Hilbert modular varieties of low dimension

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Abstract. We study in detail properties of Hilbert modular varieties of low dimension in positive characteristic $p$; in particular, the local and global properties of certain stratifications. To carry out this investigation we develop some new tools in the theory of displays, intersection theory on a singular surface and Hecke correspondences at $p$.

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1. Introduction

This paper studies Hilbert modular varieties of low dimension. Besides the interesting geometric problems it raises, we also feel that such a detailed study is bound to play a valuable role in future applications to number theory. For example, the Hilbert modular varieties of dimension one are the modular curves that have been studied extensively, and their geometric properties are intimately connected with the theory of modular forms. We consider here mainly the case of dimension 2 and 3.

To carry out this study we had to develop further existing tools and these results are of independent interest. One is intersection theory on a surface with isolated normal singularities, developed in § 7; the other is methods to calculate the universal display of a PEL problem. Regarding the latter, some of the details will appear, under a much more general setting, in a future work [AG4].

Let $L$ be a totally real field of degree $g$ over $\mathbb{Q}$, let $\mathcal{O}_L$ be its ring of integers, let $p$ be a rational prime and let $\mathfrak{M}$ be the moduli space parameterizing abelian varieties of dimension $g$, in characteristic $p$, endowed with an action of $\mathcal{O}_L$. Some further conditions are imposed - see § 2. The properties of $\mathfrak{M}$ that we study are mostly defined using the Frobenius morphism on various objects that are $\mathcal{O}_L \otimes \mathbb{F}_p$-modules. For example, the Hodge bundle $E$ and the cohomology group $H^1(A, \mathcal{O}_A)$ of an abelian variety $A$. Hence, the analysis is divided according to the prime decomposition of $p$ in $\mathcal{O}_L$. In § 3 we recall the stratifications defined in [AG1, GO] and their main properties.

In § 4 we discuss the singularities of Hilbert modular varieties. We recall the theory of local models, introduced by Deligne-Pappas [DP], de Jong [deJ] and Rapoport-Zink [RZ], and illustrate the results for the Hilbert and Siegel moduli varieties. The singularities in the Hilbert case are local complete intersections.
Given a closed point \( x \in \mathfrak{M}^{\text{sing}} \) we determine when is \( \hat{O}_{\mathfrak{M},x} \) parafactorial. A question of interest here is when the pair \( (\mathfrak{M}, \mathfrak{M}^{\text{sing}}) \) is parafactorial. This is motivated by the question of whether certain automorphic line bundles, initially defined on the non-singular locus in \( \mathfrak{M} \), actually extend to \( \mathfrak{M} \). We show that the parafactorial property usually does not hold for \( g < 4 \) by exploiting the explicit description of the completed local rings. It seems likely to us that \( (\mathfrak{M}, \mathfrak{M}^{\text{sing}}) \) will not be parafactorial in the presence of ramification. See Proposition 4.4.2, and the following discussion, for details.

Section 5 discusses the display of an abelian variety with real multiplication. After some preparatory work, we provide two main theorems. The first, Theorem 5.6.1, gives the universal display with real multiplication. It uses Theorem 5.6.2 that provides a criterion for a display to be universal. Both theorems can be generalized considerably, i.e., to the setting of PEL problems, (hopefully) even with level involving \( p \). Details will appear in [AG4]. The results are applied in the sequel to study the local properties of the strata. See, for instance, § 8.3.1 and § 9.

In § 6 we provide some general results concerning our stratification in the maximally ramified case. This continues our investigation in [AG1]. Some of our results are the following: In § 6.1 we show that \( W_{(j,j)} \) is equal to the central leaf, as defined by Oort [Oo2], through each of its points and the strata \( W_{(j,n)} \) are quasi-affine. In § 6.2 we show that certain of the strata \( T_a \), i.e., where the \( a \)-number is greater or equal to \( a \), are connected. In § 6.3 we show (a striking result) that the non-ordinary locus is irreducible for \( g \geq 3 \).

Section 7 develops intersection theory on a complete surface with isolated normal singularities, building on [RT1, RT2]. Our approach is very concrete and suitable for the calculations we need to perform. This approach can be developed further [Arc]. One of the applications we give is determining in Theorem 8.1.1, for \( p \) inert, which automorphic line bundles (yielding Hilbert modular forms, usually of non-parallel weight) are ample.

Finally, in § 9 we study in some detail Hilbert modular threefolds in the maximally ramified case.

### 2. Definitions and notations

Let \( L \) be a totally real field of degree \( g \) over \( \mathbb{Q} \) with ring of integers \( \mathcal{O}_L \). Let \( D_L \) be its different ideal and \( d_L \) its discriminant. Let \( p \) be a rational prime and \( \mathfrak{p} \) a prime of \( \mathcal{O}_L \) dividing \( p \). We let \( \mathbb{F}_p \) denote the residue field \( \mathcal{O}_L/\mathfrak{p} \). Let \( a_1, \ldots, a_h \) be ideals of \( L \) forming a complete set of representatives for the strict class group \( \text{cl}^+(L) \) of \( L \). By an abelian variety with RM we shall mean a triple \( (A \to S, \iota, \lambda) \) consisting of an abelian scheme \( A \) of relative dimension \( g \) over a scheme \( S \), where \( S \) is an \( \mathcal{O}_L \)-scheme; an embeddings of rings \( \iota : \mathcal{O}_L \hookrightarrow \text{End}_S(A) \); an isomorphism of \( \mathcal{O}_L \)-modules with a notion of positivity \( \lambda : a_i \to M_A := \text{Hom}_{\mathcal{O}_L}(A, A'^{\text{symm}}) \), where \( A' \) is the dual abelian variety (for some, necessarily unique, \( i \)). One imposes the
The following notation is used: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, denote the integers, rationals, $p$-adic integers and $p$-adic numbers; $\mathbb{W}(k)$ denotes the ring of infinite Witt vectors, with respect to a prime $p$. $\mathbb{W}$ belongs to $[\text{AG1, AG2, DP}]$ for details. $\mathbb{O}_L$ denotes the truncated Witt vectors (resp. $\mathbb{W}_l(A)$) denote the vectors in $\mathbb{W}(k)$ (resp. $\mathbb{W}_l(k)$) all whose coordinates belong to $A$. We denote by $\mathbb{F}_p, \mathbb{V}_w$ the Frobenius and Verschiebung maps on $\mathbb{W}(k)$. For a Dedekind ring $R$ and a prime ideal $\mathfrak{p}$, we let $f_\mathfrak{p} = \dim_{\mathfrak{p}}(R/\mathfrak{p})$. In the case of $\mathbb{O}_L$, we also let $e_p$ be the absolute ramification index of $\mathfrak{p}$ and we define $g_\mathfrak{p} = e_p f_\mathfrak{p}$. For a prime $\mathfrak{p}|\mathfrak{p}$ of $\mathbb{O}_L$, we let $\mathbb{O}_{L,\mathfrak{p}}$ be the localization of $\mathbb{O}_L$ at the multiplicative set $\mathbb{O}_L \setminus \mathfrak{p}$, we let $\hat{\mathbb{O}}_{L,\mathfrak{p}}$ be the completion, $L_\mathfrak{p}$ its field of fractions, and $\hat{\mathbb{O}}_{L,\mathfrak{p}}^{ur}$ be the ring of integers of the maximal unramified sub-extension of $L_\mathfrak{p}$ over $\mathbb{Q}_p$.

Let $k$ be a perfect field of characteristic $p$. A $p$-divisible group over $k$ is called ordinary if all its slopes are zero and one. An abelian variety over $k$ is called ordinary if its $p$-divisible group $A(p)$ is; it is called supersingular if the slopes of its Newton polygon are all equal $1/2$, equivalently, if it is isogenous to a product of supersingular elliptic curves; it is called superspecial if it is isomorphic over $k$ to a product of supersingular elliptic curves. We denote by $\mathcal{C}_k$ the category of local artinian $k$-algebras $(R, m)$ equipped with an identification $R/\mathfrak{m} = k$. We denote the closure of a set $Z$ in a topological space by $Z^c$.

3. Stratification of Hilbert modular varieties

We shall be concerned primarily with the geometry of the moduli space $\mathfrak{M}$. The moduli space $\mathfrak{M}$ will provide us with a ‘Hecke correspondence’ at $p$ that we shall utilize to study certain strata in $\mathfrak{M}$. Two particular cases will be considered in detail: when $p$ is unramified and when $p$ is maximally ramified.

3.1. $p$ unramified.

In this case $\mathbb{O}_L \otimes \mathbb{F}_p \cong \oplus_{\mathfrak{p}|p} \mathbb{F}_p[\mathfrak{p}]$ is a sum of fields. Let $A/k$ be an RM abelian variety over a perfect field $k \supseteq \mathbb{F}_p$. It is known that $H^1(A, \mathbb{O}_A)$ is a free $\mathbb{O}_L \otimes \mathbb{Z}$ $k$- or rank 1. The kernel of Frobenius $F : H^1(A, \mathbb{O}_A) \to H^1(A, \mathbb{O}_A)$ is a $k$-subspace of dimension $a = a(A)$. Let us assume that for every $\mathfrak{p}|p$ an embedding $\mathbb{F}_p \hookrightarrow k$ is
given. The action of $\mathbb{F}_p$ on every $\mathcal{O}_L$-eigenspace of $H^1(A, \mathcal{O}_A)$ is either trivial or is given by $x$ acting as multiplication by $x^p$ for some $1 \leq i \leq f_p$. The structure of the $\mathcal{O}_L \otimes k$-module $\operatorname{Ker}(F : H^1(A, \mathcal{O}_A) \rightarrow H^1(A, \mathcal{O}_A))$ is therefore uniquely determined by a vector $(\tau_p)_{|\mathcal{O}_L} = (\tau_p)_{|\mathcal{O}_p}(A)$ of sets, with $\tau_p \subset \{1, \ldots, f_p\}$. There is a natural partial order, induced from inclusion of sets in each component, on the set of possible vectors $(\tau_p)_{|\mathcal{O}_L}$.

Given any vector $(\tau_p)_{|\mathcal{O}_L}$, where each $\tau_p \subset \{1, \ldots, f_p\}$, we can define a closed subset $D_{(\tau_p)_{|\mathcal{O}_L}}$ of $\mathcal{M}$ by the property that for each geometric point $x \in D_{(\tau_p)_{|\mathcal{O}_L}}$ we have $(\tau_p)_{|\mathcal{O}_L}(A_x) \geq (\tau_p)_{|\mathcal{O}_L}$. This is a regular subvariety of codimension $\sum_{p|\mathcal{O}_L} \tau_p$. For further properties see [Gor, GO].

Example 3.1.1. For $g = 1$ (so $L = \mathbb{Q}$) the vector $(\tau_p)_{|\mathcal{O}_L}(A)$ has a single component and there are only two possibilities. Either $(\tau_p)_{|\mathcal{O}_L}(A) = (\emptyset)$, which corresponds to $A$ being an ordinary elliptic curve, or $(\tau_p)_{|\mathcal{O}_L}(A) = (\{1\})$, which corresponds to $A$ being supersingular. The locus $D_{(\emptyset)}$ is the whole moduli space (of codimension 0), and the locus $D_{(\{1\})}$ is the supersingular locus (of codimension 1).

Example 3.1.2. For $g = 2$ ($L$ is a real quadratic field) we have two cases:

- $p$ is inert in $L$. In this case the possibilities for $(\tau_p)_{|\mathcal{O}_L}(A)$ are the vectors of sets $(\emptyset), (\{1\}), (\{2\}), (\{1, 2\})$. The case $(\emptyset)$ corresponds to ordinary abelian surfaces, the cases $(\{1\}), (\{2\})$ to supersingular, but not superspecial abelian surfaces, and the case $(\{1, 2\})$ to superspecial abelian surfaces. The variety $D_{(\emptyset)}$ is the whole moduli space, the varieties $D_1 = D_{(\{1\})}, D_2 = D_{(\{2\})}$ are (usually reducible) divisors, and $D_{(\{1, 2\})} = D_1 \setminus D_2$ is the finite set of superspecial points. We also know that each $D_i$ is a disjoint union of non-singular rational curves and that $D_1$ and $D_2$ intersect transversely. See Figure 3.1 and [BG] for details.

- $p$ is split in $L$. In this case the possibilities for $(\tau_p)_{|\mathcal{O}_L}(A)$ are $(\emptyset, \emptyset), (\emptyset, \{1\}), (\{1\}, \emptyset)$, and $(\{1\}, \{1\})$. The case $(\emptyset, \emptyset)$ corresponds to ordinary abelian surfaces, the cases $(\emptyset, \{1\})$ and $(\{1\}, \emptyset)$ to non-ordinary (but not supersingular) abelian surfaces (they are in fact simple abelian surfaces), and the case $(\{1\}, \{1\})$ to superspecial abelian surfaces.

In this case, the divisors $D_1 = D_{(\emptyset, \{1\})}, D_2 = D_{(\{1\}, \emptyset)}$ are also each a disjoint union of non-singular curves but, in contrast with the situation of inert prime, we have no real information on these curves: they are not the reduction of Shimura curves, we do not know their genera. We do know, however, that $D_1$ and $D_2$ intersect transversely and that $D_1 \cup D_2 = D_{(\{1\}, \{1\})}$ is precisely the set of superspecial points, and in §8.2 we provide an argument that suggests that the components of the $D_i$ have usually genus 2.

3.2. $p$ maximally ramified.

Let $k \supseteq \mathbb{F}_p$ be a field. In this case $\mathcal{O}_L \otimes k \cong k[T]/(T^g)$, where $T$ may be chosen to be an Eisenstein element of the discrete valuation ring $\mathcal{O}_L \otimes \mathbb{Z}_p$. It is known that
$H^1_{dR}(A/k)$ is a free $k[T]/(T^g)$-module of rank 2. We have a sequence of $k[T]/(T^g)$ modules

$$0 \to H^0(A, \Omega^1_{A/k}) \to H^1_{dR}(A/k) \to H^1(A, \mathcal{O}_A) \to 0.$$

We let $i = i(A), j = j(A)$ be the elementary divisors of $H^0(A, \Omega^1_{A/k})$, normalized so that $j \leq i$. Note that $j = g - i$. Thus, there is a $k[T]/(T^g)$-basis $\alpha, \beta$ to $H^1_{dR}(A/k)$ such that

$$H^0(A, \Omega^1_{A/k}) \cong (T^i)\alpha \oplus (T^j)\beta.$$

An easy calculation shows that $a(A) \geq 2j$ and we let $n := n(A) = a(A) - j(A)$. Then $j \leq n \leq g - j$.

We let $J = \{ (j, n) \in \mathbb{Z}^2 : 0 \leq j \leq n \leq g - j \}$. For every $(j, n) \in J$ one proves [AG1] that there is a locally closed subvariety $W_{(j, n)}$ of $\mathfrak{M}$, whose geometric points parameterize the abelian varieties $A$ with $\text{RM}$ such that $(j(A), n(A)) = (j, n)$. We know [AG1] that $W_{(j, n)}$ is pure dimensional, non-singular of dimension $g - (j + n)$, that the Newton polygon is constant on $W_{(j, n)}$, consisting of two slopes $(n/g, (g - n)/g)$ with equal multiplicities (unless $n \geq g/2$ and then the Newton polygon has one slope equal to $1/2$), and that the collection $\{ W_{(j, n)} : (j, n) \in J \}$ is a stratification of $\mathfrak{M}$. The description of the order defined by “being in the closure” is complicated to write, but is easy to describe pictorially. We provide the graphs for $g = 1, 2, 3, 4$ and 8 in Diagram A. The convention is that if a point $a$ is above a point $b$ in the graph, and $a$ is connected to $b$ by a strictly decreasing path, then the strata corresponding to $a$ is in the closure of the strata corresponding to $b$. 

Figure 3.1: Hilbert modular surface - inert case.
We know that $W^c_{(1,1)} = \cup_{(j,n), j \geq 1} W_{(j,n)}$ is the singular locus of $\mathcal{M}$, and, in a sense, $j$ is a measure for severity of the singularities [DP]. More precisely, put $S_j := W^c_{(j,j)} = \cup_{(s,t), s \geq j} W_{(s,t)}$, then,

$$S_{j+1} = S_j^{\text{sing}}.$$  \quad (3.1)

We provide a diagram for the case $g = 2$; See Figure 3.2. The lower part of the diagram depicts the modular surface $\mathcal{M}$ with a description of the local structure around a point of type $(1,1)$. The completion of the local ring is a cone, and the supersingular locus, equal to $W^c_{(0,1)}$, has $p + 1$ branches at such a point.

One of the main tools used in [AG1] is the correspondence defined by the moduli space $\mathcal{M}$ and its two projections $\pi_1, \pi_2$ to $\mathcal{M}$. In fact, over $W^c_{(1,1)}$ the morphisms $\pi_i$ are $\mathbb{P}^1$-bundles. We can trace the invariants of the image $\pi_2 \pi_1^{-1}(x)$ of a point $x$ of type $(j, n)$ under this correspondence. Again, the formal description is cumbersome and we content ourselves with providing Diagram B, referring the reader to [AG1] for more details. The convention is that the invariants along $\pi_2 \pi_1^{-1}(x)$ of a point $x$ of type $(j, n)$ are the couples $(j', n')$, connected to, and in distance one from $(j, n)$ (whether above of below; a loop is considered distance 1).
Figure 3.2: Hilbert modular surface - ramified case.

