-1}([\underline{A}_x^\vee])) \\ &= \left\{ [(\underline{A}_x^\vee/K)^\vee, K^\vee] \mid K \subset A_x^\vee, O_L\text{-invariant of degree } p \right\}. \end{aligned} \quad (8.5)$$

In particular, $\pi_1(\pi_2^{-1}[\underline{A}])$ consists of points whose invariants (j, n) are *exactly* those constructed out of A via the Moret-Bailly construction in [Proposition 6.5](#) if $j(A_x) = 0$, and in [Proposition 6.6](#) if $j(A_x) > 0$. □

Proposition 8.7. The following holds:

- (1) the restrictions of the morphisms π_1 and π_2 to the Rapoport locus \mathfrak{M}^R are finite;
- (2) the restrictions of the morphisms π_1 and π_2 to the singular locus \mathcal{S}_1 are \mathbb{P}^1 -bundles. □

Proof. By [Remark 8.3](#) and [Lemma 8.5](#), the statements concerning π_2 follow from those concerning π_1 . By [Lemma 8.4](#), we know that $\pi_1 : \mathfrak{M} \rightarrow \mathfrak{M}$ is proper. Over \mathfrak{M}^R , the morphism π_1 is quasifinite. In fact, using Dieudonné theory, one proves, as in [Section 6](#), that the (reduced) fibers have two points over $\mathfrak{W}_{(0,0)}$ and one point over $\mathfrak{W}_{(0,n)}$ for $n > 0$. We conclude from [[10](#), Proposition 4.4.2] that π_1 is finite. This completes the proof of (1).

Let A be the universal abelian scheme over \mathcal{S}_1 . Let $U = \mathrm{Spec}(R)$ be the completion of \mathcal{S}_1 at a geometric point, $\mathbf{W}(R)$ the Witt vectors of R , and I_R the kernel of $\mathbf{W}(R) \rightarrow R$. Let $(P_U, Q_U, F_U, V_U^{-1})$ be the display over U associated to $A \times_{\mathcal{S}_1} U$. Note that P_U is free of rank 2 as $O_L \otimes \mathbf{W}(R)$ -module. It follows from [Theorem 4.7](#) that $\pi_1^{-1}(U)$ is isomorphic to the Grassmannian of $(O_L/p) \otimes R$ -invariant submodules of $P_U/I_R P_U$, that is, to \mathbb{P}_U^1 . In particular, we deduce from [[6](#)] that $\mathcal{T} := \pi_1^{-1}(\mathcal{S}_1)$ is locally an integral scheme. Let W be the

Table 8.1

t	$\Lambda(\{t\})$
$(0, 0)$	$\{(0, 0)\}$
$(0, g)$	$\{(1, g - 1)\}$ (if $g > 1$) $\{(0, g)\}$ (if $g = 1$)
$(0, n) \ n < g$	$\{(1, n)\}$
$(j, n) \ 0 < j < n < g - j$	$\{(j + 1, n), (j - 1, n)\}$
$(j, n) \ 0 < j = n < g - j$	$\{(j, n), (j - 1, n)\}$
$(j, n) \ 0 < j = n = g - j$	$\{(j, n), (j - 1, n + 1), (j - 1, n)\}$
$(j, n) \ 0 < j < n = g - j = j + 1$	$\{(j, n), (j - 1, n + 1), (j - 1, n)\}$
$(j, n) \ 1 < j + 1 < n = g - j$	$\{(j + 1, n - 1), (j - 1, n + 1), (j - 1, n)\}$

tangent space of $A[p]$ relative to S_1 . Since $A[p]$ is a local group scheme, using Dieudonné theory, one concludes that, geometrically, W is a locally free O_{S_1} -module of rank 2. Since S_1 is Zariski locally an integral scheme by [6], we deduce that W is a locally free O_{S_1} -module of rank 2. Let H be the universal O_L -invariant subgroup scheme of order p of $A[p]$ over \mathcal{T} . Its tangent space Z relative to \mathcal{T} defines an $(O_L/p) \otimes_{\mathbb{Z}} O_{\mathcal{T}}$ -invariant submodule of $W \otimes_{O_{S_1}} O_{\mathcal{T}}$. The Oort-Tate classification of group schemes of order p implies that there exists a covering $\{U_i = \text{Spec}(A_i)\}$ of \mathcal{T} by affine open subschemes and elements $\{a_i \in A_i\}_i$ such that $H_{U_i} \cong \text{Spec}(A_i[y]/(y^p - a_i y))$. Since H is, geometrically, a local group scheme, for every i , the element a_i lies in the intersection of every prime ideal, that is, in the nil radical of A_i . Thus, $a_i = 0$ and $\text{Lie}(H_{U_i}) \cong A_i$ for every i . We deduce that Z is locally free as $O_{\mathcal{T}}$ -module of rank 1. In particular, we get a map from \mathcal{T} to the Grassmannian of $(O_L/p) \otimes_{O_{S_1}}$ -invariant submodules of W which are locally free of rank 1 as O_{S_1} -modules. Such Grassmannian is a \mathbb{P}^1 -bundle over S_1 , and the given map is an isomorphism after completing at every point of S_1 and, hence, it is an isomorphism. This proves (2). ■

Definition 8.8. Define a function

$$\Lambda : 2^J \longrightarrow 2^J, \quad (8.6)$$

where J is as in Section 5, by defining $\Lambda(T) = \cup_{t \in T} \Lambda(\{t\})$ and by defining $\Lambda(\{t\})$ according to Table 8.1.

Given $(s, t) \in J$, we will write $\Lambda(s, t)$ for $\Lambda(\{(s, t)\})$.

Lemma 8.9. Let (j, n) and (s, t) be elements of J . Then, $\mathfrak{W}_{(s,t)} \subset \pi_2(\pi_1^{-1}(\mathfrak{W}_{(j,n)}))$ if and only if $(j, n) \in \Lambda(s, t)$. Furthermore, $\mathfrak{W}_{(s,t)} \cap \pi_2(\pi_1^{-1}(\mathfrak{W}_{(j,n)})) \neq \emptyset \Rightarrow \mathfrak{W}_{(s,t)} \subset \pi_2(\pi_1^{-1}(\mathfrak{W}_{(j,n)}))$. \square

Proof. (\Rightarrow) By assumption, we have $\pi_2^{-1}(\mathfrak{W}_{(s,t)}) \cap \pi_1^{-1}(\mathfrak{W}_{(j,n)}) \neq \emptyset$. On the other hand, the inversion [Lemma 8.6](#) implies that $\pi_2^{-1}(\mathfrak{W}_{(s,t)}) \subset \pi_1^{-1}(\mathfrak{W}_{\Lambda(s,t)})$. Hence, also $\pi_1^{-1}(\mathfrak{W}_{\Lambda(s,t)}) \cap \pi_1^{-1}(\mathfrak{W}_{(j,n)}) \neq \emptyset$. This implies that $(j, n) \in \Lambda(s, t)$. (Note that this also proves the final assertion.)