Diagram B:
4. Background on the singularities of Hilbert modular varieties

4.1. Cusps.

Let $\mathfrak{A} \to \mathfrak{M}$ be the universal abelian scheme with RM and let $e : \mathfrak{M} \to \mathfrak{A}$ be the identity section. The Hodge bundle $E$ is the rank $g$ vector bundle over $\mathfrak{M}$ defined by $e^*\Omega^1_{\mathfrak{A}/\mathfrak{M}}$. Let $\omega = \det E$; it is an ample line bundle on $\mathfrak{M}$. The Satake compactification $\mathfrak{M}^S$ of $\mathfrak{M}$ is defined as $\text{Proj}(\oplus_{n=1}^\infty \Gamma(\mathfrak{M}, \omega^n))$; it is a projective normal variety and $\mathfrak{M}^S \setminus \mathfrak{M}$ is a finite set of points, called cusps. Though $\omega$ extends to the Satake compactification, we do not know if the Hodge bundle itself extends.

The set $\mathfrak{M}^R = \mathfrak{M} \setminus \mathfrak{M}^{\text{sing}}$ is the largest open set over which the Hodge bundle is a locally free $\mathcal{O}_L \otimes \mathcal{O}_{\mathfrak{M}^{\text{et}}}^n$-module. One has $\mathfrak{M}^R = \mathfrak{M}$ if and only if $p$ is unramified [DP]. Let $k$ be a big enough finite field so that $\mathcal{O}_L \otimes k$ is a direct sum of local artinian rings with residue field $k$. Let $I \subset \mathcal{O}_L \otimes k$ be an ideal and let $E_I = "E/I\mathcal{E}"$ be the quotient bundle corresponding to $I$, defined over $\mathfrak{M}^R \otimes k$. We do not know if $E_I$ extends to the cusps.

Example 4.1.1. 1. If $p = p_1 \cdots p_g$ is a product of split primes, then the Hodge bundle is a direct sum $E = E_{p_1} \oplus \cdots \oplus E_{p_g}$ of line bundles over $\mathfrak{M}$. Since we shall refer to that case later, we introduce the simpler notation $E = L_1 \oplus \cdots \oplus L_g$.

2. If $p$ is an inert prime in $\mathcal{O}_L$ then $\mathcal{O}_L \otimes F_p = \bigoplus_{i=1}^g F_{p_i}$, and the Hodge bundle is again a direct sum of line bundles $E = L_1 \oplus \cdots \oplus L_g$ over $\mathfrak{M}$.
3. If \( p = p \) is maximally ramified, we get a quotient line bundle \( L \) of \( E \) defined over \( \mathcal{M}^R \). We remark that in this case the complement of \( \mathcal{M}^R \) is of codimension 2 in \( \mathcal{M} \) [DP] and it is not a priori clear whether \( L \) can be extended to a line bundle on \( \mathcal{M} \). We shall discuss this problem in §4.4.

4.2. Local models.

Many of the results we stated above require a detailed understanding of the local (infinitesimal) structure of the moduli space \( \mathcal{M} \). Such information may be obtained by the technique of local models. The theory of local models constructs for a moduli space \( \mathcal{B} \) of abelian varieties another scheme \( \mathcal{B}^{\text{loc}} \), typically a flag variety, such that for every geometric point \( x \in \mathcal{B} \), there exists a geometric point \( y \in \mathcal{B}^{\text{loc}} \) and an isomorphism of completed local rings

\[
\hat{\mathcal{O}}_{\mathcal{B},x} \cong \hat{\mathcal{O}}_{\mathcal{B}^{\text{loc}},y}.
\]

Below, we shall use the following notation for Grassmann varieties. Let \( k \) be an algebraically closed field, \( B \) be a \( k \)-algebra, and let \( a < b \) be positive integers. Assume that a ring homomorphism \( B \twoheadrightarrow M_b(k) \) is given. Assume also that a bilinear pairing \( \langle \cdot, \cdot \rangle \) on \( k^b \) is given. We shall use \( \text{Grass}(a,b) \) (resp. \( \text{Grass}(\langle \cdot, \cdot \rangle, a, b) \); resp. \( \text{Grass}(B, \langle \cdot, \cdot \rangle, a, b) \)) to denote the Grassmannian of \( a \)-dimensional subspaces of \( k^b \) (resp. isotropic; resp. isotropic and \( B \)-invariant). Often implicit in the notation \( \text{Grass}(B, \langle \cdot, \cdot \rangle, a, b) \) is a connection between the pairing and the action of \( B \), e.g., the elements of \( B \) are self-adjoint with respect to the pairing.

4.2.1. The idea of local models. Let \( \mathfrak{A}_g \) be the moduli space of principally polarized abelian varieties of dimension \( g \) in characteristic \( p \). We shall assume that on \( \mathfrak{A}_g \), or \( \mathcal{M} \), there is a given rigid prime to \( p \) level structure, which we omit from the notation. Given a point \( x \in \mathcal{M} \), or \( x \in \mathfrak{A}_g \), one can trivialize the locally free sheaf \( H^1_{dR} \) in a Zariski open neighborhood \( U \) of \( x \). Then, the locally free, locally direct summand of rank \( g \) given by the Hodge bundle \( E \), provides a morphism \( U \to \text{Grass}(\langle \cdot, \cdot \rangle, g, 2g) \) \( \text{Grass}(\mathcal{O}_L, \langle \cdot, \cdot \rangle, g, 2g) \), where the Grassmannian is of isotropic \( g \)-dimensional (and \( \mathcal{O}_L \)-invariant) subspaces of a \( 2g \)-dimensional space with a perfect alternating pairing. The idea of local models is to show that this is an isomorphism on the level of completed local rings. There is a shortcoming to this result in that the morphism is not canonical and therefore it is not a priori clear how to define the strata coming from the moduli space on the local model (even in the infinitesimal neighborhood of a point). The crystalline theory makes this morphism somewhat more canonical. But, in fact, the proof that this is an isomorphism on the completed local rings requires an auxiliary scheme and a dimension count.

Let \( f : A \to S \) be an abelian scheme and let \( \mathcal{D}(A) \) be the associated Grothendieck-Messing crystal, defined on the nilpotent crystalline site of \( S \). This crystal is
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deﬁned by

\[ D(A) = R^1f_{\text{crys}}(O_{A,\text{crys}}). \]

The value of this crystal on \( S \) is the de Rham sheaf \( D(A)_S = R^1f_*\langle \Omega^*_{A/S} \rangle \), hence it provides us with a locally free direct summand of rank \( g \), \( E_A \subset D(A)_S \), which is \( f_*\Omega_{A/S} \). The crucial theorem here is due to Grothendieck (cf. [deJ]):

**Theorem 4.2.1.** Let \( S \hookrightarrow S' \) be a nilpotent thickening with a divided powers structure. The ﬁltered Dieudonné functor gives an equivalence of categories between

1. the category of abelian schemes over \( S' \), and
2. the category of pairs \((A, E)\), where \( A \) is an abelian scheme over \( S \) and \( E \subset D(A)_S \) is a locally free direct summand which lifts \( E_A \subset D(A)_S \).

Let \( S \) be the spectrum of an algebraically closed ﬁeld \( k \). Let \( S \subset S' \) be a PD thickening such that \( S' \) is a local artinian \( k \)-algebra. Let \( A' \to S' \) be the deformation of \( A \) over \( S' \) for which \( D(A)_{S'} = H^1_{\text{dR}}(A/S) \otimes_k O_{S'} \). Then, given any other deformation \( A'' \) of \( A \) to \( S' \), the canonical isomorphism \( H^1_{\text{dR}}(A''/S') \cong H^1_{\text{dR}}(A/k) \otimes_k O_{S'} \) provides us with a submodule \( E_{A''} \subset H^1_{\text{dR}}(A) \otimes_k O_{S'} \) lifting \( E_A \subset H^1_{\text{dR}}(A) \). Thus we get a morphism from the functor of deformations over the nilpotent crystalline site of \( S \) to the functor \( \text{Grass}(g, H^1_{\text{dR}}(A/k)) \).

Let \( T \) be the spectrum of \( \mathcal{O}_{\mathbb{R},x} \). Trivialize \( R^1f_*\langle \Omega^*_{A_{\text{uni}}}/T \rangle \cong \hat{\mathcal{O}}^2_{\mathbb{R},x} \) with respect to a basis horizontal for the Gauss–Manin connection. Considering the submodule \( E_{\mathbb{R},x}/T \subset R^1f_*\langle \Omega^*_{A_{\text{uni}}}/T \rangle \), we obtain a morphism \( T \to \text{Grass}(\mathcal{O}_L \otimes k, (, ), g, 2g) \). Similar constructions can be made with endomorphism and polarization structures. Using this map and the crystalline theory, one obtains [DP, deJ] the following theorem (recall the tacit assumption of rigid level structure):

**Theorem 4.2.2.**

1. In the Siegel case, there is an isomorphism

\[ \hat{O}_{A, x} \cong \hat{O}_{G, y}, \]

where \( G \) is the Grassmannian variety \( \text{Grass}(, , g, 2g) \) that parameterizes \( g \)-dimensional isotropic subspaces of \( H^1_{\text{dR}}(A/k) \) and \( y \) is the point corresponding to the ﬁltration \( H^0(A, \Omega^1_{A/k}) \subset H^1_{\text{dR}}(A/k) \).

2. In the Hilbert case, there is an isomorphism

\[ \hat{O}_{M, x} \cong \hat{O}_{G, y}, \]

where \( G \) is the Grassmannian variety \( \text{Grass}((, , , ), g, 2g) \) that parameterizes \( g \)-dimensional isotropic \( \mathcal{O}_L \)-invariant subspaces of \( H^1_{\text{dR}}(A/k) \) and \( y \) is the point corresponding to the ﬁltration \( H^0(A, \Omega^1_{A/k}) \subset H^1_{\text{dR}}(A/k) \).

**4.3. Examples.**

We only consider deformations in characteristic \( p \).
4.3.1. The Siegel case. Let $V$ be a $2g$-dimensional vector space, let $\Gamma \subset V$ be a $g$-dimensional subspace of $V$ and choose a complementary subspace $W \subset V$ such that $V = \Gamma \oplus W$. Then an affine chart of Grass$(g, V)$ about $\Gamma$ is given by $\text{Hom}(\Gamma, W)$. Given $t \in \text{Hom}(\Gamma, W)$ we associate to it its graph.

Suppose that $V$ has a symplectic pairing and $\Gamma$ is isotropic. Choose a basis $a_1, \ldots, a_g$ to $\Gamma$ and complete it to a standard symplectic basis by $b_1, \ldots, b_g$. Take $W$ to be the span of $b_1, \ldots, b_g$. We may identify $t$ with a $g \times g$ matrix $(t_{i,j})$ such that

$$a_j \mapsto a_j + \sum_i t_{i,j} b_i.$$  

The graph of $t$ is isotropic if and only if for each $j, k$

$$\left(a_j + \sum_i t_{i,j} b_i\right) \land \left(a_k + \sum_i t_{i,k} b_i\right) = 0.$$  

(4.1)

Since $\left(a_j + \sum_i t_{i,j} b_i\right) \land \left(a_k + \sum_i t_{i,k} b_i\right) = t_{j,k} - t_{k,j}$, Equation (4.1) is equivalent to $(t_{i,j})$ being a symmetric matrix. This is of course in accord with $\mathfrak{A}_{g}$ (with a rigid level structure prime to $p$) being a non-singular variety of dimension $g(g+1)/2$.

4.3.2. The Hilbert case. We again consider two cases.

- The inert case. In this case we have a decomposition

$$H^1_{dR}(A/k) = \oplus_{i=1}^g D(i),$$

where each $D(i)$ is a two dimensional $k$-vector space with a perfect alternating pairing, on which $\mathcal{O}_L$ acts via $\sigma_i$. There is a compatible decomposition

$$H^0(A, \Omega_{A/k}) = \oplus_{i=1}^g H(i),$$

where each $H(i)$ is a one dimensional $k$-vector space on which $\mathcal{O}_L$ acts via $\sigma_i$. The Grassmannian is therefore isomorphic to

$$\text{Grass}(1, 2)^g \cong (\mathcal{P}_k^1)^g.$$  

Note that the completed local ring of every point $x$ on $\mathfrak{M}$ is isomorphic to the completed power series ring $k[[t_1, \ldots, t_g]]$, where $t_i$ is canonical up to an element of $k[[t_i]]^\times$.

- The maximally ramified case. In this case

$$H^1_{dR}(A/k) \cong k[T]/(T^g) \oplus k[T]/(T^g).$$

The Grassmannian Grass($\mathcal{O}_L \otimes k, \langle, , \rangle, g, 2g$) is that of parameterizing isotropic $g$-dimensional subspaces that are $\mathcal{O}_L$-invariant. One can show $[\text{DP}]$ that one can replace the $k$-valued pairing, for which the action of $\mathcal{O}_L$ is self-adjoint, by a $k[T]/(T^g)$-valued pairing, which is $k[T]/(T^g)$-linear.

Given $A/k$ we can find a basis $\alpha, \beta$ of $H^1_{dR}(A/k)$ such that

$$H^0(A, \Omega_{A/k}) = (T^\beta) \alpha \oplus (T^\alpha) \beta, \quad \alpha \land \beta = 1.$$
where $j = j(A)$, $i = g - j$. We choose the complementary subspace to be
$$
\bigoplus_{s=0}^{j-1} T^s \alpha \oplus \bigoplus_{s=0}^{i-1} T^s \beta.
$$

The deformations $f$ that are $O_L$-linear are determined by

$$
T^s \alpha \mapsto T^s \alpha + \sum_{s=0}^{j-1} a_s T^s \alpha + \sum_{s=0}^{i-1} b_s T^s \beta, \quad T^j \beta \mapsto T^j \beta + \sum_{s=0}^{j-1} c_s T^s \alpha + \sum_{s=0}^{i-1} d_s T^s \beta.
$$

We write that in shorthand notation as

$$
T^i \alpha \mapsto T^i \alpha + a_i + b_i \beta, \quad T^j \beta \mapsto T^j \beta + c_j + d_j \beta.
$$

The extra condition is

$$
(T^i \alpha + a_i + b_i \beta, T^j \beta + c_j + d_j \beta) = 0.
$$

This is equivalent to

$$
ad - bc + a T^j + d T^i = 0,
$$

with

$$
a = \sum_{s=0}^{j-1} a_s T^s, \quad b = \sum_{s=0}^{i-1} b_s T^s,
$$
$$
c = \sum_{s=0}^{i-1} c_s T^s, \quad d = \sum_{s=0}^{j-1} d_s T^s.
$$

**Example 4.3.1.** $j = 0$ (non-singular points). In this case $i = g$. We get immediately $b = d = 0$ and hence also $a = 0$. It follows that $c = \sum_{s=0}^{g-1} c_s T^s$ is unobstructed and we conclude that the completed local ring is isomorphic to

$$
k[c_0, \ldots, c_{g-1}].
$$

**Example 4.3.2.** $g = 2, i = j = 1$. In this case we find the equation

$$
a_0 d_0 - b_0 c_0 + a_0 T + d_0 T = 0.
$$

We get the relations $a_0 = -d_0$ and $a_0 d_0 - b_0 c_0 = 0$. This gives that the completed local ring is isomorphic to

$$
k[\{a_0, b_0, c_0\}]/(a_0^2 + b_0 c_0).
$$

**Example 4.3.3.** $g = 3, j = 1, i = 2$. We have

$$
a = a_0 + a_1 T, \quad b = b_0
$$
$$
c = c_0 + c_1 T, \quad d = d_0.$$
with the equation

\[(a_0 d_0 - b_0 c_0) + (a_0 + a_1 d_0 - b_0 c_1)T + (a_1 + d_0)T^2 = 0.\]

This yields \(d_0 = -a_1\), \(a_0 = a_1^2 + b_0 c_1\) and that the completed local ring \(R\) is isomorphic to

\[k[a_1, b_0, c_0, c_1]/(a_1^2 + a_1 b_0 c_1 + b_0 c_0),\]

which is 3-dimensional with a tangent cone at the origin defined by \(b_0 c_0 = 0\).

The singular locus of \(\text{Spec}(R)\) is given by \(b_0 = c_0 = 0\) (which implies \(a_1 = 0\)) and is hence one dimensional, isomorphic to \(\text{Spf}(k[c_0])\).

### 4.4. Singular points.

Using the local models one can show [DP] that \(\mathcal{M}\) is singular if and only if \(p\) is ramified in \(O_L\) and that the singular locus is of codimension 2. However, the singularities are local complete intersections, hence Cohen-Macaulay and so normal, by Serre’s criterion. We remark that, in particular, the completed local rings are domains, i.e., the moduli space is locally (formally) irreducible.

Now, in local commutative algebra a property which is subtle and of interest is the property of parafactoriality. The definition is motivated by its relation to factoriality and representability of the local Picard functor. For this we refer the interested reader to the references below and to [Lip]. A noetherian local ring \((R, m)\) is parafactorial if it is of depth at least 2 and \(\text{Pic}(R - \{m\}) = 0\). A global definition follows:

**Definition 4.4.1.** Let \((X, Z)\) be a pair consisting of a ringed space \(X\) and a closed subset \(Z\). Let \(U = X \setminus Z\). One says that \((X, Z)\) is parafactorial if, for every open set \(V\) of \(X\), the restriction functor \(M \mapsto M_{U \cap V}\), from the category of invertible \(O_V\)-modules to the category of invertible \(O_{U \cap V}\)-modules, is an equivalence of categories.

We refer the reader to [EGA IV, 21.13], [SGA 2, Exp. XI] for details. In particular, [EGA IV, 21.13.8] gives the equivalence of the definitions for local rings.

**Proposition 4.4.2.** Let \(x\) be a closed point of \(\mathcal{M}\) with local ring \(R\). Assume \(g \geq 2\).

1. The ring \(R\) is parafactorial if \(x\) is a nonsingular point or if \(g \geq 4\).

2. Let \(g = 2\). The ring \(R\) is not parafactorial if and only if \(p\) is ramified in \(O_L\) and \(x\) is a singular point.

3. Let \(g = 3\). The ring \(R\) is not parafactorial if and only if \(p\) is maximally ramified in \(O_L\) and \(x\) is a singular point.

**Proof.** It is known that regular noetherian local rings of dimension at least 2 are parafactorial - a result due to Auslander-Buchsbaum - cf. [SGA 2, Thm. 3.13], [EGA IV, 21.13.9 (ii)].
Assume first \( g \geq 4 \). Let \( S_1 = \mathfrak{M}^{\text{sing}} \) and let \( s \in S_1 \) be a closed point. By [SGA 2, Exp. XI, Cor. 3.7], to show that the local ring \( \mathcal{O}_{\mathfrak{M}, s} \) is parafactorial it is enough to show that \( \hat{\mathcal{O}}_{\mathfrak{M}, s} \) is parafactorial. For \( g \geq 4 \) the ring \( \hat{\mathcal{O}}_{\mathfrak{M}, s} \) is of dimension \( 4 \) and is a complete intersection by the theory of local models (see [DP]). It follows from [SGA 2, Exp. XI, Thm. 3.13] that it is parafactorial.

We now show that if \( p \) is maximally ramified and \( g = 2, 3 \) then the pair \( (\mathfrak{M}, \mathfrak{M}^{\text{sing}}) \) is not parafactorial. We argue as follows: Suppose that it is. Consider the line bundle \( L \) over \( \mathfrak{M}^{\text{sing}} \) defined as \( pE \). We have natural inclusions \( L \rightarrow E \rightarrow H^1_{dR} \).

Then \( L \) can be extended to a line bundle over \( \mathfrak{M} \). In particular, let \( x \in \mathfrak{M}^{\text{sing}} \) be a closed point then the line bundle \( L \), initially defined on \( \text{Spf}(\hat{\mathcal{O}}_{\mathfrak{M}, x})-\text{Spf}(\hat{\mathcal{O}}_{\mathfrak{M}, x})^{\text{sing}} \), can be extended to \( \text{Spf}(\hat{\mathcal{O}}_{\mathfrak{M}, x}) \).

Now, let \( R \) be the completed local ring of a point \( y \) corresponding to \( x \) in the local model. By construction, the Hodge filtration \( E \rightarrow H^1_{dR} \) is isomorphic to the Hodge filtration \( \mathcal{E} \rightarrow \mathcal{H} \) over the local model and \( L \) corresponds to the line bundle \( \mathcal{L} = p\mathcal{E} \). All this can be made very explicit: the ring \( R \) is

\[
k[a, b, c, d]/(ad - bc + aT^i + dT^j)
\]

(see §4.3.2). The vector bundle \( \mathcal{H} \) is the free \( \mathcal{O}_L \otimes R \)-module \( R[T]/(T^g)^2 \) with basis \( \alpha, \beta \), the Hodge bundle is the free \( R \)-module spanned by the \( \mathcal{O}_L \otimes R \) module by \( T^i \alpha + a \beta, T^j \beta + c \alpha + d \beta \) (where \( i, j \) are the invariants of the point \( x \)) and the line bundle \( \mathcal{L} \) corresponds to the \( R \)-module spanned by the elements \( T^{g-1+i} \alpha + T^{g-1}a \alpha + T^{g-1}b \beta, T^{g-1+j} \beta + T^{g-1}c \alpha + T^{g-1}d \beta \), over the open set \( U \) where at least one of the generators is not zero.

Let \( x \) be a point with invariants \( 0 < j \leq i \). Let \( Z \) be the closed subscheme of \( X = \text{Spec}(R) \) defined by the ideal \((a_0, b_0, c_0, d_0)\) and let \( U = \text{Spf}(R) - Z \). Then \( \mathcal{L} \) is generated over \( U \) by \( T^{g-1}a_0 \alpha + T^{g-1}b_0 \beta, T^{g-1}c_0 \alpha + T^{g-1}d_0 \beta \).

Let \( U_{ab} \) (resp. \( U_{cd} \)) denote the open set where either \( a_0 \) or \( b_0 \) (resp. \( c_0 \) or \( d_0 \)) are not zero. We have a trivialization of \( \mathcal{L} \) over \( U_{ab} \) \((T^{g-1}a_0 \alpha + T^{g-1}b_0 \beta \) is a basis\) and \( U_{cd} \) \((T^{g-1}c_0 \alpha + T^{g-1}d_0 \beta \) is a basis\). Note that \( R - (U_{ab} \cup U_{cd}) = \mathcal{A}_1 \) is the singular locus of \( R \). Note also that on \( R \) we have the relation \( a_0 d_0 = b_0 c_0 \). The transition function between the trivializations is \( d_0/b_0 = c_0/a_0 \).

Let \( D \) be the divisor on \( X \) that is the closure of the divisor defined by \((\text{the transition function of}) \ \mathcal{L} \) on \( U \). Since \( X \) is normal and \( U \) has codimension 2, if \( \mathcal{L} \) extends as an invertible sheaf to \( X \), then it must be the sheaf \( \mathcal{O}_X(-D) \) and, in particular, \( D \) must be locally principal. We show this is not the case for \( g = 2, 3 \) and \( p \) maximally ramified.

For \( g = 2 \) the ring \( R \) is, by Example 4.3.2,

\[
k[b_0, c_0, d_0]/(d_0^2 + b_0 c_0).
\]

Since \( d_0 = -a_0 \), we find that the divisor \( D \) is defined by the ideal \((b_0, d_0)\). We remark that it generates \( \text{Cl}(R) = \mathbb{Z}/2\mathbb{Z} \) (cf. [Har II 6.5.2]). We only check it is of exact order 2. First, note that the element \( d_0 \) is a uniformizer for the local ring
of \( D \). The divisor \( 2D \) is \((b_0)\). To see that \( D \) is not principal we follow loc. cit., since \( R \) is normal, it is enough to show that the ideal \((b_0, d_0)\) is not principal.

Consider the closed point \( m = (b_0, c_0, d_0) \). Its tangent space \( m/m^2 \) is three dimensional over \( k \) with basis \( \{b_0, c_0, d_0\} \). Since the image of \((b_0, d_0)\) in \( m/m^2 \) is two dimensional, the ideal \((b_0, d_0)\) cannot be principal.

Consider now the case \( g = 3 \). Example 4.3.3 gives that the local ring at a point \( x \) with \( j = 1 \) is

\[
k[[b_0, c_0, c_1, d_0]]/(d_0^3 + b_0c_1d_0 - b_0c_0),
\]

and \( a_0 = d_0^3 + b_0c_1 \). The divisor \( D \) is defined on \( U_{ab} \) by \( b_0 \) and on \( U_{cd} \) by \( d_0 \). Note that if \( b_0 = 0 \) then \( d_0 = 0 \) by \((4.2)\) and so \( a_0 = 0 \). That is, \( D \) is the empty divisor on \( U_{ab} \). If \( d_0 = 0 \) then either \( b_0 = 0 \) or \( c_0 = 0 \) by \((4.2)\). Thus, on \( U_{cd} \) we get the divisor \((b_0, d_0)\). It defines an irreducible divisor on \( X \).

To verify \( D \) is not locally principal one argues as above, calculating its image in \( m/m^2 \), where \( m = (b_0, c_0, c_1, d_0) \). In fact, the order of \( D \) in \( Cl(X) \) is precisely 3. The divisor of the function \( b_0 \) is supported on \( D \), and since \( d_0 \) is a uniformizer for the local ring of \( D \), the identity \( b_0(c_0 - c_1d_0) = d_0^3 \) gives that \( b_0 \) vanishes to order 3 along \( D \).

It remains to consider the case \( g = 3 \) and \( p = p^2q \). In this case, the completed local ring of any non-singular point is of the form \( k[x, y, z]/(z^2 + xy) \oplus k[t] \). Such a ring is parafactorial by [Bou, III, Prop. 1.2].

Our initial motivation to study the question of parafactoriality was to determine if the vector bundles \( \mathbb{E}/p\mathbb{E} \), defined on \( \mathcal{M}^R \), extend across \( \mathcal{M} \). This suggests the question of whether \((\mathcal{M}, \mathcal{M}^{\text{sing}})\) is parafactorial. By [SGA 2, Prop. 3.3] this is the case if and only if for all points \( x \in \mathcal{M}^{\text{sing}} \) the local ring \( \mathcal{O}_{\mathcal{M},x} \) is parafactorial. Here points must be taken in the scheme theoretic sense. Namely, one can easily give examples of a pair \((X, Z)\) where the rings \( \mathcal{O}_{X,z} \) are parafactorial for \( z \) a closed point of \( Z \) and yet the pair \((X, Z)\) is not parafactorial. (Say, \( X \) is a smooth projective surface and \( Z \) and an irreducible hyperplane section. A more interesting example is obtained by taking \( X = C \times \mathbb{A}^1 \), where \( C \) is the cone \( x_1^2 + x_2x_3 = 0 \) and \( Z = \{(0, 0, 0)\} \times \mathbb{A}^1 \).

Proposition 4.4.2 above shows that “in most cases” the local rings \( \mathcal{O}_{\mathcal{M},x} \) are parafactorial for all closed points \( x \). It seems likely, though we do not have a proof, that the pair \((\mathcal{M}, \mathcal{M}^{\text{sing}})\) is not parafactorial. Our reasoning is the following:

It is easy to see that for \( x \in \mathcal{M}^{\text{sing}} \)- not necessarily a closed point - the local ring \( \mathcal{O}_{\mathcal{M},x} \) is Cohen-Macaulay and of codimension at least 2. Hence, for all \( x \in \mathcal{M}^{\text{sing}} \) the local ring \( \mathcal{O}_{\mathcal{M},x} \) is of depth \( \geq 2 \). It follows from [EGA IV, 21.13.3-4] that the restriction functor \( \mathcal{E} \mapsto \mathcal{E} \mid \mathcal{M}^R \), from the category of locally free sheaves on \( \mathcal{M} \) to the category of locally free sheaves on \( \mathcal{M}^R \), is faithfully flat.

Now, the Hodge bundle \( \mathbb{E} \) certainly extends to a locally free sheaf over \( \mathcal{M} \). To fix ideas, suppose that \( p = p^2 \) is maximally ramified and \( g > 1 \). Suppose that the line bundle \( \mathbb{L} \) on \( \mathcal{M}^R \) did extend to \( \mathcal{M} \) (as parafactoriality of the pair \((\mathcal{M}, \mathcal{M}^{\text{sing}})\) would
have implied) and denote the extension by $L$ again. Then the canonical surjective morphism $E \to L$ over $\mathcal{M}^R$ would extend to a morphism $E \to L$ over $\mathcal{M}$. This morphism must be surjective because $\mathcal{M}$ is normal and $\mathcal{M}^{\text{sing}}$ is of codimension 2. We conclude that there is a section $\mathcal{M} \to \mathcal{M}$. This is the fact we find highly unlikely (for the case $g = 2$ see Proposition 8.3.1).

Finally, we remark that it is easy to see that $L^g$ does extend to $\mathcal{M}$ (and similar statements hold in the case where $p$ in not maximally ramified). Indeed $L^g \cong \det(E)$. Note that this is in accordance with the computations above.

5. The display of an abelian variety with RM

Let $x \in \mathcal{M}$ be a $k$-valued point, where $k$ is an algebraically closed field of characteristic $p$. The theory of local models allows us to determine the ring $\mathcal{O}_{\mathcal{M},x}$, and even the behavior of the strata $S_j$, but falls short of describing the behavior of the strata $W_{(j,n)}$.

As we shall explain, the local deformation theory factors according to the prime ideals dividing $p$ in $\mathcal{O}_L$ and that allows us, essentially, to assume that the $p$-divisible group $A_x(p)$ is either ordinary, or local-local. The first case is studied very effectively using Serre-Tate coordinates but is of no interest to us in this paper. In order to study the second case, we make use of the theory of displays as reformulated and developed by Zink [Zin].

Our main idea is the following. Suppose, for simplicity, that the abelian variety $A_x$ has a local-local $p$-divisible group. Then, the display associated to the abelian scheme $A \to \text{Spec}(\mathcal{O}_{\mathcal{M},x})$, whose fiber over the closed point is $A_x$, is universal with respect to the problem of deformations over local artinian $k$-algebras $(R,m)$ with $R/m = k$ of the polarized $\mathcal{O}_L$-display associated to $A_x$. Indeed, the universality is one of Zink's main results.

On the other hand, the theory of local models provides us with a concrete model $R$ for $\mathcal{O}_{\mathcal{M},x}$, which is the completion of the local ring of a point on a suitable Grassmann variety. We view the universal display $\mathcal{P}^{\text{uni}}$ as lying over $R$. We explicitly construct a display $\mathcal{P}$ over $R$ that we want to show is universal. By the universal property, $\mathcal{P}$ is obtained from $\mathcal{P}^{\text{uni}}$ by base change coming from a unique map $\varphi: R \to R$. At least over $R/m_R^2$, the Hodge filtrations defined by $\mathcal{P}^{\text{uni}}$ and $\mathcal{P}$ produce two maps (that are unique) $\psi_1, \psi_2: R \to R$, coming from the interpretation of $R$ as a completed local ring of a point on a Grassmannian, and the crystalline nature of displays. One gets a commutative diagram $\varphi \circ \psi_1 = \psi_2$. We then argue that, in fact, $\psi_1$ and $\psi_2$ are isomorphism, hence so is $\varphi$. The universality of $\mathcal{P}$ ensues.
5.1. Recall.

In this section we review the theory of displays, developed in [Zin], discussing a variant where a real multiplication is considered. For details we refer to [AG1, §4]. Having in mind applications to local models, we recall the connection between displays and crystals as developed in [Zin].

5.1.1. Let $R$ be an $\mathbb{F}_p$-algebra. Let $\mathcal{W}(R)$ be the Witt vectors over $R$ and let $I_R$ be the kernel of the ring homomorphism $\mathcal{W}(R) \to R$ given by projection on the first coordinate.

A polarized display $\mathcal{P}$ over $R$ with real multiplication by $\mathcal{O}_L$, a RM display for short, is a quintuple $(P, Q, V^{-1}, F, \langle \cdot, \cdot \rangle)$ consisting of:

1. a projective $\mathcal{O}_L \otimes \mathcal{W}(R)$-module $P$ of rank 2;
2. a finitely generated $\mathcal{O}_L \otimes \mathcal{W}(R)$-submodule $Q$ of $P$ such that $I_R P \subset Q \subset P$ and $P/Q$ is a direct summand of the $R$-module $P/I_R P$;
3. additive maps $F: P \to P$ and $V^{-1}: Q \to P$, which are linear with respect to $\mathcal{O}_L$ and $\sigma$-linear with respect to $\mathcal{W}(R)$, and satisfy $V^{-1}(V^{-1}(y)) = wF(y)$ for any $w \in \mathcal{W}(R)$ and any $y \in P$. One imposes a further nilpotence condition [Zin, Def. 2];
4. an $\mathcal{O}_L \otimes \mathcal{W}(R)$-bilinear map $\langle \cdot, \cdot \rangle: P \times P \to D^{-1}_L \otimes \mathcal{W}(R)$ satisfying the identity $V^{-1}(x, y) = \langle x, y \rangle$ for every $x$ and $y$ in $Q$.

Define

$$D_{\mathcal{P}} := P/I_R P, \quad H_{\mathcal{P}} := Q/I_R P.$$ 

The filtration $H_{\mathcal{P}} \subset D_{\mathcal{P}}$ is called the Hodge filtration of $\mathcal{P}$.

Replacing $\mathcal{O}_L$ with $\mathcal{O}_{L,p}$ (or its completion $\mathcal{O}_{L,p}$) and $D_L$ with its localization at $p$ (resp. completion at $p$), one gets the notion of a polarized display with $\mathcal{O}_{L,p}$-action (resp., $\mathcal{O}_{L,p}$-action).