(\Leftarrow) Let $y \in \mathfrak{W}_{(s,t)}$. Since $(j, n) \in \Lambda(s, t)$, using the inversion [Lemma 8.6](#), we conclude that there exists a point $x \in \pi_2^{-1}(y) \cap \pi_1^{-1}(\mathfrak{W}_{(j,n)})$ as wanted. \blacksquare

Proposition 8.10. Let $(j, n) \in J$. Then, $\pi_2(\pi_1^{-1}(\mathfrak{W}_{(j,n)})) = \mathfrak{W}_{\Lambda(j,n)}$. \square

Proof. The inclusion \subset follows from [Propositions 6.5](#) and [6.6](#). To prove the inclusion \supset , we make use of [Lemma 8.9](#). It suffices to verify that for any $(s, t) \in \Lambda(j, n)$, we have $(j, n) \in \Lambda(s, t)$. This follows from a direct computation. We give one example. Assume that $0 < j < n < g - j$ and take $(s, t) = (j + 1, n) \in \Lambda(j, n)$. We have the following possibilities:

- (i) $j + 1 < n < g - j - 1$, in which case $\Lambda(j + 1, n) = \{(j + 2, n), (j, n)\}$;
- (ii) $j + 1 = n < g - j - 1$, in which case $\Lambda(j + 1, n) = \{(j + 1, n), (j, n)\}$;
- (iii) $j + 1 = n = g - j - 1$, in which case $\Lambda(j + 1, n) = \{(j + 1, n), (j, n + 1), (j, n)\}$;
- (iv) $j + 1 < n < g - j - 1 = j + 2$, in which case $\Lambda(j + 1, n) = \{(j + 1, n), (j, n + 1), (j, n)\}$;
- (v) $j + 2 < n = g - j - 1$, in which case $\Lambda(j + 1, n) = \{(j + 2, n - 1), (j, n + 1), (j, n)\}$.

One readily checks that indeed $(j, n) \in \Lambda(j + 1, n)$. \blacksquare

As a consequence of the proof of [Proposition 8.10](#), we get the following.

Corollary 8.11. Let (j, n) and (s, t) be elements of J . Then,

$$(s, t) \in \Lambda(j, n) \iff (j, n) \in \Lambda(s, t). \quad (8.7)$$

\square

Lemma 8.12. Let $(j, n) \in J$ and $j > 0$. Then, $\pi_2(\pi_1^{-1}(\overline{\mathfrak{W}_{(j,n)}})) = \overline{\mathfrak{W}_{(j-1,n)}}$. \square

Proof. Since, over the stratum \mathcal{S}_1 , the morphism π_1 is a \mathbb{P}^1 -bundle by [Proposition 8.7](#), we have $\pi_1^{-1}(\overline{\mathfrak{W}_{(j,n)}}) = \overline{\pi_1^{-1}(\mathfrak{W}_{(j,n)})}$. For any $x \in \mathfrak{W}_{(j,n)}$, the generic invariants of $\pi_2(\pi_1^{-1}(x))$ are $(j - 1, n)$. In particular, it follows from [Proposition 8.10](#) that $\mathfrak{W}_{(j-1,n)}$ is contained in $\pi_2(\pi_1^{-1}(\mathfrak{W}_{(j,n)}))$. Hence,

$$\overline{\mathfrak{W}_{(j-1,n)}} = \overline{\pi_2(\pi_1^{-1}(\mathfrak{W}_{(j,n)}))} = \pi_2(\overline{\pi_1^{-1}(\mathfrak{W}_{(j,n)})}) = \pi_2(\pi_1^{-1}(\overline{\mathfrak{W}_{(j,n)}})) \quad (\pi_2 \text{ proper}). \quad (8.8)$$

\blacksquare

Lemma 8.13. Let $0 \leq j \leq g/2$. Then, $\overline{\mathfrak{W}}_{(j,j)} = \mathcal{S}_j$ and $\overline{\mathfrak{W}}_{(j,g-j)} = \mathfrak{W}_{(j,g-j)}$. \square

Proof. The generic points of \mathcal{S}_j have singularity index j by [6] and slope j by Corollary 7.4. The set $\mathfrak{W}_{(j,g-j)}$ consists of superspecial abelian varieties and, hence, has dimension 0. \blacksquare

Theorem 8.14. There exists a unique function $\Delta : 2^J \rightarrow 2^J$ such that

$$\overline{\mathfrak{W}}_{(j,n)} = \mathfrak{W}_{\Delta(j,n)} \quad (8.9)$$

(where $\Delta(j, n)$ stands for $\Delta(\{(j, n)\})$). The function Δ satisfies the following properties and is characterized by them:

- (1) for any integer $0 \leq j \leq g/2$, we have $\Delta(j, j) = \{(j', n') \in J \mid j \leq j'\}$;
- (2) for any integer $0 \leq j \leq g/2$, we have $\Delta(j, g-j) = \{(j, g-j)\}$;
- (3) for any integer $1 \leq j \leq g/2$, we have $\Delta(j-1, n) = \Delta(j, n)$. \square

Proof. Claims (1) and (2) follow from Lemma 8.13. Given a pair (j, n) with $j > 0$, we distinguish two cases:

- (i) $n \leq g/2$. By (1), we have $\overline{\mathfrak{W}}_{(n,n)} = \mathfrak{W}_{\Delta(n,n)}$. We proceed by induction on $n - j$. If $n = j$, we are done. Assume that $\overline{\mathfrak{W}}_{(n-h,n)} = \mathfrak{W}_{\Delta(n-h,n)}$ for some integer $0 \leq h < n - j$. By Lemma 8.12, $\overline{\mathfrak{W}}_{(n-h-1,n)} = \pi_2(\pi_1^{-1}(\overline{\mathfrak{W}}_{(n-h,n)}))$. By Proposition 8.10, we have $\pi_2(\pi_1^{-1}(\mathfrak{W}_{\Delta(n-h,n)})) = \mathfrak{W}_{\Delta(n-h,n)}$;
- (ii) $n > g/2$. By (2), we have the equality $\overline{\mathfrak{W}}_{(g-n,n)} = \mathfrak{W}_{\Delta(g-n,n)}$. We proceed by induction on $(g-n) - j$. If $j = g-n$, we are done. Assume that $\overline{\mathfrak{W}}_{(g-n-h,n)} = \mathfrak{W}_{\Delta(n-h,n)}$ for some integer $0 \leq h < (g-n) - j$. By Lemma 8.12, $\overline{\mathfrak{W}}_{(g-n-h-1,n)} = \pi_2(\pi_1^{-1}(\overline{\mathfrak{W}}_{(g-n-h,n)}))$. We conclude, by Proposition 8.10, that $\pi_2(\pi_1^{-1}(\mathfrak{W}_{\Delta(g-n-h,n)})) = \mathfrak{W}_{\Delta(g-n-h,n)}$. \blacksquare

9 Newton polygons

In this section, we determine the Newton polygon of an abelian variety with real multiplication by O_L over a field of characteristic p , where $p = p^g$ is totally ramified in L . The result we obtain is phrased in terms of the two invariants j and n we associated to such an abelian variety in Section 5.

From the point of view of Dieudonné modules, we restrict our attention to the classification of F -isocrystals of rank 2 over the discrete valuation ring $O_L \otimes \mathbf{W}(k)$, which is the case relevant to our situation. Our methods are somewhat ad hoc and follow the exposition in [7]. Indeed, the classification of F -crystals up to isomorphism over this ring can be carried out along lines similar to Manin's seminal paper [11]. Details will appear in [14].