5.1.2. The following is a consequence of [Zin, Thm. 44]. Let $S \to R$ be a surjective homomorphism of rings such that $p$ is nilpotent in $S$ and its kernel $\mathfrak{a}$ is equipped with divided powers. Let $\mathcal{P} := (P, Q, V^{-1}, F, \langle \cdot, \cdot \rangle)$ be a RM display (or a polarized display with $\mathcal{O}_{L,p}$-action) over $R$. Let $\mathcal{P}_1 = (P_1, Q_1, V^{-1}_1, F_1, \langle \cdot, \cdot \rangle_1)$ and $\mathcal{P}_2 = (P_2, Q_2, V^{-1}_2, F_2, \langle \cdot, \cdot \rangle_2)$ be RM displays (or a polarized displays with $\mathcal{O}_{L,p}$-action) over $S$ reducing to $\mathcal{P}$. Let $\tilde{Q}_1$ (resp. $\tilde{Q}_2$) be the inverse image of $Q$ via $P_1 \to P$ (resp. $P_2 \to P$). Then, $V^{-1}_1$ (resp. $V^{-1}_2$) extends uniquely to $\tilde{Q}_1$ (resp. $\tilde{Q}_2$). The theorem states that there is a unique isomorphism,

$$\alpha: (P_1, \tilde{Q}_1, V^{-1}_1, F_1) \to (P_2, \tilde{Q}_2, V^{-1}_2, F_2),$$

reducing to the identity on $\mathcal{P}$ and commuting with the $\mathcal{O}_L$-action (or $\mathcal{O}_{L,p}$-action). Hence, the sheaf $P(\text{Spec}(R) \subset \text{Spec}(S)) := P_1$ on the crystalline site (of
Factorizing according to primes.

5.2.1. The local deformation theory and displays.

Lemma 5.2.1. Let \( k \) be an algebraically closed field of positive characteristic \( p \). Let \( x \in \mathbb{M} \) be a \( k \)-valued point. Then,

1. the RM \( p \)-divisible group \( A_x(p) \) factors canonically as the product of the \( \mathcal{O}_{L,p} \)-polarized \( p \)-divisible groups, denoted \( A_x(p) \);

2. for each \( p \), the \( \mathcal{O}_{L,p} \)-polarized \( p \)-divisible group \( A_x(p) \) is either ordinary or local-local. Its Dieudonné module is a free \( \mathcal{O}_{L,p} \otimes_{\mathbb{Z}} \mathbb{W}(k) \)-module of rank 2;

3. the functor of deformations of \( A_x(p) \) on \( \mathcal{C}_k \) as an \( \mathcal{O}_{L,p} \)-polarized \( p \)-divisible group is naturally equivalent to the direct product, over \( p \) dividing \( \mathfrak{p} \), of the functors of deformations of \( A_x(p) \) on \( \mathcal{C}_k \) as an \( \mathcal{O}_{L,p} \)-polarized \( p \)-divisible group.

One considers RM displays as in §5.1 and polarized displays with \( \mathcal{O}_{L,p} \) action. It is easy to see that the first category is naturally isomorphic to the direct product of the categories of polarized displays with \( \mathcal{O}_{L,p} \) action, where \( \mathfrak{p} \) runs over primes factor of \( p \mathcal{O}_L \).

Under the equivalence of categories between deformations of connected \( p \)-divisible groups and displays, the decomposition according to primes is respected.

5.2.2. The associated local model. Let \( D_0 \) be the \( \mathcal{O}_L \)-module \( \mathcal{O}_L \otimes_k \mathcal{O}_L \otimes k \), let \( \langle , \rangle : D_0 \times D_0 \to \mathbb{C} \), \( D_0 = \mathcal{O}_L \otimes k \) be the wedge product, and let \( H_0 \subset D_0 \) be an isotropic \( \mathcal{O}_L \otimes k \)-submodule of \( D_0 \) having dimension \( g \) over \( k \). Let \( R \) be the complete local ring pro-representing the moduli problem of associating to a local artinian \( k \)-algebra \( (S, \mathfrak{m}_S) \) an \( \mathcal{O}_L \otimes S \)-submodule \( H \) of \( D := D_0 \otimes_k S \), such that \( H \) is free as a \( S \)-module, is a direct summand of \( D \), is totally isotropic with respect to the pairing \( \langle , \rangle \), and reduces to \( H_0 \) modulo \( \mathfrak{m}_S \). The ring \( R \) is isomorphic to the completion of the local ring of the point corresponding to \( (H_0, D_0) \) in the appropriate Grassmann variety \( \text{Grass}(\mathcal{O}_L \otimes k, \langle , \rangle, g, 2g) \).

The Grassmann variety \( \text{Grass}(\mathcal{O}_L \otimes k, \langle , \rangle, g, 2g) \) is canonically isomorphic to the product, over \( p \mathfrak{p}_0 \), of the Grassmann varieties \( \text{Grass}(\mathcal{O}_{L,p} \otimes k, \langle , \rangle_p, g_p, 2g_p) \). In particular, writing \( D_0 = \bigoplus_p D_0(p), H_0 = \bigoplus_p H_0(p) \), using the decomposition \( \mathcal{O}_L \otimes k = \bigoplus_k \mathcal{O}_{L,p} \otimes k \), and noting that the pairing decomposes accordingly, we find that \( R = \hat{\bigotimes}_k R(p) \), where \( R(p) \) is the completed local ring of the point \( (H_0(p), D_0(p)) \) on the Grassmann variety \( \text{Grass}(\mathcal{O}_{L,p} \otimes k, \langle , \rangle_p, g_p, 2g_p) \).
5.3. The setting in which the theorems are proved.

Using the decomposition above, one sees that the construction of the universal RM display (for deformations of a given RM display over \( k \)) may be considered “one prime at a time”, and therefore, for notational convenience, one may assume that \( p\mathcal{O}_L = p^r \). The results in this section will be formulated under this assumption, from which the more general assertions follow immediately.

We set the following notation: \( p\mathcal{O}_L = p^r \), \( f = [\mathcal{O}_L/p : \mathbb{F}_p] \). Let \( \sigma_1, \ldots, \sigma_f \) denote the embeddings of \( \hat{\mathcal{O}}_{L,p} \to \mathbb{W}(k) \), ordered such that \( \mathcal{O} \circ \sigma_i = \sigma_{i+1} \).

Note that \( \mathcal{O}_L \otimes \mathbb{W}(k) = \bigoplus_{i=1}^f B(i) \), where the decomposition is induced by the isomorphism \( \mathbb{W}(\mathbb{F}_{p^r}) \otimes_{\mathbb{Z}_p} \mathbb{W}(k) \cong \bigoplus \mathbb{W}(k), a \otimes \lambda \mapsto (\ldots, \sigma_i(a) \lambda, \ldots) \). We also have \( \mathcal{O}_L \otimes k = \bigoplus_i \mathcal{B}(i) \) with the obvious notation. Note that \( \mathcal{B}(i) \cong k[T]/(T^e) \), where \( T \) is the reduction of an Eisenstein element for the extension \( \hat{\mathcal{O}}_{L,p}/\hat{\mathcal{O}}_{L,p}^+ \). For any \( k \)-algebra \( S \) denote by \( F \) and \( V \), the maps on \( \mathcal{O}_L \otimes \mathbb{W}(S) \) given by \( F(\ell \otimes w) \mapsto \ell \otimes Fw \) and \( V(\ell \otimes w) \mapsto \ell \otimes Vw \) for all \( \ell \in \mathcal{O}_L \) and \( w \in \mathbb{W}(S) \).

5.4. Further decomposition of the local model.

Let \( r = 1, \ldots, f \). Let

\[
D_0(r) := \mathcal{B}(r) \oplus \mathcal{B}(r)
\]

and denote by \( \langle , \rangle : D_0(r) \times D_0(r) \to \mathcal{B}(r) \) the wedge product. Let \( H_0(r) \subset D_0(r) \) be an isotropic \( \mathcal{B}(r) \)-submodule of \( D_0(r) \) having dimension \( e \) over \( k \). There exist a basis \( \{ \alpha(r), \beta(r) \} \) of \( D_0(r) \), as a \( \mathcal{B}(r) \)-module, such that \( \langle \alpha(r), \beta(r) \rangle = 1 \) and

\[
H_0(r) = (T^{i(r)})\alpha(r) \oplus (T^{j(r)})\beta(r)
\]

for integers \( i \geq i(r) \geq j(r) \geq 0 \) satisfying \( i(r) + j(r) = e \). Let \( R(r) \) be the complete local ring pro-representing the moduli problem of associating to an object \( (S, m_S) \) of \( \mathcal{E}_k \) a \( \mathcal{B}(r) \otimes S \)-submodule \( H(r) \) of \( D_0(r) \otimes_k S \) such that \( H(r) \) is free as a \( S \)-module, is a direct summand of \( D(r) \), is totally isotropic with respect to the pairing \( \langle , \rangle \), and reduces to \( H_0(r) \) modulo \( m_S \). Then,

\[
R(r) \cong k[a(r)_0, \ldots, a(r)_{i(r)-1}, b(r)_0, \ldots, b(r)_{j(r)-1},
\begin{align*}
c(r)_0, & c(r)_{i(r)-1}, d(r)_0, \ldots, d(r)_{j(r)-1}\rangle/ \\
( & a(r)d(r) - b(r)c(r) + a(r)T^{i(r)} + d(r)T^{j(r)}),
\end{align*}
\]

where \( a(r) := a(r)_0 + \ldots + a(r)_{i(r)-1}T^{i(r)-1}, b(r) := b(r)_0 + \ldots + b(r)_{j(r)-1}T^{j(r)-1}, c(r) := c(r)_0 + \ldots + c(r)_{i(r)-1}T^{i(r)-1} \) and \( d(r) := d(r)_0 + \ldots + d(r)_{j(r)-1}T^{j(r)-1} \).

The universal flag \( H(r) \subset D(r) \) over \( R(r) \) is defined by the \( \mathcal{B}(r) \)-span of \( T^{i(r)}\alpha(r) + a(r)\alpha(r) + b(r)\beta(r) \) and \( T^{j(r)}\beta(r) + c(r)\alpha(r) + d(r)\beta(r) \). Note that the Grassmann variety \( \text{Grass}(\mathcal{O}_L \otimes k, \langle , \rangle, g, 2g) \) decomposes as the product of the Grassmann varieties \( \text{Grass}(\mathcal{B}(r), \langle , \rangle_r, e, 2e) \). Hence, \( R \cong \bigotimes_{r=1}^f R(r) \).
5.5. The display over the special fiber and its trivial extension.

Let \( \mathcal{O}_0 := (P_0, Q_0, F_0, V_0^{-1}, (, )_0) \) be a RM display over \( k \) with an \( \mathcal{O}_L \otimes k \)-linear isomorphism of the Hodge filtration \( H_{\mathcal{O}_0} \subset D_{\mathcal{O}_0} \) with \( H_0 \subset D_0 \), compatible with the pairings on \( P_0 \) and \( D_0 \). Choose a decomposition \( P_0 = \oplus_r (B(r)\alpha(r) \oplus B(r)\beta(r)) \) as \( \mathcal{O}_L \otimes \mathbb{W}(k) \)-module so that \( P_0/pP_0 = D_0, Q_0/pP_0 = H_0 \) and \( \langle \alpha(r), \beta(r) \rangle_0 = 1 \). Note that \( F_0 = \oplus F_0(r) \), a direct sum of \( \mathbb{F} \)-linear maps, and

\[
F_0(r) [B(r)\alpha(r) \oplus B(r)\beta(r)] \subset [B(r+1)\alpha(r+1) \oplus B(r+1)\beta(r+1)].
\]

The matrix of \( F_0(r) \) with respect to the bases \( \{\alpha(r), \beta(r)\} \) and \( \{\alpha(r+1), \beta(r+1)\} \) is of the form

\[
F_0(r) := \begin{pmatrix} T^{j(r)}g_{1,1}(r) & T^{i(r)}g_{1,2}(r) \\ T^{j(r)}g_{2,1}(r) & T^{i(r)}g_{2,2}(r) \end{pmatrix}.
\]

To state the main theorem of this section we need some more notation. Let \( \hat{a}(r)s, \hat{b}(r)s, \hat{c}(r)s \) and \( \hat{d}(r)s \) be the Teichmüller lifts in \( \mathbb{W}(R(r)) \) of \( a(r)s, b(r)s, c(r)s \) and \( d(r)s \) for \( 1 \leq r \leq f \), \( 0 \leq s \leq \hat{i}(r) - 1 \) and \( 0 \leq t \leq j(r) - 1 \). Define \( \hat{a}(r) := \sum_{s=0}^{\hat{i}(r)-1} \hat{a}(r)sT^s \), \( \hat{b}(r) := \sum_{s=0}^{\hat{j}(r)-1} \hat{b}(r)sT^s \), \( \hat{c}(r) := \sum_{s=0}^{\hat{i}(r)-1} \hat{c}(r)sT^s \) and \( \hat{d}(r) := \sum_{s=0}^{\hat{j}(r)-1} \hat{d}(r)sT^s \); these are elements of \( B(r) \otimes \mathbb{W}(k) \mathbb{W}(R(r)) \). Let

\[
n(r) := \hat{a}(r)\hat{d}(r) - \hat{b}(r)\hat{c}(r) + \hat{a}(r)\hat{d}(r)T^{\hat{i}(r)} + \hat{d}(r)T^{\hat{j}(r)}.
\]

**Lemma 5.5.1.** Let \( M(r) \) be the maximal ideal of \( R(r) \). Then, the element \( F_n(r) \) lies in \( T^n B(r) \otimes \mathbb{W}(k) \mathbb{W}(M(r)) \). Let

\[
u_r := 1 + T^{-c} F n(r);
\]

it is a unit in \( B(r) \otimes \mathbb{W}(k) \mathbb{W}(R(r)) \).

**Proof.** Note that \( T^n B(r) \otimes \mathbb{W}(k) \mathbb{W}(M(r)) \) is equal to \( pB(r) \otimes \mathbb{W}(k) \mathbb{W}(M(r)) \). Since multiplication by \( p \) coincides with the composition of Verschiebung and Frobenius, we conclude that \( p\mathbb{W}(M(r)) \) consists of the Witt vectors \((a_0, a_1, \ldots)\) with \( a_0 = 0 \) and \( a_i \in F M(r) \). The assertion concerning \( F_n(r) \) follows. Note that \( u(r) \) lies in \( 1 + B(r) \otimes \mathbb{W}(k) \mathbb{W}(M(r)) \). It is a unit by Lemma 5.5.2. \( \square \)

**Lemma 5.5.2.** Let \( S \) be a \( k \)-algebra. Let \( v \in B(r) \otimes \mathbb{W}(k) \mathbb{W}(S) \). Assume that the image \( \overline{v} \) of \( v \) via \( B(r) \otimes \mathbb{W}(k) \mathbb{W}(S) \to B(r) \otimes \mathbb{W}(k) S \to S \) is a unit. Then, \( v \) is a unit.

**Proof.** Let Norm on \( B(r) \otimes \mathbb{W}(k) \mathbb{W}(S) \) (resp. \( B(r) \otimes \mathbb{W}(k) S \)) be the norm as a \( \mathbb{W}(S) \)-module (resp. a \( S \)-module). Then, \( v \) (resp. \( \overline{v} \)) is a unit if and only if Norm(\( v \)) (resp. Norm(\( \overline{v} \)) is a unit. Hence, we may assume \( \mathcal{O}_L = \mathbb{Z} \). Let \( u \) be an element of \( \mathbb{W}(S) \) such that \( uv = 1 - i \) with \( i \equiv 0 \in \mathbb{W}_1(S) = S \). Note that \( i^n \equiv 0 \) in \( \mathbb{W}_n(S) \). Since \( \mathbb{W}(S) = \lim \mathbb{W}_n(S) \), we get that the element \( z = \sum_n i^n \) exists in \( \mathbb{W}(S) \). Hence, \( v(uz) = 1 \). \( \square \)
Let

\[ F(r) : B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}(R) \oplus B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}(R) \to B(r+1) \otimes_{\mathbb{W}(k)} \mathbb{W}(R) \]

be the \( F \)-linear operator whose matrix with respect to the bases \( \{ \alpha(r), \beta(r) \} \) and \( \{ \alpha(r+1), \beta(r+1) \} \) is:

\[
F(r) := u(r)^{-1} \begin{pmatrix}
T^{(r)}g_{1,1}(r) + F(\hat{d}(r))g_{1,1} - F(\hat{b}(r))g_{1,2} & T^{(r)}g_{1,2}(r) - F(\hat{c}(r))g_{1,1} + F(\hat{a}(r))g_{1,2} \\
T^{(r)}g_{2,1}(r) + F(\hat{d}(r))g_{2,1} - F(\hat{b}(r))g_{2,2} & T^{(r)}g_{2,2}(r) - F(\hat{c}(r))g_{2,1} + F(\hat{a}(r))g_{2,2}
\end{pmatrix}.
\]

(5.2)

5.6. The main results on displays.

Theorem 5.6.1. Let \( P := P_0 \otimes_{\mathbb{W}(k)} \mathbb{W}(R) \) and let \( \langle , \rangle \) be the base change of \( \langle , \rangle_0 \) via \( \mathbb{W}(k) \to \mathbb{W}(R) \). Let \( Q \) be the inverse image of \( H \) via the projection \( P \to D \). Let \( F : P \to P \) be the \( F \)-linear map whose matrix form with respect to the decomposition \( P = \oplus_r B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}(R) \alpha(r) \oplus B(r) \otimes_{\mathbb{W}(k)} \mathbb{W}(R) \beta(r) \) is

\[
\begin{pmatrix}
0 & 0 & \ldots & 0 & F(g) \\
F(1) & 0 & \ldots & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & \ldots & F(g-1) & 0
\end{pmatrix},
\]

with \( F(r) \) given in Equation (5.2). Then, there exists a unique \( \mathbb{F} \)-linear homomorphism \( V^{-1} : Q \to P \) so that \( \mathcal{P} := (P, Q, V^{-1}, \langle , , \rangle) \) is a RM display. Moreover,

- its base change via \( R/m = k \) coincides with \( \mathcal{P}_0 \) as RM display;
- \( (R, \mathcal{P}) \) is the universal pro-representing object and the universal RM display for the moduli problem of deforming \( \mathcal{P}_0 \) to objects of \( \mathcal{E}_k \) as a RM display;
- the projection \( P \to D \) identifies \( H_{\mathcal{P}} \subset D_{\mathcal{P}} \) with \( H \subset D \) compatibly with the pairings on \( P \) and \( D \).

Proof. Note that \( F \) is the composition \( F_0 \circ \psi^{-1} \) of the \( F \)-linear base change of \( F_0 \) to \( P = P_0 \otimes_{\mathbb{W}(k)} \mathbb{W}(R) \) with the inverse in \( P \otimes_{\mathbb{Q}} Q \) of the map \( \psi : P \otimes_{\mathbb{Q}} Q \to P \otimes_{\mathbb{Q}} Q \) defined by the diagonal matrix \( \text{diag} \{ \psi(1), \ldots, \psi(f) \} \), where the map \( \psi(r) \) is defined with respect to the basis \( \{ \alpha(r), \beta(r) \} \) by the matrix

\[
\begin{pmatrix}
1 + \hat{a}(r)T^{-i(r)} & \hat{c}(r)T^{-j(r)} \\
\hat{b}(r)T^{-i(r)} & 1 + \hat{d}(r)T^{-j(r)}
\end{pmatrix}.
\]

One proves that, indeed, \( F \) is well defined. One defines \( V^{-1} := \tilde{F}_P \) on \( P \otimes Q \) and one proves that \( V^{-1} \) restricted to \( Q \) is well defined, it is compatible with \( \langle , , \rangle \) and \( V^{-1}(Q) \) spans \( P \). By definition \( V^{-1} \) is compatible with \( F \). See [AG4] for
details. Claims (1) and (3) follow immediately from the construction. Claim (2) follows from the following theorem.

**Theorem 5.6.2.** Let $\mathcal{P} := (P, Q, F, V^{-1}, \{\cdot, \cdot\})$ be a RM display over $R$ and let $\tau: D_\mathcal{P} \to D$ be an isomorphism as $\mathcal{O}_L \otimes R$-modules, compatible with pairings, such that $\tau(H_\mathcal{P}) = H$ and $\tau$ is a horizontal map mod $m^2$. Here, we consider the connection on $D_\mathcal{P} \otimes_R R/m^2$ induced by the fact that $D_\mathcal{P}$ is a crystal 5.1 and we consider on $D \otimes_R R/m^2$ the connection having $D_0 \subset D$ as horizontal sections. Then, $(R, \mathcal{P})$ is the universal pro-representing object and the universal RM display for the moduli problem of deforming the special fiber $\mathcal{P}_0$ of $\mathcal{P}$ to local artinian $k$-algebras as RM display.

**Proof.** Let $\mathcal{P}^{\text{uni}} := (P^{\text{uni}}, Q^{\text{uni}}, F^{\text{uni}}, (V^{\text{uni}})^{-1}, \{\cdot, \cdot\}^{\text{uni}})$ be the universal RM display deforming the special fiber $\mathcal{P}_0$ of $\mathcal{P}$ to local artinian $k$-algebras. By the theory of local models [DP, Thm. 3.3] and the equivalence of categories between deformations of displays and of formal $p$-divisible groups [Zin, Thm. 9] it exists over $R$.

Let $\phi: \text{Spec}(R) \to \text{Spec}(R)$ be the unique homomorphism such that $\mathcal{P} = \phi^*(\mathcal{P}^{\text{uni}})$. Since $R$ pro-represents a Grassmannian moduli problem, we get unique maps $\psi_i: \text{Spec}(R/m^2) \to \text{Spec}(R/m^2)$, such that $\psi_1^*(H \subset D) \cong (H_{\mathcal{P}^{\text{uni}}} \subset D_{\mathcal{P}^{\text{uni}}})$ and $\psi_2^*(D) \cong D_{\mathcal{P}^{\text{uni}}}$ is horizontal, and $\psi_2$ such that $\psi_2^*(H \subset D) = (H_{\mathcal{P}} \subset D_{\mathcal{P}})$ and $\psi_2^*(D) \cong D_{\mathcal{P}}$ is horizontal. Moreover, $\psi_1 \circ \phi = \psi_2$ - all the maps appearing being canonical. By [DP, Lemma 3.5] the map $\psi_1$ is an isomorphism. Hence, $\phi$ is an isomorphism on tangent spaces.

Let $\text{Gr}(R)$ be the graded ring $\oplus_n m^n/m^{n+1}$ associated to $R$. The induced map $\text{Gr}(\phi^*): \text{Gr}(R) \to \text{Gr}(R)$ is then surjective on each graded piece and, hence, by dimension considerations it is injective. Since $\text{Gr}(\phi^*)$ is an isomorphism, we conclude that $\phi^*$ is an isomorphism as well [AM, Lem. 10.23]. Hence, $\phi$ is an isomorphism as claimed.

**Corollary 5.6.3.** Let $p$ be maximally ramified. Let $x \in \mathfrak{M}$ be a geometric point of type $(j, n)$.

1. The deformations to $S_{j'}$, where $j' \leq j$, are parameterized by the closed subscheme defined by the ideal $\langle a_i, b_i, c_i, d_i : 0 \leq i \leq j' - 1 \rangle$.

2. The deformations to $W_{(j', n')}$, where $j' \leq j$, are parameterized by the closed subscheme of deformations to $S_{j'}$ intersected with the closed subscheme (with the reduced structure) given by the relations $T^{j'+n'}|F^2$. 
6. Some general results concerning strata in the maximally ramified case

6.1. Foliations of Newton polygon strata.

Definition 6.1.1. ([Oo2, §2]) Let $\mathcal{P}$ be a RM display over a perfect field $K$ of positive characteristic $p$. Let $S$ be a noetherian scheme over $K$. Let $A \to S$ be a RM abelian scheme. Define

$$C_P(A \to S)$$

as the subset of $S$ consisting of the geometric points $s \in S$ for which there exists an isomorphism of RM displays between $P$ and the display associated to $A_s$.

If $P$ is the RM display associated to a geometric point $x \in S$, we write $C_x$ instead of $C_P$ and we call it the central leaf at $x$.

Note that the Newton polygon and the type, in the sense of [Oo1], of the geometric points of $C_P$ are those of $P$ and, hence, they are constant. By [Oo2, Thm 2.2] the set $C_P$ is a closed subset of the locally closed subscheme $S$ defined by the points having the same Newton polygon as $P$.

Definition 6.1.2. Let $k$ be a perfect field and let $S$ be a $k$-algebra. Let $s = t(s)g + r(s)$ with $t(s) \in \mathbb{N}$ and $0 \leq r(s) \leq g - 1$. Consider the exact sequence of $\mathbb{W}(S)$-modules

$$0 \to S \overset{\varphi}{\longrightarrow} \mathbb{W}_{t(s)+1}(S) \to \mathbb{W}_{t(s)}(S) \to 0.$$ 

The map $\varphi$ is the $t(s)$-th power of Verschiebung $r \in \mathbb{F}_p^\times /\{0, \ldots, 0, r\}$. It identifies $S$ with the $\mathbb{W}(S)$-module whose additive structure is that of $S$ and multiplication of $r \in S$ by $a = (a_0, a_1, \ldots) \in \mathbb{W}(S)$ is given by $a \cdot r = a_0^{p^{t(s)}} r$. Since $\mathcal{O}_L$ is a free $\mathbb{Z}$-module, the sequence

$$0 \to \mathcal{O}_L \otimes_{\mathbb{Z}} S \overset{1 \otimes \varphi}{\longrightarrow} \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}_{t(s)+1}(S) \to \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}_{t(s)}(S) \to 0$$

is an exact sequence of $\mathcal{O}_L \otimes \mathbb{W}(S)$-modules. Since $S$ is of characteristic $p$, we have that $\mathcal{O}_L \otimes_{\mathbb{Z}} S \cong \mathbb{F}_p[T]/(T^g) \otimes_{\mathbb{F}_p} S$. Consider the $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}(S)$-submodule of $\mathcal{O}_L \otimes_{\mathbb{Z}} S$ defined by $I_s := T^{t(s)} S \oplus \cdots \oplus T^{g-1} S$. Let

$$Z_s(S) := (\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}_{t(s)+1}(S))/(1 \otimes \varphi)(I_s).$$

By construction we have an exact sequence of $\mathcal{O}_L \otimes \mathbb{W}(S)$-modules

$$0 \to S \to Z_{s+1}(S) \to Z_s(S) \to 0,$$

where $S$ is a $\mathbb{W}(S)$-module as above and $\mathcal{O}_L$ acts on $S$ via the quotient $\mathcal{O}_L/(T)$. We note that

$$\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{W}(S) = \lim_{\to} Z_s(S).$$
Remark 6.1.3. Let \( A \) be an abelian variety with RM by \( \mathcal{O}_L \) over a perfect field of characteristic \( p \), totally ramified in \( L \). It is proven in [AG1] that one can choose an \( \mathcal{O}_L \otimes W(k) \) basis \( \alpha, \beta \) for the Dieudonné module (or “display the display”) of \( A \) such that, if \( A \) is not superspecial, Frobenius is given with respect to this basis by a matrix
\[
\begin{pmatrix}
T^n & c_3 T^n \\
T^j & 0
\end{pmatrix},
\]
with \( c_3 \in (\mathcal{O}_L \otimes W(k))^\times \). If \( A \) is superspecial then, in fact, one can choose the basis so that the matrix is
\[
\begin{pmatrix}
0 & T^n \\
T^j & 0
\end{pmatrix}.
\]

The following proposition tells us that one can get the same on the level of \( p\)-torsion (for any \( s \)) by passing to a suitable finite cover depending on \( s \).

Proposition 6.1.4. Let \( k \) be a perfect field of positive characteristic \( p \). Let \( R \) be a \( k \)-algebra. Let \( (P, Q, V^{-1}, F) \) be an \( \mathcal{O}_L \)-display over \( R \) such that
\[
F(\alpha_s) = T^j \alpha_s + T^j \beta_s, \quad F(\beta_s) = c_3 T^i \alpha_s,
\]
with \( i > j \) and \( c_3 \in (\mathcal{O}_L \otimes W(R))^\times \). Then, there exist ring extensions \( R = R_0 \subset \cdots \subset R_s \subset R_{s+1} \subset \cdots \) and elements \( A_s \) and \( B_s \) in \( Z_s(R_s) \) such that:

1. for the elements \( \alpha_s := A_s \alpha + B_s \beta \) and \( \beta_s := (A_s^e - A_s + B_s c_3 T^{i-j}) \alpha + (A_s^e - B_s) \beta \) of \( P \otimes W(R) Z_s(R_s) \) the following equalities hold:
   \[
   F(\alpha_s) = T^j \alpha_s + T^j \beta_s, \quad F(\beta_s) = T^i \alpha_s;
   \]
2. the matrix
   \[
   \begin{pmatrix}
   A_s & A_s^e - A_s + B_s c_3 T^{i-j} \\
   B_s & A_s^e - B_s
   \end{pmatrix}
   \]
   is invertible;
3. \( A_{s+1} \) and \( B_{s+1} \) map to \( A_s \) and \( B_s \) resp., viewing \( A_s \) and \( B_s \) in \( Z_s(R_{s+1}) \) via the inclusion \( Z_s(R_s) \subset Z_s(R_{s+1}) \);
4. for every \( s \) the extension \( R_s \subset R_{s+1} \) is universal for the existence of \( A_s \) and \( B_s \) satisfying (1) and (2);
5. the extension \( R_s \subset R_{s+1} \) is defined by iterated monogenic extensions. In particular, \( R_{s+1} \) is finite and free over \( R_s \).

Proof. From the definition of \( \beta_s \) we get
\[
F(\alpha_s) = A_s^e F(\alpha) + B_s F(\beta) = A_s^e T^j \alpha + A_s^e T^j \beta + B_s c_3 T^i \alpha
= A_s T^j \alpha + B_s T^j \beta + T^j \beta_s = T^j \alpha_s + T^j \beta_s.
\]
Hence, condition (1) is equivalent to
\[ F(\beta_s) = (A_s^2 - A_s^c + B_s^2 c_3^T T^{i-1}) F(\alpha) + (A_s^2 - B_s^2) F(\beta) \]
\[ = ([A_s^2 - A_s^c + B_s^2 c_3^T T^{i-1}] + A_s^2 c_3 T^{i-1} - B_s^2 c_3 T^{i-1}) T^i \alpha \]
\[ + [A_s^2 - A_s^c + B_s^2 c_3^T T^{i-1}] T^i \beta \]
\[ = T^i \alpha_s \]
\[ = A_s T^i \alpha + B_s T^i \beta \]

This is implied by the following two equations in \( Z_s(R_s) \):
\[ A_s^2 - A_s^c + B_s^2 c_3^T T^{i-1} - B_s T^{i-1} = 0, \quad (6.4) \]
\[ A_s^2 c_3 - B_s^2 c_3 - A_s + B_s = 0. \quad (6.5) \]

Condition (2) is equivalent to
\[ \det \begin{pmatrix} A_s & A_s^c - A_s + B_s^2 c_3 T^{i-1} \\ B_s & A_s^c - B_s \end{pmatrix} = A_s(A_s^c - B_s) - B_s(A_s^c - A_s + B_s^2 c_3 T^{i-1}) = \text{unit}. \quad (6.6) \]

We proceed by induction on \( s \). Start with \( s = 1 \). Let \( \bar{\pi} \) be the reduction of \( c_3 \) in \( R = \bar{Z}_1(R) \). Let \( R_1 := R[[A_1, u]]/((A_1^p - 1)^p, u^{p-1} - \bar{c_3}) \). Since \( \bar{Z}_1(S) = S \) for any \( k \)-algebra \( S \), Equations (6.4) and (6.5) have solutions \( A_1 \) and \( B_1 \equiv A_1 + u^{-1} \) in \( R_1 \). Note that the element \( A_1 (A_1^p - B_1) - B_1 (A_1^p - A_1 + B_1^p c_3 T^{i-1}) \) is equivalent to \( -u A_1 \) in \( R_1/(A_1^p - 1) \). Hence, it is invertible in \( R_1 \). By 5.5.2, condition (6.6) is satisfied. Condition (4) is clearly satisfied for \( R \subset R_1 \).

Assume that the proposition holds for \( s \). Let \( A_{s+1}', B_{s+1}' \) be elements in \( Z_{s+1}(R_s) \) reducing to \( A_s \) and \( B_s \) respectively in \( Z_s(R_s) \). Let \( R_{s+1}' \) be the polynomial ring \( R_s[\lambda, \mu] \). Let \( \lambda_s := (0, \ldots, 0, \lambda) \) and \( \mu_s := (0, \ldots, 0, \mu) \) in \( \ker(Z_{s+1}(R_s') \to Z_s(R_s')) \). Let \( A_{s+1} := A_{s+1}' + \lambda_s \) and \( B_{s+1} := B_{s+1}' + \mu_s \). Then, (6.4) becomes
\[ \lambda^p = \lambda^p + P_s = 0, \]
where \( P_s \) is the element of the ring \( R_s \) defined by \( P_s = ((B_1')^{\sigma_2} c_3 - B_1') T^{i-1} + (\mu^p - \mu) T^{i-1} = ((B_1')^{\sigma_2} c_3 - B_1') T^{i-1} \). We used the fact that \( T^{i-1} \) kills the module \( \ker(Z_{s+1}(R_s') \to Z_s(R_s')) \). For the same reason \( P_s \) is independent of the choice of \( B_1' \). Finally, (6.5) becomes
\[ \mu^p c_3 - \mu + Q_s = 0, \]
where \( Q_s \) is the element of \( R_s \) defined by \( Q_s = (B_1')^{\sigma_2} c_3 - B_1 - A_{s+1}' c_3 + A_{s+1}' \). These two equations define an ideal \( J_s \) in \( R_{s+1}' \). Let \( R_{s+1} := R_{s+1}'/J_s \). It is an extension of \( R_s \), finite and free as \( R_s \)-module. By construction \( A_{s+1} \) and \( B_{s+1} \) reduce to \( A_s \) and \( B_s \) in \( Z_s(R_{s+1}) \). Since (6.6) holds for \( s = 1 \), we deduce from (5.5.2) that (6.6) holds for \( A_{s+1} \) and \( B_{s+1} \).