Lemma 9.1. Let $\varepsilon = 1$ if $g\lambda(n)$ is an integer, and let $\varepsilon = 1/2$ otherwise. We extend the automorphism σ to $O_L \otimes \mathbf{W}(k)[T^\varepsilon]$ by requiring $\sigma(T^\varepsilon) = T^\varepsilon$. The extra commutation rule on $O_L \otimes \mathbf{W}(k)[T^\varepsilon][F]$ is $FT^\varepsilon = T^\varepsilon F$. Then, there exist elements $b_0, b_1, u \in O_L \otimes \mathbf{W}(k)[T^\varepsilon]$, with u a unit, such that

$$(b_0 F + b_1)(F - T^{g\lambda(n)})u = F^2 - T^m F - c_3 T^g. \quad (9.1)$$

□

Proof. We write λ instead of $\lambda(n)$. Note that

$$(b_0 F + b_1)(F - T^{g\lambda})u = b_0 u^{\sigma^2} F^2 + (b_1 - b_0 T^{g\lambda})u^\sigma F - u b_1 T^{g\lambda}. \quad (9.2)$$

Let $v = u^{-1}$, $b_0 = v^{\sigma^2}$, and $b_1 = v^{\sigma^2} T^{g\lambda} - v^\sigma T^m$. Then, the above factorization holds if and only if v can be chosen to be a unit satisfying

$$v^{\sigma^2} - v^\sigma T^{m-g\lambda} - v c_3 T^{g(1-2\lambda)} = 0. \quad (9.3)$$

We proceed by constructing a converging sequence of elements $v_n \in O_L \otimes \mathbf{W}(k)[T^\varepsilon]$ such that v_n solves (9.3) modulo $T^{\varepsilon n}$ for all $n \in \mathbb{N}$.

Let $n = 1$. Note that either $g(1 - 2\lambda) = 0$ or $m = n$ and $m - g\lambda = 0$. Hence, (9.3) admits a nonzero solution $v_1 \bmod T^\varepsilon$ since k is algebraically closed.

Suppose that for $n \in \mathbb{N}$, we have constructed a unit $v_n \in O_L \otimes \mathbf{W}(k)[T^\varepsilon]$ such that

$$v_n^{\sigma^2} - v_n^\sigma T^{m-g\lambda} - v_n c_3 T^{g(1-2\lambda)} = z T^{\varepsilon n} \quad (9.4)$$

for some $z \in O_L \otimes \mathbf{W}(k)[T^\varepsilon]$. Let $v_{n+1} := v_n + T^{\varepsilon n} x$ with $x \in O_L \otimes \mathbf{W}(k)[T^\varepsilon]$. Then, v_{n+1} solves (9.3) mod $T^{(n+1)\varepsilon}$ if and only if

$$x^{p^2} - x^p T^{m-g\lambda} - c_3 x T^{g(1-2\lambda)} + z = 0 \bmod T^\varepsilon. \quad (9.5)$$

Since k is algebraically closed, such x exists. ■

Theorem 9.2. Let k be an algebraically closed field of positive characteristic p . Let (A, ι) be an abelian variety with real multiplication by O_L over k satisfying (DP) and of type (j, n) . The slopes of the Newton polygon of A are $\lambda(n)$ and $1-\lambda(n)$, each with multiplicity g .

□

Proof. Let P be the polarized O_L -display associated to A . By Proposition 4.10, there exist α and β in P such that $P = (O_L \otimes \mathbf{W}(k))\alpha \oplus (O_L \otimes \mathbf{W}(k))\beta$, the submodule $(O_L \otimes \mathbf{W}(k))\alpha$ is maximal isotropic with respect to the O_L -polarization on P , and the matrix of Frobenius

in the basis $\{\alpha, \beta\}$ is given by

$$\begin{pmatrix} T^m & c_3 T^i \\ T^j & 0 \end{pmatrix}. \quad (9.6)$$

Let

$$\beta' = F(\alpha) = T^m \alpha + T^j \beta. \quad (9.7)$$

Recall that $m \geq n$ with equality if $n < i$. The case $m = \infty$ is allowed, and then $T^m = 0$ and $\beta' = T^j \beta$. Notice that $\{\alpha, \beta'\}$ is an $L \otimes \mathbf{W}(k)$ -basis of $P \otimes \mathbb{Q}$ that $(L \otimes \mathbf{W}(k))\alpha$ is a maximal isotropic submodule of $P \otimes \mathbb{Q}$ and that F is given in this basis by

$$\begin{pmatrix} 0 & c_3 T^g \\ 1 & T^m \end{pmatrix}. \quad (9.8)$$

The slopes of P are those of the $O_L \otimes \mathbf{W}(k)[F]$ -module

$$\widetilde{M} := (O_L \otimes \mathbf{W}(k))\alpha \oplus (O_L \otimes \mathbf{W}(k))\beta'. \quad (9.9)$$

Notice that $F^2(\alpha) = F(\beta') = c_3 T^g \alpha + T^m F(\alpha)$. Hence, the surjective map of Dieudonné modules with O_L -action

$$\Phi : O_L \otimes \mathbf{W}(k)[F] \longrightarrow \widetilde{M}, \quad (9.10)$$

given by $\Phi(1) = \alpha$ and $\Phi(F) = \beta'$, induces an isomorphism

$$M := O_L \otimes \mathbf{W}(k)[F]/(F^2 - T^m F - c_3 T^g) \xrightarrow{\sim} \widetilde{M}. \quad (9.11)$$

By [Lemma 9.1](#), the $O_L \otimes \mathbf{W}(k)[F]$ -linear map

$$\Psi : M \longrightarrow O_L \otimes \mathbf{W}(k)[T^\varepsilon][F]/(F - T^{g\lambda(n)}) =: N \quad (9.12)$$

given by

$$\Psi(1) = u^{-1} \quad (9.13)$$

is well defined. See [Lemma 9.1](#) for the definitions of ε and u . Moreover, its image contains

$$\Psi(\Phi^{-1}(O_L \otimes \mathbf{W}(k)\alpha)) = (O_L \otimes \mathbf{W}(k))u^{-1}, \quad (9.14)$$

and hence has $\mathbf{W}(k)$ -rank at least g . In particular, the slopes of F -isocrystal $P \otimes \mathbb{Q}$ contain the slopes of N . Define

$$N_0 := O_L \otimes \mathbb{Z}_p[T^\varepsilon], \quad F_0 = \text{multiplication by } T^{g\lambda(n)}. \quad (9.15)$$

Then,

$$N \xrightarrow{\sim} N_0 \otimes \mathbf{W}(k), \quad F = F_0 \otimes \sigma. \quad (9.16)$$

In particular, the slopes of N are the slopes of N_0 . Since on N_0 the operator F_0 is linear and equal to multiplication by $T^{g\lambda(n)}$, the slopes of N_0 are the p -adic valuations of the eigenvalues of $T^{g\lambda(n)}$ on N_0 . Since $T \in O_L$ is the zero of an Eisenstein polynomial, we conclude that the slopes of N_0 are all equal to $\lambda(n)$. \blacksquare

10 The main theorem

Let J , $\mathfrak{W}_{(j,n)}$, \mathcal{S}_j , and $\lambda(n)$ be as in [Section 5](#). Let k be a field of characteristic p .

Theorem 10.1. Let $(j, n) \in J$. The following properties hold:

- (1) $\mathfrak{W}_{(j,n)}$ is a nonempty, regular, locally closed subscheme of $\mathfrak{M}(k, \mu_N)$ of dimension equal to $g - (j + n)$. Moreover, every irreducible component $\mathfrak{M}(k, \mu_N, \mathcal{I})$ of $\mathfrak{M}(k, \mu_N)$ contains a point of $\mathfrak{W}_{(j,n)}$;
- (2) the slopes of the Newton polygon of every point x of $\mathfrak{W}_{(j,n)}$ are $(\lambda(n), 1 - \lambda(n))$, each appearing with multiplicity g ;
- (3) there exists a function $\Delta : 2^J \rightarrow 2^J$ such that the Zariski closure $\overline{\mathfrak{W}_{(j,n)}}$ of $\mathfrak{W}_{(j,n)}$ is equal to $\mathfrak{W}_{\Delta(j,n)}$. In particular, the subsets $\{\mathfrak{W}_{(j,n)} \mid (j, n) \in J\}$ define a stratification of $\mathfrak{M}(k, \mu_N)$. See [Theorem 8.14](#) for the properties of Δ ;
- (4) there exists a function $\Lambda : 2^J \rightarrow 2^J$ such that the image of $\mathfrak{W}_{(j,n)}$ via the $\Gamma_0(p)$ -Hecke correspondence is $\mathfrak{W}_{\Lambda(j,n)}$. See [Definition 8.8](#) for the properties of Λ . \square

We remark again that

$$\mathcal{S}_j = \bigcup_{j \leq n \leq g-j} \mathfrak{W}_{(j,n)}. \quad (10.1)$$

We define the Newton stratification. For any $\ell \in \{i/g \mid 0 \leq i \leq [g/2]\} \cup \{1/2\}$, let β_ℓ be the Newton polygon with slopes $(\ell, 1 - \ell)$, each appearing with multiplicity g . Let \mathcal{N}_ℓ be the closed reduced subscheme of $\mathfrak{M}(k, \mu_N)$ universal for the property that the Newton polygon lies above or is equal to β_ℓ .

Corollary 10.2. We have

$$\mathcal{N}_\ell = \bigcup_{\{(j,n)|\lambda(n)\geq\ell\}} \mathfrak{W}_{(j,n)}. \quad (10.2)$$

In particular,

$$\dim(\mathcal{N}_\ell) = g - \lceil g \cdot \ell \rceil. \quad (10.3)$$

□

Proof. The corollary follows from the theorem noticing that $\overline{\mathcal{N}_\ell} = \bigcup_{\{(j,n):\lambda(n)=\ell\}} \overline{\mathfrak{W}_{(j,n)}} = \overline{\mathfrak{W}_{(0,\lceil g \cdot \ell \rceil)}}$. ■

Proof of Theorem 10.1

$\mathfrak{W}_{(j,n)} \cap \mathfrak{M}(k, \mu_N, \mathfrak{I})$ is nonempty. For any fractional ideal \mathfrak{I} with its natural notion of positivity, we construct, in several steps, a \mathfrak{I} -polarized abelian variety with real multiplication by O_L and invariants j and n .

Let E be a supersingular elliptic curve with a μ_N -level structure. Then, $E \otimes_{\mathbb{Z}} D_L^{-1}$ satisfies (R), carries a $\mu_N \otimes D_L^{-1}$ -level structure, and is parameterized by a point x of $\mathfrak{W}_{(0,g)} \cap \mathfrak{M}(k, \mu_N, D_L)$. Choose a prime q , relatively prime to Nd_L , and a prime ideal \mathfrak{q} dividing q such that $q^{-1}D_L = \mathfrak{I}$ in $\text{Cl}(L)^+$. Consider the image of x under the finite Hecke correspondence T_q sending any geometric point $[A]$ to $\sum_{H \subset A[\mathfrak{q}]} [A/H]$, where the sum ranges over all nontrivial, O_L -invariant, subgroup schemes H of A . Using arguments similar to Section 3, one checks that, indeed, T_q is a well-defined correspondence on $\mathfrak{M}(k, \mu_N)$ (in particular, condition (DP) is preserved). To calculate how the polarization module changes under T_q , one may focus on Tate objects.

Let $\mathbb{T}(D_L, O_L) = (\mathbf{G}_m \otimes O_L) / (\underline{q}(O_L))$. Then, the Tate object $\mathbb{T}(q^{-1}, O_L)$ belongs to $T_q(\mathbb{T}(D_L, O_L))$ and has polarization module $q^{-1}D_L = \mathfrak{I}$. This proves point (a) below.

(a) $\mathfrak{W}_{(0,g)} \cap \mathfrak{M}(k, \mu_N, \mathfrak{I})$ is nonempty.

(b) Let $x \in \mathfrak{W}_{(0,n)} \cap \mathfrak{M}(k, \mu_N, \mathfrak{I})$, where $1 \leq n \leq g-1$. Then, $\mathfrak{W}_{(1,n)} \cap \mathfrak{M}(k, \mu_N, \mathfrak{p}\mathfrak{I})$ is nonempty by Proposition 6.5.

(c) Let $x \in \mathfrak{W}_{(j,n)} \cap \mathfrak{M}(k, \mu_N, \mathfrak{I})$, where $0 < j < n < g-j$. Then, $\mathfrak{W}_{(j+1,n)} \cap \mathfrak{M}(k, \mu_N, \mathfrak{p}\mathfrak{I})$ is nonempty by Proposition 6.6.

(d) Let $x \in \mathfrak{W}_{(j,n)} \cap \mathfrak{M}(k, \mu_N, \mathfrak{I})$. Then, $\mathfrak{W}_{(j,n')} \cap \mathfrak{M}(k, \mu_N, \mathfrak{I})$ is nonempty for any $j \leq n' \leq n$ by Section 7.2.1 and Corollary 7.4.

Proof of (1). The assertion that $\mathfrak{W}_{(j,n)} \cap \mathfrak{M}(k, \mu_N, \mathfrak{I})$ is nonempty follows from (a), (b), (c), and (d) above. The rest of the claim follows from Corollary 7.4.

Proof of (2). This is Theorem 9.2.

Proof of (3). This is [Theorem 8.14](#). We will give in [Section 10.2](#) a second proof outside the supersingular locus using the de Jong-Oort purity theorem.

Proof of (4). This is [Proposition 8.10](#). ■

10.1 The Grothendieck conjecture

The Grothendieck conjecture was initially asserted in terms of deformations of p -divisible groups and can be reformulated for abelian varieties, possibly with extra structure; see [\[16\]](#). In our context, it corresponds to the following corollary.