\textbf{Corollary 6.1.5.} Let \( \mathbf{A} \to W_{(j,j)} \) be the restriction of the universal RM abelian scheme to \( W_{(j,j)} \). Let \( s \in W_{(j,j)} \) be a geometric point. For every \( m \in \mathbb{N} \), there...
exists a scheme $W_{(j,j)}^{[m]}$ finite and dominant over $W_{(j,j)}$, such that
\[ A[p^m] \times W_{(j,j)}^{[m]} \cong A_s[p^m] \times_{k(s)} W_{(j,j)}^{[m]} \]

Proof. Consider the functor associating to a $W_{(j,j)}$-scheme $T$ the group
\[ \text{Isom}(A[p^m] \times W_{(j,j)}^{[m]}, T, A_s[p^m] \times_{k(s)} T) \]

of isomorphisms as group schemes over $T$ endowed with an $O_L$-action. It is represented by a scheme $W_{(j,j)}^{[m]}$, affine and of finite type over $W_{(j,j)}$ (see [Oo2, Lemma 2.4]). It follows from 6.1.4 that for every geometric point $x$ of $W_{(j,j)}$ one can trivialize Frobenius on the Dieudonné module of $A_x[p^{m+1}]$. Hence, one can trivialize the Dieudonné module of $A_x[p^m]$. We conclude that the reduced fiber of $W_{(j,j)}^{[m]}$ over $x$ is non-empty. Analogously, 6.1.4 implies that $W_{(j,j)}^{[m]}$ is quasi-finite over $W_{(j,j)}$. Finally, using the valuative criterion of properness and 6.1.4, we conclude that the map $W_{(j,j)}^{[m]} \rightarrow W_{(j,j)}$ is proper. □

Remark 6.1.6. Let $j \geq g/2$ be an integer. The locus $W_{(j,g-j)}$ is zero dimensional. Since Frobenius of the Dieudonné module of each of its points has the canonical form described by the matrix (6.2), it follows that for every $m \in \mathbb{N}$ the $O_L$-group scheme $A[p^m] \times_{\mathbb{B}} W_{(j,g-j)}$ is constant. In this case, we define $W_{(j,g-j)}^{[m]} := W_{(j,g-j)}$.

Corollary 6.1.7. The $p$-divisible group, with RM by $O_L$, associated to the universal abelian scheme over $W_{(j,j)}$ is geometrically constant for $j < g/2$. In particular, the central leaf $\mathcal{A}_x$ at any point $x$ of $W_{(j,j)}$ coincides with $W_{(j,j)}$.

Proof. Let $x$ be a geometric point of $W_{(j,j)}$. Let $G_x$ be the $p$-divisible group defined by $x$. Apply 6.1.4 to the $O_L$-display over $R = k(x)$ associated to $G_x$. The $k(x)$-algebras $R_x$ are finite as $k(x)$-modules. Since $k(x)$ is an algebraically closed field, there exist compatible sections $R_x \rightarrow k(x)$. Note that $O_L \otimes_k W(k(x)) \cong \lim \mathbb{Z}_s(k(x))$. Hence, $\alpha = \lim \alpha_s$ and $\beta = \lim \beta_s$ are well defined and form an $O_L \otimes_k W(k(x))$-basis of the Dieudonné module of $G_x$ such that $F(\alpha) = T^2 \alpha + T \beta$ and $F(\beta) = T \alpha$. Since $F \circ V = p$, we deduce that also Verschiebung $V$ has a canonical form with respect to the basis $\alpha$ and $\beta$ independent of $x$. Since the category of connected $p$-divisible groups and the category of displays are equivalent over perfect fields, we conclude. □

Corollary 6.1.8. Let $0 \leq j \leq g/2$. The scheme $W_{(j,j)}$ is quasi-affine.

Proof. If $j = g/2$, then $\text{dim}(W_{(j,j)}) = 0$ and $W_{(j,j)}$ consists of superspecial points. The corollary is trivial in this case. Suppose $j < g/2$. By 6.1.5 there exists a finite covering $W_{(j,j)}^{[1]}$ of $W_{(j,j)}$ over which the $p$-torsion of the universal RM abelian scheme can be trivialized. It follows from Raynaud’s trick that the pull-back of the Hodge bundle to $W_{(j,j)}^{[1]}$ is torsion [Oo1, Proof Thm 4.1]. Since the Hodge bundle is ample on $\mathcal{M}$, it follows that $W_{(j,j)}^{[1]}$ is quasi-affine. Hence, the conclusion. □
Let $\alpha_p$ be the group scheme over $k$ defined as the kernel of Frobenius on the additive group $\mathbb{G}_{a,k}$. We make $\mathcal{O}_L$ act on it via its quotient $\mathcal{O}_L/T = \mathbb{F}_p$.

**Proposition 6.1.9.** Let $0 \leq m \leq j \leq g/2$ be two integers. Let $j'$ be either $j$ or $g-j$. There exists a smooth, connected affine scheme $U_m$ over $k$ of dimension $m$ and a finite, surjective map

$$\psi_m : W_{(j,j')}^{[m]} \times_k U_m \to W_{(j-m,j')}^{[m]}$$

such that:

- for every $u \in U_m$ the image of $W_{(j,j')}^{[m]} \times_k k(u)$ is contained in the central leaf through any point of $\psi_m \left( W_{(j,j')}^{[m]} \times \{u\} \right)$;

- for every $s \in W_{(j,j')}^{[m]}$ the image of $k(s) \times_k U_m$ is the image of $A_s$ via iterated $\alpha_p$-Hecke correspondences.

**Proof.** Let $s \in W_{(j,j')}^{[m]}$. Define the schemes $U_n$ for $0 \leq n \leq j$ by induction on $n$ as follows. Let $U_0 := \text{Spec}(k)$. Suppose that $U_n$ has been defined and it is a smooth, connected affine scheme of dimension $n$ and that every $u \in U_n$ defines an iterated $\alpha_p$-quotient $A_s \to A_n$ of invariants $(j-n,j')$. Let $U_{n+1}$ be the scheme over $U_n$ whose fiber over any geometric point $u \in U_n$ is the subscheme of $\text{Hom}_{\mathcal{O}_n} (\alpha_p, A_n[p])$ of those maps for which the quotient $A_n/\alpha_p$ has invariants $(j-(n+1),j')$. By [AG1, Prop. 6.8] the morphism $U_{n+1} \to U_n$ is a smooth, connected affine scheme of dimension $n$.

Fix $m$. Define the map

$$\psi_m : W_{(j,j')}^{[m]} \times_k U_m \to \mathfrak{M}$$

as follows. By 6.1.5 or 6.1.6 we have a canonical isomorphism $\tau_m : A[p^m] \times_{\mathfrak{M}} W_{(j,j')}^{[m]} \cong A_s[p^m] \times W_{(j,j')}^{[m]}$. View $U_m$ as classifying suitable subgroup schemes of $A_s[p^m]$. Then, $\psi_m$ is the unique map such that the pull-back of the universal RM abelian scheme coincides with the quotient of $A \times_{\mathfrak{M}} (W_{(j,j')}^{[m]} \times_k U_m)$ by the inverse image via $\tau_m$ of the universal subgroup scheme of $A_s[p^m] \times_k U_m$ defined by $U_m$. Note that such a quotient is a RM abelian scheme by [AG1, Cor. 3.2]; in particular, the definition of $\psi_m$ makes sense. By construction the image of $\psi_m$ lies in $W_{(j-m,j')}^{[m]}$. To conclude it suffices to prove that $\psi_m$ is finite and surjective. We proceed by induction on $m$. By Corollary 6.1.8 and since $\psi_0$ is the identity, this holds true for $m = 0$. Suppose that $\psi_{m-1}$ is finite and surjective. Consider the diagram

$$\begin{array}{ccc}
W_{(j,j')}^{[m]} \times_k U_m & \xrightarrow{\delta} & \pi_2^{-1} (W_{(j-m,j')}^{[m]}) \cap \pi_1^{-1} (W_{(j-m+1,j')}^{[m]}) \\
\gamma \downarrow & & \downarrow \pi_1 \\
W_{(j,j')}^{[m-1]} \times_k U_{m-1} & \xrightarrow{\psi_{m-1}} & W_{(j-m+1,j')}^{[m]}
\end{array}$$
where $\gamma$ is the product of the the natural maps $\gamma_1: W^{[m]}_{(j,j')} \to W^{[m-1]}_{(j,j')}$ and $\gamma_2: U_m \to U_{m-1}$ and the top horizontal arrow is the unique making the diagram commute and such that $\gamma_2 \circ \delta = \psi_m$. By construction of $U_m$, for every point $s = (s_1, s_2) \in W^{[m-1]}_{(j,j')} \times_k U_{m-1}$ and any point $t$ in the finite scheme $\gamma_1^{-1}(s_1)$ the map $(t) \times_k \gamma_2^{-1}(s_2) \to \pi_1^{-1}(\psi_m(s))$ is an isomorphism. Hence, $\delta$ is quasi-finite, proper (by the valuative criterion) and surjective.

By [AG1, Lemma 8.6] the image via $\pi_1$ of $\pi_2^{-1}(W_{(j-m,j')})$ has invariants $(j - m + 1, j')$ and, if $j - m > 0$, also $(j - m - 1, j')$. Since the maps $\pi_1$ and $\pi_2$ are proper by [AG1, Lemma 8.6] and the intersection with $\pi_1^{-1}(W_{(j-m+1,j')})$ of the fiber of $\pi_2$ over a point of $W_{(j-m,j')}$ is non-empty and finite by [AG1, Prop. 6.8], we conclude that the composite $\psi_m = \pi_2 \circ \delta$ is quasi-finite, proper and surjective as claimed.

\[\square\]

**Corollary 6.1.10.** For every $j$ and $n$ the scheme $W_{(j,n)}$ is quasi-affine.

**Proof.** By Corollary 6.1.8 the claim holds for the loci $W_{(j,n)}$. The locus $W_{(j,g-j)}$ is zero dimensional and, hence, quasi affine. By 6.1.9 the locus $W_{(j,n)}$ is the image via a finite map of a quasi-affine scheme. Hence, the conclusion. \[\square\]

### 6.2. Connectedness of $T_0$, $T_1$ and $T_2$.

**Definition 6.2.1.** Let $0 \leq a \leq g$ be an integer. Let $T_a$ be the closed subscheme of $\mathfrak{M}$ defined by

$$T_a := \{ [A] \in \mathfrak{M}(k) | a(A) \geq a \}.$$ 

**Remark 6.2.2.** By [AG1, Lemma 4.4.2] we have $T_a = \Pi_{j,n} W_{(j,n)}$ where the union is taken over all pairs of integers $(j, n)$ such that $0 \leq j \leq n/2$ and $j \leq n \leq g - j$ and $a \leq j + n$. It also follows from op. cit. that $T_a$ has dimension $g - a$.

**Theorem 6.2.3.** The intersection of $T_0$ with any irreducible component of $\mathfrak{M}$ is connected. The same holds for $T_1$ if $g > 1$, and for $T_2$ if $g > 2$.

**Proof.** If $a = 0$, the statement follows since the ordinary locus is dense in every component of $\mathfrak{M}$. Suppose that $a = 1$. Then, $T_1$ is the complement of the ordinary locus in $\mathfrak{M}$. Hence, it is the zero locus of the Hasse invariant $h$. Since $h$ is a section of the Hodge bundle over $\mathfrak{M}$, and the Hodge bundle is ample, it follows that $T_1$ is connected if $g > 1$ (cf. [Har, III, Cor. 7.9]).

Assume now that $g > 2$. Let $E$ be the set of connected components of the intersection of $T_2$ with an irreducible component of $\mathfrak{M}$. Let $\pi_1, \pi_2: \mathfrak{M} \to \mathfrak{M}$ be as in § 2.2. The Hecke correspondence $\pi_2 \circ \pi_1^{-1}$ preserves properties such as being closed, or being irreducible, or being connected, for closed subschemes not intersecting the non-singular $(j = 0)$ locus of $\mathfrak{M}$, see [AG1, Prop 8.7]. For every $(j, n)$, it sends an irreducible component of $W_{(j,n)}$ to the union of irreducible components of loci $W_{(j', n')}$ with $(j', n')$ in a given set $A(j, n)$ depending only
on \((j,n)\) [AG1, Def 8.8]. Moreover, for every \((j',n') \in \Lambda (j,n)\) we have \(j' + n' \geq j + n - 1\). The Hecke correspondence has the additional property of sending each component of \(\mathfrak{M}\) into a single component of \(\mathfrak{M}\).

Fix a component \(C \in \mathcal{C}\). By 6.2.2 the irreducible components of \(C\) consist of irreducible components of strata \(W_{(j,n)}^{c}\) with \(j + n \geq 2\). We conclude that locus \(\pi_{2}(\pi_{1}^{-1}(C))\) is closed and connected, it lies in \(\mathcal{T}_1\) and its irreducible components consist of union of irreducible components of loci \(W_{(j,n)}^{c}\) for suitable pairs \((j,n)\) with \(j + n \geq 1\).

Suppose that \(|\mathcal{C}| > 1\). Since \(\pi_{2}(\pi_{1}^{-1}(\mathcal{T}_2)) = \mathcal{T}_1\), the inductive hypothesis implies that there exist distinct connected components \(C_1\) and \(C_2\) such \(E := \pi_{2}(\pi_{1}^{-1}(C_1)) \cap \pi_{2}(\pi_{1}^{-1}(C_2))\) is non-empty. If two irreducible components of the loci \(W_{(j,n)}^{c}\) and \(W_{(j',n')}^{c}\) intersect, then \((j,n) = (j',n')\) and they must coincide, because \(W_{(j,n)}^{c}\) is smooth [AG1, Thm 11.1(1)]. Hence, \(E\) is closed and consists of irreducible components of loci of type \(W_{(j,n)}^{c}\) for suitable \((j,n)\) with \(j + n \geq 1\). By Corollary 6.1.10 the loci \(W_{(j,n)}^{c}\) do not contain any complete curve. We conclude that \(E\) contains a point \([A]\) of type \((j,n)\) with \(j + n \geq 3\) and \(j \geq 1\). Note that \([A]\) is of type \((j,n)\) by [AG1] and \(\pi_{2}(\pi_{1}^{-1}(A))\) lies in \(\mathcal{T}_2\). Hence, its image \(\pi_{2}(\pi_{1}^{-1}(A))^{\vee}\) via the map \([A] \mapsto [A^{\vee}]\) lies in \(\mathcal{T}_2\). Since \(j \not= 0\) the image consists of a Moret-Bailly family. In particular, it is connected. For \(i = 1, 2\) let \([A_i]\) be a point of \(C_i\) and let \(H_i \subset A_i\) be an \(\mathcal{O}_{\mathcal{L}}\)-invariant subgroup scheme of rank \(p\) such that \(A_i \cong A_i / H_i\). Then, the moduli point corresponding to \(A_i^{\vee} / H_i^{\vee} \cong A_i^{\vee}\) lie in \(\pi_{2}(\pi_{1}^{-1}(A_i^{\vee}))\). Hence, \([A_1]\) and \([A_2]\) lie in the connected subscheme \(\pi_{2}(\pi_{1}^{-1}(A_i^{\vee}))^{\vee}\) of \(\mathcal{T}_2\). This contradicts the assumption that \(C_1\) and \(C_2\) were distinct. \(\square\)

**Remark 6.2.4.** The argument in the proof of 6.2.3 shows that if \(a\) is odd and \(\mathcal{T}_a\) is connected, then \(\mathcal{T}_{a+1}\) is connected. This is used in the proof in the claim that \(\pi_{2}(\pi_{1}^{-1}(\mathcal{T}_{a+1})) = \mathcal{T}_a\); a claim which is false for a even, cf. Diagram B in §3.2. It is an interesting question to know whether the loci \(\mathcal{T}_a\) are connected for all \(a \geq g-1\) or not. An affirmative answer would have strong consequences (perhaps too strong).

### 6.3. Irreducibility results.

The singularity strata \(S_j\) were defined in §3.2.

**Lemma 6.3.1.** Let \(g/2 \geq s \geq j \geq 0\) be integers. Let \(x \in S_s\). The completed local ring \(\hat{\mathcal{O}}_{S_j,x}\) of \(S_j\) at \(x\) is a complete intersection, regular in codimension 2. In particular, \(\hat{\mathcal{O}}_{S_j,x}\) is a normal domain.