Corollary 10.3. Let \underline{A}_0 be a g -dimensional abelian variety with real multiplication by O_L and satisfying (DP) over an algebraically closed field k of characteristic p . Let β_1 be a Newton polygon with two slopes $\{\lambda(n_1), 1 - \lambda(n_1)\}$ for some $0 \leq n_1 \leq g$, each of multiplicity g , lying below the Newton polygon β_0 of \underline{A}_0 . Then, there exist a complete equicharacteristic local ring (R, \mathfrak{m}) with isomorphism $R/\mathfrak{m} \cong k$ and a g -dimensional abelian scheme with real multiplication by O_L and satisfying (DP) over $\mathrm{Spec}(R)$ whose special fiber is \underline{A}_0 and whose generic fiber has Newton polygon β_1 . □

Proof. Let (j_0, n_0) be the invariants of \underline{A}_0 . Its Newton polygon has only two slopes $\{\lambda(n_0), 1 - \lambda(n_0)\}$ with multiplicity g depending on n_0 ([Theorem 9.2](#)). By [Proposition 7.2](#), we can deform \underline{A}_0 to an abelian variety with invariants (j, n') for every $j \leq n' \leq n_0$. By [\[6\]](#), one can deform an abelian variety with invariants (j, j) for $j > 0$ to an abelian variety with singularity index $j - 1$. Using, repeatedly, the two arguments, we conclude that we can deform \underline{A}_0 to an abelian variety with slope invariant n_1 . ■

10.2 Connection to purity

This subsection is devoted to a different proof of part (3) of our main theorem, at least outside the supersingular locus $\mathfrak{M}(\mathbb{F}_p, \mu_N)^{\mathrm{ss}}$, based on the de Jong-Oort purity theorem.

Let $n < g/2$. Note that

$$\overline{\mathfrak{M}_{(j,n)}} \setminus \mathfrak{M}(\mathbb{F}_p, \mu_N)^{\mathrm{ss}} \subset \bigcup_{j \leq j', n \leq n' < g/2} \mathfrak{M}_{(j', n')} \quad (10.4)$$

because j goes up under specialization and, for $n' \leq g/2$, the Newton polygon determines n' and goes up under specialization. The converse follows from the following two assertions:

- (1) for all $0 \leq a \leq g/2$ and all $a \leq b < g - a$, we have $\mathfrak{M}_{(a, b+1)} \subset \overline{\mathfrak{M}_{(a, b)}}$;

(2) $\mathfrak{W}_{(j+1,n)} \subset \overline{\mathfrak{W}_{(j,n)}}$ for $n \leq g/2$.

The assertion (1) follows from [Corollary 7.4](#).

We now prove that $\mathfrak{W}_{(j+1,n)} \subset \overline{\mathfrak{W}_{(j,n)}}$ for $n \leq g/2$. Let $x = [\underline{A}_x]$ be a geometric point of $\mathfrak{W}_{(j+1,n)}$. Consider the formal neighborhood U of x in S_j , representing all infinitesimal deformations of \underline{A}_x with singularity index $\geq j$. By Deligne-Pappas, it is of pure dimension $g - 2j$. Let $\eta = [\underline{A}_\eta]$ be a generic point of an irreducible component of U . It has invariants (j, j) . Note that $j < n \leq g/2$. Therefore, the Newton polygon β_j of \underline{A}_η is strictly below that of \underline{A}_x . By de Jong-Oort purity theorem [5], there exists a closed integral subscheme U_1 of U with generic point η_1 , such that $\dim(U_1) = g - 2j - 1$ and the Newton polygon of \underline{A}_{η_1} is strictly above β_j . Let $a = j(\underline{A}_{\eta_1})$ and $b = n(\underline{A}_{\eta_1})$ be the invariants of \underline{A}_{η_1} . They satisfy $j \leq a$, $j < b$, and $a + b \leq 2j + 1$ (because $\dim(U_1) \leq \dim(\mathfrak{W}_{(a,b)})$). Hence, the invariants of \underline{A}_{η_1} are $(j, j + 1)$.

If $n = j + 1$, we are done. If not, assume that for some integer $1 \leq h < n - j$, we have constructed a closed integral subscheme U_h of U with generic point η_h such that $\dim(U_h) = g - 2j - h$ and \underline{A}_{η_h} has invariants $(j, j + h)$. Since $j + h < n \leq g/2$, the Newton polygon of \underline{A}_{η_h} is strictly below β_n . By the purity theorem, there exists a closed integral subscheme U_{h+1} of U_h with generic point η_{h+1} , such that $\dim(U_{h+1}) = g - 2j - h - 1$ and the Newton polygon of $\underline{A}_{\eta_{h+1}}$ is strictly above that of \underline{A}_{η_h} . Let $a = j(\underline{A}_{\eta_{h+1}})$ and $b = n(\underline{A}_{\eta_{h+1}})$ be the invariants of $\underline{A}_{\eta_{h+1}}$. They satisfy $j \leq a$, $j + h < b$, and $a + b \leq 2j + h + 1$ (because $\dim(U_{h+1}) \leq \dim(\mathfrak{W}_{(a,b)})$). Hence, the invariants of $\underline{A}_{\eta_{h+1}}$ are $(j, j + h + 1)$.

Again, if $n = j + h + 1$, we are done. Else, continue in the same fashion.

Remark 10.4. The inclusion $\mathfrak{W}_{(j+1,n)} \subset \overline{\mathfrak{W}_{(j,n)}}$ need not hold for $n > g/2$. See [Section 11](#) for examples.

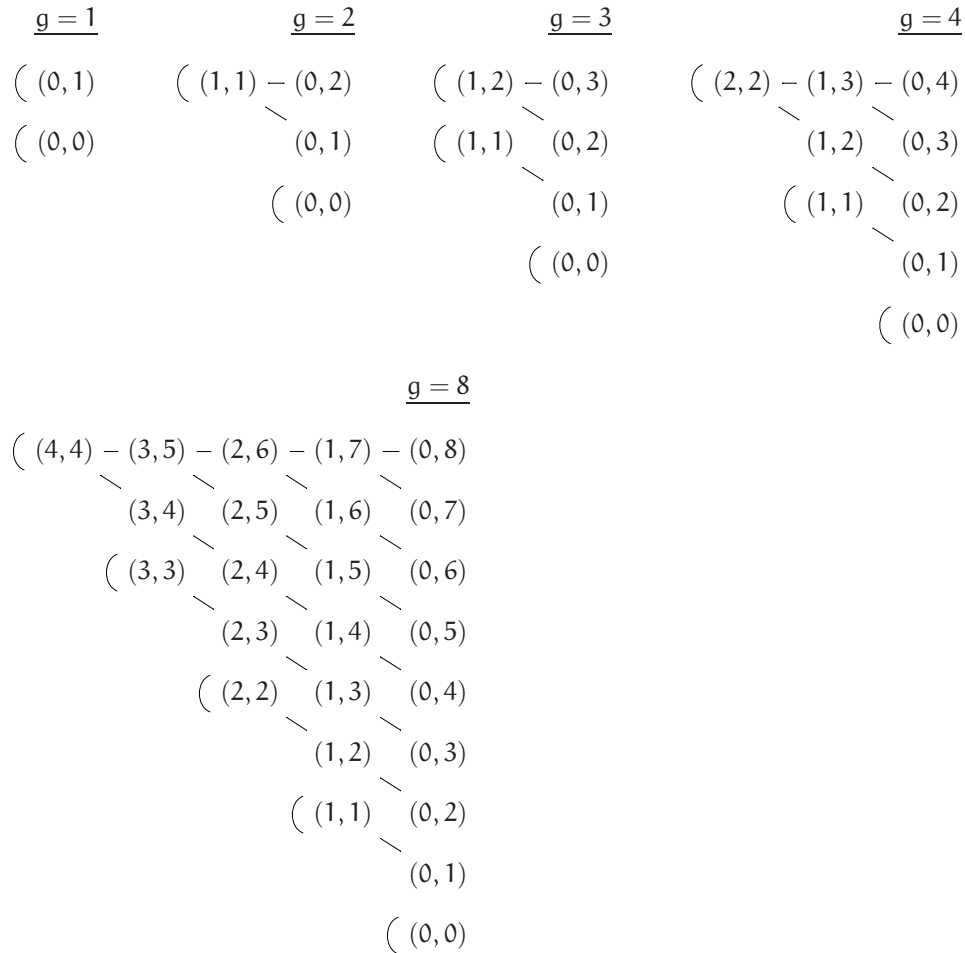
11 Examples

In this section, we provide examples of the functions Λ and Δ defined in [Definition 8.8](#) and [Theorem 8.14](#), respectively. Some of the properties of these functions can be easily guessed from the diagrams and were used above ([Corollary 8.11](#) and [Section 10.2](#)).

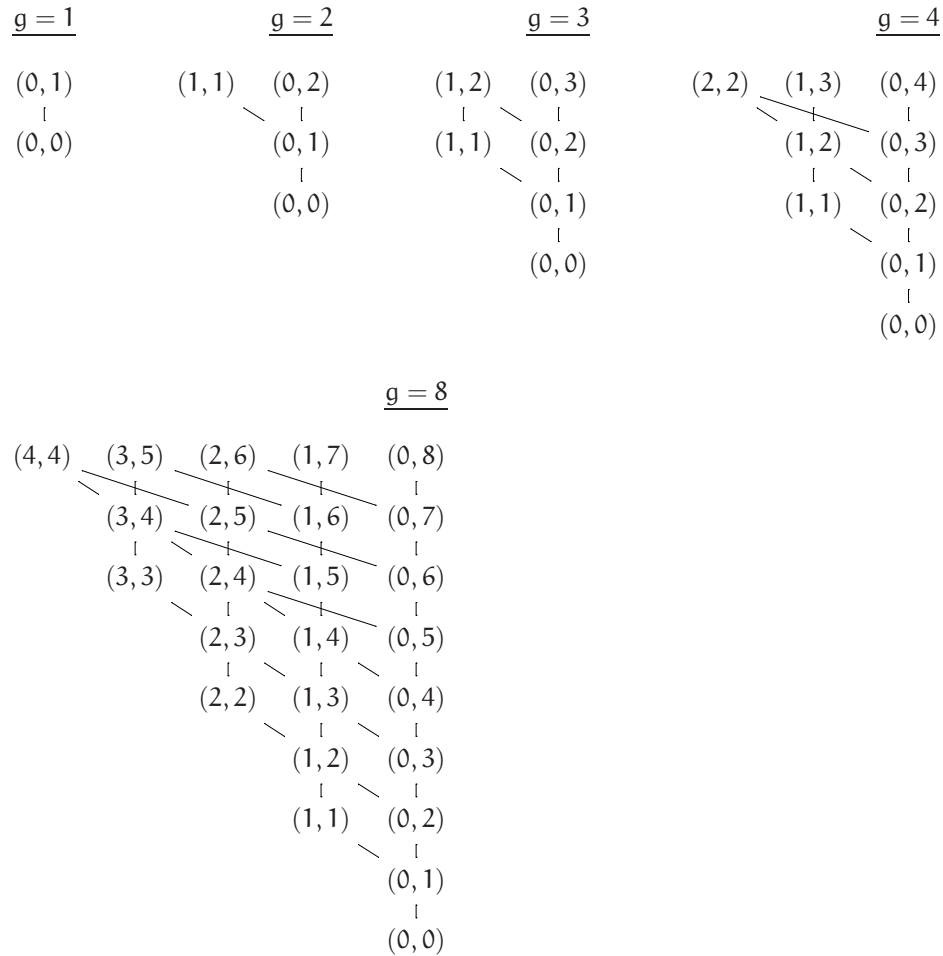
The diagrams should be read as follows. In the diagrams for the function Λ , the image of the function on a couple (j, n) is the set of all the couples attached to it (the property is symmetric! see [Corollary 8.11](#)). Thus, for example, for $g = 4$, we have $\Lambda(2, 2) = \{(2, 2), (1, 3), (1, 2)\}$ and $\Lambda(1, 2) = \{(2, 2), (0, 2)\}$. For the function Δ , our convention that if a point x lies above a point y and is connected to y by a descending path, then the stratum corresponding to x lies in the boundary of the stratum corresponding to y . Thus, for example, for $g = 4$, the closure of $(0, 3)$ is $\Delta(0, 3) = \{(2, 2), (0, 4)\}$ and the closure of $(0, 2)$ is given by $\Delta(0, 2) = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (2, 2)\}$.

We remark that the diagrams for Δ give a convenient way to see the structure of the moduli space. The columns describe the singularities: the rightmost column is the nonsingular part, the further the column is to the left the worse are the singularities. The rows correspond to the α -number: the r th row from below corresponds to abelian varieties with α -number equal to r , and we see that the α -number gives a stratification. If T_r denotes the abelian varieties with α -number greater than or equal to r , then every component of T_r has codimension $g - r$. Finally, the Newton strata is defined by “wedges.” We immediately see that the smooth locus of the moduli space is dense in every Newton strata (but not in the α -number strata).

11.1 The function Λ



11.2 The function Δ



Appendix

A pathology

Let k be an algebraically closed field of characteristic p . We consider Dieudonné modules over k , that are self-dual, of rank 4 and have a -number equals one. We call them modules of type x . These are precisely the Dieudonné modules arising from supersingular, but not superspecial, abelian surfaces having a separable polarization. Every such Dieudonné module D is generated by an element $\alpha_0 \in D$ satisfying $F^2\alpha_0 = V^2\alpha_0$. The elements

$$\alpha_0, F\alpha_0, V\alpha_0, F^2\alpha_0 = V^2\alpha_0 \tag{A.1}$$

form a k -basis for D . Note that every other generator α of D as a $k[F, V]$ -module is of the form $\alpha = r\alpha_0 + sF\alpha_0 + tV\alpha_0 + uF^2\alpha_0$, where $r \in k^\times$ and $s, t, u \in k$. We let $r(\alpha)$ be defined by the formula

$$r(\alpha) = F^2\alpha/V^2\alpha. \quad (\text{A.2})$$

Let L be a quadratic field in which p is ramified. Given a module D of type x , we wish to consider an $O_L = k[T]/(T^2)$ -action on D such that the action commutes with F and V and the kernel of T has rank 2. We call such modules the modules of type y . The modules of type y may appear as the Dieudonné modules of the p -torsion of a supersingular, and not superspecial, abelian surface over k with real multiplication by O_L .