**Proof.** One deduces as in [DP, §4.3 ], cf. §4.3.2, that the completed local ring of \(W_{(j,j)}^{c}\) at \(x\) has the presentation \(k[a,b,c,d]/(ad - bc + aT^2 + dT^{g-s})\) with \(a := a_jT^j + \ldots + a_{j-s-1}T^{j-s-1}, \ b := b_jT^j + \ldots + b_{j-1}T^{j-1}, \ c := c_jT^j + \ldots + c_{j-s-1}T^{j-s-1}\) and \(d := d_jT^j + \ldots + d_{j-1}T^{j-1}\). Hence, \(\hat{\mathcal{O}}_{S_j,x}\) is defined by \(g - 2j\) equations in \(2g - 4j\) variables. By [DP, §4.2 ] the dimension of \(\hat{\mathcal{O}}_{S_j,x}\) is \(g - 2j\). Hence, \(\hat{\mathcal{O}}_{S_j,x}\) is a complete intersection and, in particular, Cohen-Macaulay. By
loc. cit. \( \mathcal{O}_{S_j,x} \) is smooth in codimension 2. Using Serre’s criterion for normality we deduce that \( \mathcal{O}_{S_j,x} \) is a normal domain. \( \square \)

**Corollary 6.3.2.** For every \( j \), the irreducible components of \( W_{(j,j)}^c \) are disjoint.

**Proof.** Recall that \( S_j = W_{(j,j)}^c \). The Lemma implies that for every \( x \in W_{(j,j)}^c \) the ring \( \mathcal{O}_{S_j,x} \) is a domain. In particular, \( \mathcal{O}_{S_j,x} \) is a domain. Hence, there exists only one irreducible component of \( W_{(j,j)}^c \) containing \( x \). \( \square \)

**Proposition 6.3.3.** Let \( g > 2 \). Every irreducible component of \( \mathfrak{M} \) contains exactly one irreducible component of the non-ordinary locus \( W_{(0,1)}^c \). The same holds for the locus \( W_{(1,1)}^c \).

**Proof.** Let \( x \in W_{(0,1)}^c \). The locus \( W_{(0,1)}^c \) is defined by the zeroes of a section (the Hasse-Witt matrix) of an ample line bundle (the Hodge bundle). Hence, every irreducible component of \( \mathfrak{M} \) contains exactly one connected component of \( W_{(0,1)}^c \). The completed local ring of \( \mathfrak{M} \) at \( x \) is Cohen-Macaulay of dim \( g \). Hence, the completed local ring of \( W_{(0,1)}^c \) at \( x \) is Cohen-Macaulay as well of dim \( g - 1 \) by [Eis, Prop. 18.13].

Let \( C \) be a connected component of \( W_{(0,1)}^c \). Let \( \{T_i\} \) be the set of irreducible components of \( C \). Assume its cardinality is \( > 1 \). Let \( Z \) be the union of all the intersections \( T_i \cap T_j \) for \( i \neq j \). Then, \( C \setminus Z \) is disconnected. Hence, by Hartshorne’s connectedness theorem, see [Eis, Thm. 18.12], there must exist indices \( i \) and \( j \) and an irreducible component \( T \) of \( T_i \cap T_j \) of codimension 1 in \( C \) and, hence, of dimension \( g - 2 \). Since the locus \( \bigcup_n W_{(0,n)} \) is smooth, \( T \) consists of points with singularity index \( > 0 \). Since the types define a stratification and \( W_{(j,n)} \) is pure dimensional of dimension \( < g - 2 \) for \( j > 0 \) and \( n > 1 \), \( T \) consists of a full irreducible component of the locus \( W_{(1,1)}^c \). Hence, it contains a full component of the locus \( W_{(1,2)}^c \). By Lemma 6.3.4 the nilradical of the completed local ring of the locus \( W_{(0,1)}^c \) at a closed point of type \( (1,2) \) is a prime ideal. This implies that the prime ideals defined by \( T_i \) and \( T_j \) in the local ring of the locus \( W_{(0,1)}^c \) at a closed point of \( W_{(1,2)} \cap T_i \cap T_j \) are equal. Hence, \( T_i = T_j \). Contradiction. This proves the first part of the proposition.

Since \( \pi_2(\pi_1^{-1}(W_{(0,1)}^c)) = W_{(1,1)}^c \), the second claim follows. \( \square \)

**Lemma 6.3.4.** Let \( g > 2 \). Let \( x \) be a closed point of \( W_{(1,2)} \). Let \( \mathfrak{D} = W_{(0,1)}^c \) be the non-ordinary locus of \( \mathfrak{M} \). Then, the nilradical of the completed local ring \( \mathcal{O}_{\mathfrak{D},x} \) of \( \mathfrak{D} \) at \( x \) is a prime ideal.

**Proof.** By § 4.3.2 the completed local ring \( \mathcal{O}_{\mathfrak{M},x} \) of \( \mathfrak{M} \) at \( x \) is isomorphic to the quotient of the ring \( k[[a_0, \ldots, a_{g-2}, b_0, \ldots, c_{g-2}, d_0]] \) by the relations \( ad - bc + aT + dT^{g-1} = 0 \), viz.,
\[ a_0 d_0 - b_0 c_0 = 0, \]
\[ a_i d_0 + a_{i-1} - b_0 c_i = 0, \quad 1 \leq i \leq g - 2, \]
\[ a_g + d_0 = 0. \]

Eliminating the variables \( a_i \), using these equations, we get
\[
\hat{O}_{2g, 2} \cong \frac{k[b_0, c_0, \ldots, c_{g-2}, d_0]}{(b_0 c_0 - d_0 b_0 c_1 + d_0^2 b_0 c_2 - d_0^3 b_0 c_3 + \ldots + (-1)^{g-2} d_0^{g-2} b_0 c_{g-2} + (-1)^{g-2} d_0^g)}.
\]

The equations of the non-ordinary locus can be deduced as in § 9.1 and coincide with equations (Eq1)–(Eq4) given there with \( a_0 := b_0 (c_1 - d_0 c_2 + d_0^2 c_3 + \ldots - (-1)^{g-2} d_0^{g-3} c_{g-2}) - (-1)^{g-2} d_0^g \). If \( d_0 = 0 \) then it follows from those equations that a power of \( b_0 \) and \( c_0 \) (and hence also \( a_0 \)) is zero. The reduced ring defined by the vanishing of \( a_0, b_0, c_0 \) and \( d_0 \) coincides with the completion of \( W_{(1,1)} \) at \( x \).

Thus, to study the components of \( \mathcal{O} \) we may invert \( d_0 \). If a power of \( b_0 \) is 0 in \( \hat{O}_{2g, 2}[d_0^{-1}] \), then a power of \( a_0 \) and a power of \( c_0 \) are 0. It follows that a power of \( d_0 \) is 0 (contradiction). Thus, we may localize also by \( b_0 \).

Let \( h := c_1 - d_0 c_2 + d_0^2 c_3 + \ldots - (-1)^{g-2} d_0^{g-3} c_{g-2} \). Then, \( a_0 = h b_0 - (-1)^{g-2} d_0^g \).

As in § 9.1 the lemma reduces to prove that there exists a unique minimal prime ideal associated to the ideal \( I \) in \( k[b_0, c_0, c_1, d_0][b_0^{-1}, d_0^{-1}] \), where \( I \) is defined by

- \( b_0^{g+1} + h b_0 d_0^g - (-1)^{g-2} d_0^{g+g-1} = 0; \)
- \( b_0 c_0 - h b_0 d_0 + (-1)^{g-2} d_0^g = 0. \)

Consider the ideal \( J \) in the ring \( k[b_0, c_0, \ldots, c_{g-2}, d_0][b_0^{-1}, d_0^{-1}] \) defined by

- \( b_0^g + c_0 d_0^{g-1} = 0 \) (obtained dividing by \( b_0 \) the sum of the first equation and the second equation multiplied by \( d_0^{g-1} \));
- \( b_0^g (c_0^g - d_0^g h^g) + (-1)^{g-2} d_0^{g-2} = 0 \) (obtained by raising to the \( p \)-th power the second equation).

Then, the minimal primes associated to \( I \) and to \( J \) in \( k[b_0, c_0, \ldots, c_{g-2}, d_0][b_0^{-1}, d_0^{-1}] \) are the same. We can write the second equation as \( c_0 (c_0 - d_0 h)^g = (-1)^{g-2} d_0^g d_0^{g-2}. \)

Let \( f(X) := X^{g+1} + h^g X - (-1)^{g-2} d_0 d_0^{g-2} \). Define the rings
\[
R_0 := k[c_1, \ldots, c_{g-2}, d_0], \quad R_1 := R_0[X]/(f(X)), \quad R_2 := R_1[b_0]/(b_0^g + X d_0^g).
\]

Since \( R_2 \) is \((d_0 X, c_1, \ldots, c_{g-2}, d_0, b_0)\)-adically complete and separated, the map of \( k[c_1, \ldots, c_{g-2}, d_0]-\)algebras from \( k[b_0, c_0, \ldots, c_{g-2}, d_0][b_0^{-1}, d_0^{-1}]/J \) to \( R_2[b_0^{-1}, d_0^{-1}] \) given by \( c_0 \mapsto d_0 X \) and \( b_0 \mapsto b_0 \) is well defined and it is an isomorphism. Let \( P \) be a prime ideal of \( R_1 \) containing \( 0 \). Then,

- either \( X \) is not a \( p \)-th power in \( \text{Frac}(R_1/P) \) and then, since \( R_2 \) is a flat \( R_1 \)-algebra, it follows that \( P \) is a prime ideal of \( R_2; \)

• or $X$ is a $p$-th power in $\text{Frac}(R_1/P)$ and then $Xd_0^p = t^p$ for some $t \in \text{Frac}(R_1/P)$. In this case let $P_2$ be a minimal prime ideal of $R_2$ containing $P$. By the going down theorem we must have $P_2 \cap R_1 = P$. Hence, $P_2$ defines a prime ideal in $(R_2/P) \otimes_{R_1} \text{Frac}(R_1/P) \cong \text{Frac}(R_1/P)[b_0]/(b_0 + t)^p$. Hence, $P_2$ must be the kernel of $R_2 \to R_2/P \to \text{Frac}(R_1/P)$ the latter map being $b_0 \mapsto -t$. Hence, $P_2$ is unique.

In any case the map $\text{Spec}(R_2) \to \text{Spec}(R_1)$ defines a one to one correspondence between the irreducible components of $\text{Spec}(R_2)$ and those of $\text{Spec}(R_1)$. Thus, we reduced the lemma to the irreducibility of $\text{Spec}(R_1)$.

Assume that the polynomial $X^{p+1} - c_1 X - (-1)^{p+1} d_0^{p(g-2)}$ factors as the product of the polynomials $f_1(X) = X^{m_1} + \ldots + \alpha_1 X + \alpha_0$ and $f_2(X) = X^{m_2} + \ldots + \beta_1 X + \beta_0$ over $k[c_1, d_0]$. Then, we have $\alpha_0 \beta_0 = -(1)^{p+1} d_0^{p(g-2)}$. Without loss of generality, we may assume that $\alpha_0 = u_0d_0^m$ for some integer $p(g-2) \geq m > 0$ and some $u_0 \in R_0$ not divisible by $d_0$. Since $f(X) = X^{p+1} - c_1 X = (X - c_1)^m X$ in the polynomial ring over $R_0/(d_0, c_2, \ldots, c_{g-2}) \cong k[c_1]$ (which is factorial), we must have $\beta_0 = \pm c_1 + v_0d_0$ for some integer $n > 0$ and $v_0 \in R_0$. In particular, $\beta_0 \equiv 0 \mod (d_0, c_1)$. Since $\alpha_0 \beta_0 = u_0d_0^{m}(\pm c_1 + v_0d_0)$, we get $\pm u_0c_1d_0^m = -(1)^{p+1} d_0^{p(g-2)} - u_0v_0d_0^{m+1}$. Since $u_0$ and $c_1$ are not divisible by $d_0$, we must have $m = p(g-2)$. Hence, $\beta_0u_0 = -(1)^{p(g-2)}$ i.e., $\beta_0$ is a unit (contradiction). This implies that the polynomial $f(X)$ is irreducible over $R_0$. Since $R_0$ is local and regular, it is also factorial and, in particular, normal. It follows from [Eis, Cor 4.12] that $R_1$ is an integral domain. Since it is a finite and flat extension of $R_0$, we conclude.

The following lemma shows that the situation is different if we start with a closed point $x$ of $W^{(1,1)}$.

**Lemma 6.3.5.** The notation is as in Lemma 6.3.4. The completed local ring $\hat{\mathcal{O}}_{D, x}$ of $\mathfrak{D}$ at $x$ has exactly two minimal associated prime ideals. Each of them has height 1 in $\hat{\mathcal{O}}_{\mathfrak{M}, x}$.

**Proof.** As in the proof of Lemma 6.3.4 the completed local ring $\hat{\mathcal{O}}_{\mathfrak{M}, x}$ of $\mathfrak{M}$ at $x$ is isomorphic to

$$k[[b_0, c_0, \ldots, c_{g-2}, d_0]]/(b_0c_0 - d_0b_0c_1 + d_0^3b_0c_2 - d_0^3b_0c_3 + \ldots + (-1)^{g-2}d_0^{g-2}b_0c_{g-2} + (-1)^{g-2}d_0^p).$$

The equations of the non-ordinary locus can be deduced as in § 9.2 and coincide with equations (Eq1)–(Eq4) given there. The reduced subscheme defined by $d_0 = 0$ coincides with the $W^{(1,1)}$ locus. Inverting $d_0$, we get that the non-ordinary locus in $k[[b_0, c_0, \ldots, c_{g-2}, d_0]][d_0^{-1}]$ is defined by the ideal $I$:

- $b_0c_0 - d_0b_0c_1 + d_0^3b_0c_2 - d_0^3b_0c_3 + \ldots + (-1)^{g-2}d_0^{g-2}b_0c_{g-2} + (-1)^g d_0^p = 0$;
- $-b_0^2 + d_0^2 - c_0^p d_0^{2-p} = 0$. 

The following lemma shows that the situation is different if we start with a closed point $x$ of $W^{(1,1)}$. 

**Lemma 6.3.5.** The notation is as in Lemma 6.3.4. The completed local ring $\hat{\mathcal{O}}_{D, x}$ of $\mathfrak{D}$ at $x$ has exactly two minimal associated prime ideals. Each of them has height 1 in $\hat{\mathcal{O}}_{\mathfrak{M}, x}$.
Let \( h := c_1 - d_0 c_2 + d_0^2 c_3 + \ldots - (-1)^{g-2} d_0^{g-3} c_{g-2} \). Consider the ideal \( J \) in the ring \( k[b_0, c_0, \ldots, c_{g-2}, d_0][d_0^{-1}] \) defined by

- \(-b_0^p + d_0^p - c_0 d_0^{p-1} = 0\);
- \(b_0^p (c_0^p - d_0^p h^p) + (-1)^p d_0^p = 0\).

Then, the minimal primes associated to \( I \) and to \( J \) in \( k[b_0, c_0, \ldots, c_{g-2}, d_0][d_0^{-1}] \) are the same. We can write the second equation as \((c_0 - d_0)(c_0 - d_0 h)^p = (-1)^p d_0^{p-1} \).

Let \( f(X) := X^{p+1} - X^p - h^p X + h^p - (-1)^g d_0^{p(g-2)} = 0 \). Define the rings

\[ R_0 := k[c_1, \ldots, c_{g-2}, d_0], \quad R_1 := R_0[X]/(f(X)), \quad R_2 := R_1[b_0]/(b_0^p - d_0^p + X d_0^p). \]

Since \( R_2 \) is \((d_0 X, c_1, \ldots, c_{g-2}, d_0, b_0)\)-adically complete and separated, the map of \( k[c_1, \ldots, c_{g-2}, d_0] \)-algebras from \( k[b_0, c_0, \ldots, c_{g-2}, d_0][d_0^{-1}]/J \) to \( R_2[d_0^{-1}] \) given by \( c_0 \mapsto d_0 X \) and \( b_0 \mapsto b_0 \) is well defined. It is easily checked that it is an isomorphism. As in the proof of Lemma 6.3.4 one concludes that the map \( \text{Spec}(R_2) \to \text{Spec}(R_1) \) defines a one to one correspondence between the irreducible components of \( \text{Spec}(R_2) \) and those of \( \text{Spec}(R_1) \).

By Hensel’s lemma, \( f(X) \) admits a root \( x \in R_0 \) which is congruent to 1 modulo \( R \). Write \( f(X) = (X - x)q(X) \) with \( q(X) \) prime to \( X - x \). Let \( R := R_0[X]/(q(X)) \). We claim that \( R \) is a domain. Since \( k[c_1, \ldots, c_{g-2}, d_0] \) is local and regular, it is also factorial and, in particular, normal. Therefore, the integrality of \( R \) is equivalent by [Eis, Cor 4.12] to the irreducibility of the polynomial \( q(X) \). It suffices to check the irreducibility of the reduction \( s(X) \) of \( q(X) \) modulo \((c_1, \ldots, c_{g-2})\). Let \( V \) be a normal, local, noetherian extension of \( k[d_0] \) such that \( s(X) \) admits a root \( z \in V \). Let \( y \) be the image of \( x \) in \( V \). Since \((X - y)s(X) = X^{p+1} - X^p - (-1)^g d_0^{p(g-2)}\) with \( y \) a unit, the element \( z \) is not a unit and \( z = 1 + (-1)^g d_0^{p(g-2)} z^{-p} \). Hence, \( z = (z')^p \) where \( z' \) satisfies \((z')^{p+1} - (z')^p - (-1)^g d_0^{p(g-2)} \). Applying inductively the same trick, we find that there exists a positive integer \( r \) prime to \( p \) and an element \( w \) in the maximal ideal of \( V \) such that \( w^{p+1} - w^p - (-1)^g d_0^p = 0 \). Hence, \( \text{val}_V(w) = \text{val}_V(w^{p+1} - w^p) = \text{val}_V(d_0) \). Therefore, \( \text{val}_V(d_0) \) is a multiple of \( p \) and so the degree of \( k[d_0] \subset V \) is \( \geq p \) and it must then be equal to \( p \) proving that \( s(X) \) is irreducible as claimed.