Given a module D of type x , choose any $k[F, V]$ -generator α to D . Giving an action of $k[T]$ on D is equivalent to giving A', A, B , and C in k . The image $T\alpha$ is

$$T\alpha := A'\alpha + AF\alpha + BV\alpha + CF^2\alpha. \quad (\text{A.3})$$

The action is extended then by the formulas

$$TF\alpha = FT\alpha, \quad TV\alpha = VT\alpha, \quad TF^2\alpha = F^2T\alpha. \quad (\text{A.4})$$

A necessary condition that the action factors through the quotient by the ideal (T^2) is $A' = 0$ (and then the action automatically commutes with F and V). Hence,

$$T\alpha = AF\alpha + BV\alpha + CF^2\alpha. \quad (\text{A.5})$$

Since

$$\begin{aligned} T^2\alpha &= T(AF\alpha + BV\alpha + CF^2\alpha) \\ &= AFT\alpha + BVT\alpha \\ &= A^{p+1}F^2\alpha + B^{1+p^{-1}}V^2\alpha \\ &= (A^{p+1}r(\alpha) + B^{1+p^{-1}})V^2\alpha, \end{aligned} \quad (\text{A.6})$$

we get the additional condition

$$A^{p+1}r(\alpha) + B^{1+p^{-1}} = 0. \quad (\text{A.7})$$

The further condition that the rank of T is 2 is equivalent to

$$AB \neq 0. \quad (\text{A.8})$$

Conversely, if (A.5) and (A.7) hold, then the action factors through T^2 .

Define an action k^\times on the set

$$\mathcal{J} = \left\{ (a, b, c) \in (k^\times)^3 : a^{p+1}c + b^{1+p^{-1}} = 0 \right\} \quad (\text{A.9})$$

by

$$r(a, b, c) = (r^{1-p}a, r^{1-p^{-1}}b, r^{p^2-p^{-2}}c). \quad (\text{A.10})$$

Lemma A.1. Giving a Dieudonné module of type y is equivalent to giving an orbit of k^\times on \mathcal{J} . The association is as follows. Given a module D , choose a generator α to D as a $k[F, V]$ -module; write $T\alpha = AF\alpha + BV\alpha + CF^2\alpha$, let $r(\alpha) = F^2\alpha/V^2\alpha$, and associate to the module D the invariants $(A, B, r(\alpha))$.

Conversely, given a triple (a, b, c) , associate to it the cyclic Dieudonné module on the generator α satisfying $F^2\alpha = cV^2\alpha$ and define the action of T by $T\alpha = AF\alpha + BV\alpha$. \square

Proof. One checks that changing the generator for D from α to $r\alpha + sF\alpha + tV\alpha + uF^2\alpha$ has the effect of changing A to $r^{1-p}A$, B to $r^{1-p^{-1}}B$, and $r(\alpha)$ to $r^{p^2-p^{-2}}r(\alpha)$. Hence, the orbit associated to D is well defined.

Now, suppose that the same orbit (A, B, r) is associated to two modules D_1 and D_2 . We may reduce to the case where we have a single module D and two generators α and α' for D such that $r = r(\alpha) = r(\alpha')$ and two actions of T

$$T *_1 \alpha' = AF\alpha' + BV\alpha', \quad T *_2 \alpha = AF\alpha + BV\alpha + CF^2\alpha. \quad (\text{A.11})$$

One checks that under a change of generator $\alpha' \mapsto \alpha'' := \alpha' + dF\alpha'$, we have $r(\alpha') = r(\alpha'')$ and

$$\begin{aligned} T\alpha'' &= T(\alpha' + dF\alpha') \\ &= AF\alpha' + BV\alpha' + dF(AF\alpha' + BV\alpha') \\ &= AF\alpha' + BV\alpha' + dA^pF^2\alpha' \\ &= AF(\alpha' + dF\alpha') + BV(\alpha' + dF\alpha') + (dA^p - Ad^p)F^2\alpha' \\ &= AF\alpha'' + BV\alpha'' + (dA^p - Ad^p)F^2\alpha''. \end{aligned} \quad (\text{A.12})$$

It is enough to find $d \in k$ such that $dA^p - Ad^p = C$. This is possible since $A \neq 0$ and k is algebraically closed. \blacksquare

Remark A.2. (a) Note that the relation $A^{p+1}r(\alpha) + B^{1+p^{-1}} = 0$ determines r and, in fact, $\mathcal{J} \cong (k^\times)^2$, where the action of k^\times is given by $(A, B) \mapsto (r^{1-p}A, r^{1-p^{-1}}B)$. However, it is

sometimes convenient to keep r . Indeed, we may then identify \mathcal{J}/k^\times with $\mathcal{J}'/\mathbb{F}_{p^4}^\times$, where $\mathcal{J}' = \{(A, B) \in (k^\times)^2 : A^{p+1} + B^{1+p^{-1}} = 0\}$.

(b) The function

$$(A, B, c) \longmapsto AB^p \quad (\text{A.13})$$

is well defined on the orbits of k^\times in \mathcal{J} ; it induces an isomorphism $\mathcal{J}'/\mathbb{F}_{p^4}^\times \cong k^\times$.

Corollary A.3. There are infinitely many nonisomorphic Dieudonné modules of type y over k . \square

In the rest of this section, we provide examples for $g = 2$ showing that there are infinitely many nonisomorphic polarized group schemes with O_L -action arising as p -torsion of abelian surfaces with RM by O_L . To show this, we compute the invariants for a certain Moret-Bailly family lying in the moduli space $\mathfrak{M}(\mathbb{F}_p, \mu_N)$ for the field L . We put ourselves in the case of [Proposition 6.6](#), case (2c). To calculate initial data for this case—an abelian surface of invariants $(i, j) = (1, 1)$ —we start with a point of invariants $(0, 2)$ and use [Proposition 6.5](#). Such a point may be obtained from $E \otimes O_L$, where E is a supersingular elliptic curve. It provides us with a display

$$\begin{aligned} P &= O_L \otimes W(k)\alpha \oplus O_L \otimes W(k)\beta \supset Q \\ &= pO_L \otimes W(k)\alpha \oplus O_L \otimes W(k)\beta. \end{aligned} \quad (\text{A.14})$$

The action of F is given by the matrix $\begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix}$. Let T be a uniformizer of $O_L \otimes W(k)$. The lattice providing initial data for case (2c) is given by

$$\begin{aligned} P_\gamma &= P + p^{-1}T\beta \\ &= O_L \otimes W(k)\alpha \oplus O_L \otimes W(k)T^{-1}\beta \supset Q_\gamma \\ &= O_L \otimes W(k)T\alpha \oplus O_L \otimes W(k)\beta. \end{aligned} \quad (\text{A.15})$$

Note that T , indeed, kills P_γ/Q_γ . The action of F , with respect to the basis $\alpha' = \alpha$ and $\beta' = T^{-1}\beta$, is given by $\begin{pmatrix} 0 & T^{-1}p \\ T & 0 \end{pmatrix}$.

We reset our notation assuming that $T^2 = p$ and putting

$$\begin{aligned} P &= O_L \otimes W(k)\alpha \oplus O_L \otimes W(k)\beta \supset Q \\ &= TO_L \otimes W(k)\alpha \oplus TO_L \otimes W(k)\beta; \end{aligned} \quad (\text{A.16})$$

the operators F and V are given by $\begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$. Take a vector $\gamma = a_1 T\alpha + a_2 T\beta$ as in [Definition 6.1](#).

Generically, $p^{-1}\gamma$ generates P_γ/pP_γ as a Dieudonné module and, to simplify the computations, we may as well assume that $a_1 = 1$. We proceed to calculate the invariants (A, B, r) for $(P_\gamma/pP_\gamma, p^{-1}\gamma)$.

We put $\delta = p^{-1}\gamma$. We note that, in the above notation, if $T\delta = AF\delta + BV\delta + CF^2\delta$ modulo pP_γ , then $FT\delta = A^pF^2\delta$ modulo pP_γ and $VT\delta = B^{p^{-1}}V^2\delta$ modulo pP_γ . To find A and B , we therefore make the following calculation. First, we calculate that

$$\begin{aligned} T\delta &= \alpha + a_2\beta, \\ FT\delta &= a_2^\sigma T\alpha + T\beta, & VT\delta &= a_2^{\sigma^{-1}} T\alpha + T\beta, \\ F^2\delta &= T\alpha + a_2^{\sigma^2} T\beta, & V^2\delta &= T\alpha + a_2^{\sigma^{-2}} T\beta. \end{aligned} \tag{A.17}$$

We first find u and z such that

$$FT\delta - uF^2\delta - zp\delta \in pP, \tag{A.18}$$

that is, expanding this expression in terms of α and β , every coefficient is divisible by p . Solving, we find that $u = a_2^\sigma - (a_2^{\sigma^2+\sigma} - 1)/(a_2^{\sigma^2} - a_2)$ which gives

$$A = a_2 - \frac{a_2^{p+1} - 1}{a_2^p - a_2^{p^{-1}}} = \frac{1 - a_2^{1+p^{-1}}}{a_2^p - a_2^{p^{-1}}} \pmod{p}. \tag{A.19}$$

Similarly, we find u and z such that

$$VT\delta - uV^2\delta - zp\delta \in pP. \tag{A.20}$$

Solving, we find that $u = a_2^{\sigma^{-1}} - (a_2^{\sigma^{-1}+\sigma^{-2}} - 1)/(a_2^{\sigma^{-2}} - a_2)$ which gives

$$B = a_2 - \frac{a_2^{1+p^{-1}} - 1}{a_2^{p^{-1}} - a_2^p} = \frac{1 - a_2^{p+1}}{a_2^{p^{-1}} - a_2^p} \pmod{p}. \tag{A.21}$$

To show that there is indeed a variation in the isomorphism type of the Dieudonné modules, it is enough to check that the values AB^p are not constant as a function of a_2 . This is immediate.

References

- [1] F. Andreatta and E. Z. Goren, *Hilbert Modular Varieties of Low Dimension*, to appear in *Geometric Aspects of Dwork's Theory*. A volume in memory of Bernard Dwork, Alan Adolphson, Francesco Baldassarri, Pierre Berthelot, Nicholas Katz, and Francois Loeser (eds.).

- [2] E. Bachmat and E. Z. Goren, *On the non-ordinary locus in Hilbert-Blumenthal surfaces*, Math. Ann. **313** (1999), no. 3, 475–506.
- [3] C.-L. Chai, *Arithmetic minimal compactification of the Hilbert-Blumenthal moduli spaces*, Ann. of Math. (2) **131** (1990), no. 3, 541–554.
- [4] ———, *Newton polygons as lattice points*, Amer. J. Math. **122** (2000), no. 5, 967–990.
- [5] A. J. de Jong and F. Oort, *Purity of the stratification by Newton polygons*, J. Amer. Math. Soc. **13** (2000), no. 1, 209–241.
- [6] P. Deligne and G. Pappas, *Singularités des espaces de modules de Hilbert, en les caractéristiques divisant le discriminant* [*Singularities of Hilbert modular varieties in characteristics dividing the discriminant*], Compositio Math. **90** (1994), no. 1, 59–79 (French).
- [7] M. Demazure, *Lectures on p-Divisible Groups*, Lecture Notes in Mathematics, vol. 302, Springer-Verlag, Berlin, 1972.
- [8] E. Z. Goren, *Hasse invariants for Hilbert modular varieties*, Israel J. Math. **122** (2001), 157–174.
- [9] E. Z. Goren and F. Oort, *Stratifications of Hilbert modular varieties*, J. Algebraic Geom. **9** (2000), no. 1, 111–154.
- [10] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11 (French).
- [11] Ju. I. Manin, *Theory of commutative formal groups over fields of finite characteristic*, Uspehi Mat. Nauk **18** (1963), no. 6 (114), 3–90 (Russian), translated in Russian Math. Surveys **18** (1963), no. 6, 1–83.
- [12] B. Moonen, *Group schemes with additional structures and Weyl group cosets*, Moduli of Abelian Varieties (Texel Island, 1999) (C. Faber, G. van der Geer, and F. Oort, eds.), Progr. Math., vol. 195, Birkhäuser, Basel, 2001, pp. 255–298.
- [13] L. Moret-Bailly, *Familles de courbes et de variétés abéliennes sur \mathbf{P}^1* , Séminaire sur les Pinceaux de Courbes de Genre au Moins Deux (L. Szpiro, ed.), Astérisque, vol. 86, Société Mathématique de France, Paris, 1981, pp. 109–140.
- [14] M.-H. Nicole, Ph.D. thesis, McGill University, Canada, in progress.
- [15] F. Oort, *Moduli of abelian varieties and Newton polygons*, C. R. Acad. Sci. Paris Sér. I Math. **312** (1991), no. 5, 385–389.
- [16] ———, *Newton polygon strata in the moduli space of abelian varieties*, Moduli of Abelian Varieties (Texel Island, 1999) (C. Faber, G. van der Geer, and F. Oort, eds.), Progr. Math., vol. 195, Birkhäuser, Basel, 2001, pp. 417–440.
- [17] ———, *A stratification of a moduli space of abelian varieties*, Moduli of Abelian Varieties (Texel Island, 1999) (C. Faber, G. van der Geer, and F. Oort, eds.), Progr. Math., vol. 195, Birkhäuser, Basel, 2001, pp. 345–416.
- [18] M. Rapoport, *Compactifications de l'espace de modules de Hilbert-Blumenthal*, Compositio Math. **36** (1978), no. 3, 255–335 (French).
- [19] G. van der Geer, *Cycles on the moduli space of abelian varieties*, Moduli of Curves and Abelian Varieties (C. Faber and E. Looijenga, eds.), Aspects Math., vol. E33, Vieweg, Braunschweig, 1999, pp. 65–89.

- [20] T. Wedhorn, *The dimension of Oort strata of Shimura varieties of PEL-type*, Moduli of Abelian Varieties (Texel Island, 1999) (C. Faber, G. van der Geer, and F. Oort, eds.), Progr. Math., vol. 195, Birkhäuser, Basel, 2001, pp. 441–471.
- [21] T. Zink, *The display of a formal p -divisible group*, Astérisque (2002), no. 278, 127–248, Cohomologies p -adiques et applications arithmétiques, I.

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