It follows that

\[ R_1 \cong R_0 \times R \]

is the product of two integral domains of dimension \( g - 1 \) which are flat \( R_0 \)-algebras. Since minimal associated primes behave nicely under localization [Eis, Thm 3.10(d)], the zero ideal in \( R_2[d_0^{-1}] \cong k[b_0, c_0, \ldots, c_{g-2}, d_0][d_0^{-1}]/J \) is contained in exactly two minimal prime ideals, each of codimension 1. \( \square \)
7. Intersection theory on a singular surface

We survey here intersection theory on complete surfaces with isolated normal singularities. The main reference for this theory is [RT1], [RT2]; see also [Mum, Arc].

By a singular surface we mean in this section an irreducible projective algebraic surface over an algebraically closed field whose only singularities are a finite number of isolated multiple points.

7.1. Definition of the intersection number.

Recall the notion of multiplicity. The multiplicity of a variety at a point $x$ is defined in terms of the local ring $\mathcal{O}_x$. Let $n = \dim(\mathcal{O}_x)$. There exists a polynomial $P$ such that $P(\ell) = \text{length}(\mathcal{O}_x / \mathfrak{m}_x^\ell)$ for $\ell \gg 0$. The multiplicity of $x$ is defined to be $n!$ times the leading coefficient of $P$ [Har, Ex. V 3.4]. For example, if $Y \subseteq \mathbb{P}^n$ is a variety of degree $d$, then the vertex of the cone over $Y$ is of multiplicity $d$.

Given a singular surface $V$, one can find a resolution of singularities, $\pi : V^* \rightarrow V$, such that $V^*$ is non-singular, $\pi$ is an isomorphism over the set $V^\circ := V \setminus V^{\text{sing}}$, $\pi^{-1}(V^{\text{sing}}) := \Upsilon$ (the “fundamental manifold”) is of pure dimension 1, each irreducible component of it is non-singular, every two irreducible components have at most simple intersections, no three components have a common point. Cf. [RT1]

Let $C \subseteq V$ be an irreducible curve. Define $\tilde{C}$, the strict transform, as the closure in $V^*$ of $\pi^{-1}(C \cap V^\circ)$. One says that $C_1 \equiv_Q C_2$ on $V$, and calls this relation algebraic equivalence with division, if for some $m > 0$ and some $\pi : V^* \rightarrow V$, $m(\tilde{C}_1 - \tilde{C}_2)$ is algebraically equivalent to a divisor supported on $\Upsilon$. This notion is independent of $V^*$ and defines an equivalence relation. Given a resolution of singularities $\pi : V^* \rightarrow V$ as above, let $\mu_1, \ldots, \mu_s$

be the irreducible components of $\Upsilon$. Let $d = (\mu_i \cdot \mu_j)_{i,j=1,\ldots,s}$, be the intersection matrix. It is an invertible, symmetric, negative definite matrix with no negative elements except on the diagonal. It follows that $k = -d^{-1}$ is a symmetric, positive definite matrix with no negative elements.

Let $C, D$ be two curves on $V$. One can find $V^*$ as above such that in addition: $\tilde{C}, \tilde{D}$ have no common point on $\Upsilon$, neither passes through a point of $\mu_i \cdot \mu_j$ and they intersect each $\mu_i$ simply. The contribution to the intersection multiplicity
coming from $V^{\text{sing}}$ is then
\[
\sum_{i,j} k_{ij} (\tilde{C} \cdot \mu_i)[\tilde{D} \cdot \mu_j] = (\ldots, \tilde{C} \cdot \mu_i, \ldots) k^t (\ldots, \tilde{D} \cdot \mu_i, \ldots).
\]

It will be convenient to denote the vector $(\ldots, \tilde{C} \cdot \mu_i, \ldots)$ by $C^\mathbb{T}$. The total intersection number is
\[
C \cdot D = \tilde{C} \cdot \tilde{D} + C^\mathbb{T} k^t D^\mathbb{T} \quad (7.1)
\]

One can prove [RT1] that this defines a symmetric bilinear pairing on divisor classes modulo $\equiv_{\mathbb{Q}}$.

### 7.2. Pull-back and intersection.

Let $\mu_1, \ldots, \mu_s$ be the irreducible components of $\mathbb{Y}$. We want to define for an irreducible curve $C$ in $V$ a divisor $C^*$ in $V^*$, such that
\[
C^* = \tilde{C} + \sum_{i=1}^s \gamma_i \mu_i, \quad (7.2)
\]

and such that
\[
C^* \cdot \mu_j = 0, \quad \forall j. \quad (7.3)
\]

Since $C^* \cdot \mu_j = \tilde{C} \cdot \mu_j + \sum_{i=1}^s \gamma_i \mu_i \mu_j$, we see that we need to solve the equation $d^t (\gamma_1, \ldots, \gamma_s) = -t \tilde{C}^\mathbb{T}$. This has a unique solution given by
\[
t^t (\gamma_1(C), \ldots, \gamma_s(C)) = k^t C^\mathbb{T}. \quad (7.4)
\]

The definition of $C^*$ extends by linearity to any divisor.

**Proposition 7.2.1.** The following identities hold.

1. Let $C$ be a divisor on $V$, then $C^* \cdot \mu_j = 0$ for all $j = 1, \ldots, s$.

2. Let $C, D$ be divisors on $V$, then $C^* \cdot D^* = C \cdot D$.

3. Let $C$ be a divisor on $V$ and $D$ a divisor on $V^*$, then $C^* \cdot D = C \cdot \pi_* D$.

**Proof.** The first part follows from the definition and the calculation above. For part (2), on the one hand, we have
\[
C \cdot D = \tilde{C} \cdot \tilde{D} + C^\mathbb{T} k^t D^\mathbb{T},
\]
and on the other hand

\[
C^* \cdot D^* = \left( \tilde{C} + \sum \gamma_i(C) \mu_i \right) \cdot \left( \tilde{D} + \sum \gamma_i(D) \mu_i \right)
\]

\[
= \left( \sum \gamma_i(C) \mu_i \right) \cdot D^* + C^* \cdot \left( \sum \gamma_i(D) \mu_i \right) + \tilde{C} \cdot \tilde{D}
\]

\[
- \left( \sum \gamma_i(C) \mu_i \right) \cdot \left( \sum \gamma_i(D) \mu_i \right)
\]

\[
= \tilde{C} \cdot \tilde{D} - \left( \sum \gamma_i(C) \mu_i \right) \cdot \left( \sum \gamma_i(D) \mu_i \right)
\]

\[
= \tilde{C} \cdot \tilde{D} - \sum_{i,j} \gamma_i(C) \gamma_j(D) \mu_i \cdot \mu_j
\]

\[
= \tilde{C} \cdot \tilde{D} - (\gamma_1(C), \ldots, \gamma_s(C)) d^i(\gamma_1(D), \ldots, \gamma_s(D))
\]

\[
= \tilde{C} \cdot \tilde{D} - (C^T k) d(k^i D^T)
\]

\[
= \tilde{C} \cdot \tilde{D} + C^T k^i D^T.
\]

For part (3), we calculate that

\[
C^* \cdot D = C^* \cdot (\pi_* D)^* - C^* \cdot (D - (\pi_* D)^*)
\]

\[
= C^* \cdot (\pi_* D)^*
\]

\[
= C \cdot \pi_* D.
\]

\[\square\]

### 7.3. Adjunction.

Let \( K[V^*] \) be the canonical divisor of \( V^* \) and let

\[
K = \pi_* K[V^*].
\]

We note that \( K \) is the unique extension of the canonical divisor on \( V^\circ \) and hence is independent of the choice of \( V^* \). We call it the canonical divisor of \( V \). One may ask if \( K \) satisfies the adjunction formula. The answer is NO as we show by a simple example:

Suppose that \( Y = \mu \) is irreducible and \( \mu^2 = -n \). This happens for example in the case of the blow-up at the origin of the curve over the curve \( x^n + y^n = z^n \).

Then \( \mu \cdot (\mu + K[V^*]) = 2g(\mu) - 2 \) and therefore \( \mu \cdot K[V^*] = 2g(\mu) - 2 + n \). Let \( C \) be a nonsingular curve passing simply through the point \( \pi(\mu) \) then \( C^* = \tilde{C} + \frac{1}{n} \mu \).
We find that
\[ C \cdot (C + K) = C^* \cdot (C^* + K[V^*]) \]
\[ = \left( \tilde{C} + \frac{1}{n} \mu \right) \cdot \left( \tilde{C} + \frac{1}{n} \mu + K[V^*] \right) \]
\[ = \tilde{C}^2 + \frac{1}{n} + \tilde{C} \cdot K[V^*] + \frac{1}{n} \mu \cdot K[V^*] \]
\[ = \tilde{C}^2 + \tilde{C} \cdot K[V^*] + \frac{1}{n} \left( \mu^2 + \mu \cdot K[V^*] + n + 1 \right) \]
\[ = 2g(\tilde{C}) - 2 + \frac{2g(\mu) + n - 1}{n} \]
\[ = 2g(C) - 2 + \frac{2g(\mu) + n - 1}{n}. \]

Since the term \( (2g(\mu) + n - 1)/n \) is not zero in general we see that adjunction does not hold in the same way.

**Proposition 7.3.1.** Define a vector \( \kappa^\mathcal{Y} \) as
\[
\kappa^\mathcal{Y} = -k \cdot (2g(\mu_1) - 2 - \mu_1^2, \ldots, 2g(\mu_s) - 2 - \mu_s^2) \\
= -k \cdot (\mu_1 \cdot K[V^*], \ldots, \mu_s \cdot K[V^*]) \\
= -k \cdot K[V^*]^\mathcal{Y}.
\]

Then
\[ K[V^*] = K^* + \sum_i \kappa_i \mu_i, \quad (7.5) \]
and
\[ C \cdot (C + K) = 2g(C) - 2 + C^\mathcal{Y} \cdot k \cdot (C^\mathcal{Y} + K[V^*]^\mathcal{Y}). \quad (7.6) \]

**Proof.** Write \( K[V^*] = K^* + \sum_i \kappa_i \mu_i \), where the \( \kappa_i \) need to be calculated. We have
\[ 2g(\mu_i) - 2 - \mu_i^2 = K[V^*] \cdot \mu_i \]
\[ = K^* \cdot \mu_i + \sum_j \kappa_j \mu_j \cdot \mu_i \]
\[ = \sum_j \kappa_j \mu_j \cdot \mu_i. \]

We conclude that \( k \cdot (2g(\mu_1) - 2 - \mu_1^2, \ldots, 2g(\mu_s) - 2 - \mu_s^2) = d \cdot (\kappa_1, \ldots, \kappa_s) \).

Write \( C^* = \tilde{C} + \sum_i \gamma_i(C) \mu_i \) and use
\[ C \cdot (C + K) = C^* \cdot (C^* + K^*) = \tilde{C} \cdot (C^* + K^*). \]
We get,
\[ C \cdot (C + K) = \tilde{C} \cdot \left( \tilde{C} + \sum_i \gamma_i(C)\mu_i + K[V^\ast] - \sum_i \kappa_i\mu_i \right) \]
\[= \tilde{C}^2 + \sum_i \gamma_i(C)\tilde{C} \cdot \mu_i + \tilde{C} \cdot K[V^\ast] - \sum_i \kappa_i\tilde{C} \cdot \mu_i \]
\[= \tilde{C}^2 + C^\ast \text{k} C^\ast + \tilde{C} \cdot K[V^\ast] - C^\ast \text{k} C^\ast \]
\[= 2g(C) - 2 + C^\ast \text{k} C^\ast - C^\ast \text{k} C^\ast \]
\[= 2g(C) - 2 + C^\ast \text{k} (C^\ast + K[V^\ast]^\ast). \]

\[ \square \]

**Remark 7.3.2.** Observe that if \( C \) passes through none of the singular points than adjunction holds in the usual sense.

### 8. Hilbert modular surfaces

Let \( L \) be a real quadratic field. We let \( \mathfrak{M} = \mathfrak{M}(\mu_N) \) be the moduli space with \( \mu_N \) level structure, where \( N \geq 4, (N, p) = 1. \)

#### 8.1. The inert case.

**8.1.1. Calculation of some intersection numbers.** Assume \( p > 2 \) in this section. To conform with the notation in §7 we let \( V \) be the Satake compactification of \( \mathfrak{M}, V^\ast \) a smooth toroidal compactification of \( \mathfrak{M}, V^\ast \pi : V^\ast \longrightarrow V \) the projection, \( V^\circ = \mathfrak{M} - \) the non-singular locus of \( V. \) We also let \( D_i = W_{(i+1)}. \) Let \( C(N) \) be the degree of \( \mathfrak{M} \) over the coarse moduli space of abelian surfaces with RM and no level structure.

Let \( \eta = \frac{1}{2}L(-1)C(N). \) We know [BG] that each \( D_i \) is a disjoint union of \( \eta \) non-singular rational curves, that \( D_1 \) and \( D_2 \) intersect transversely, the set of intersection points is the set of superspecial points, and that
\[ D_1 \cdot D_2 = \eta(p^2 + 1). \]

Let \( h \) be the total Hasse invariant [Gor]. It is a section of \( \mathcal{L}^p_{(-1)} \otimes \mathcal{L}_{(-1)}^2. \) Over \( V^\circ \) we know by Kodaira-Spencer that \( \det \Omega^1_{V/k} \cong \mathcal{L}_{(-1)}^2 \otimes \mathcal{L}_{(-1)}^2, \) thus
\[ K \sim \frac{2}{p - 1}(h) = \frac{2}{p - 1}(D_1 + D_2), \]
hence this also holds over \( V \) (since \( V \) is normal and \( V - V^\circ \) is of codimension 2. Note also that over \( V^\circ \) we have \( \mathcal{L}^p_{(-1)} \cong \mathcal{O}_{V^\circ}(D_1), \) as follows from the properties of the partial Hasse invariants [Gor]. Since \( D_i \) is closed in \( V^\circ \) we conclude that...
\( \mathbb{L}_i^p \mathbb{L}_{i+1}^{-1} \) extends to \( V \) and therefore we may define unique classes \( \ell_i \in CH(V) \otimes \mathbb{Q} \) so that
\[
c_1(\mathbb{L}_i^p \mathbb{L}_{i+1}^{-1}) = p\ell_i - \ell_{i+1}.
\]

Now,
\[
D_1 \cdot K = \sum_{C \in D_1} C \cdot K
\]
\[
= -2\eta - \sum_{C \in D_1} C^2
\]
\[
= -2\eta - D_1^2
\]
(adjunction, each \( C \cong \mathbb{P}^1 \))
\[
(D_1 \text{ is a disjoint union of its components}).
\]

On the other hand,
\[
D_1 \cdot K = \frac{2}{p-1} D_1 \cdot (D_1 + D_2) \quad \text{(Equation (8.2))}
\]
\[
= \frac{2}{p-1} D_1^2 + \frac{2}{p-1} \eta (p^2 + 1) \quad \text{(Equation (8.1)).}
\]

This yields
\[
D_1^2 = -2p\eta, \quad D_2^2 = -2p\eta.
\]

Solving for \( \ell_1, \ell_2 \), one finds
\[
\ell_1^2 = 0, \quad \ell_2^2 = 0, \quad \ell_1 \ell_2 = \eta. \quad \text{(8.3)}
\]

8.1.2. **On ampleness.** The sections of the line bundle \( \mathbb{L}_1^{a_1} \mathbb{L}_2^{a_2} \) are Hilbert modular forms of weight \((a_1, a_2)\). This motivates our interest in its ampleness.

**Theorem 8.1.1.** The class \( a_1 \ell_1 + a_2 \ell_2 \) is ample if and only if \( pa_1 > a_2 > \frac{1}{p} a_1 \).

**Proof.** We prove the claim by using the Nakai-Moishezon criterion. We first make some preliminary calculations.

Let \( C \) be a component of \( D_1 \). We have \( C^2 = -2 - C \cdot K \) by adjunction. On the other hand, \( C \cdot K = \frac{2}{p-1} C \cdot (D_1 + D_2) = \frac{2}{p-1} (C^2 + p^2 + 1) \), where we have used that \( D_1 \) is a disjoint union of its components, one of which is \( C \), and that \( C \cdot D_2 \) is the set of superspecial points on \( C \), which has cardinality \( p^2 + 1 \) [BG]. Therefore, \( C^2 = -2 - \frac{2}{p-1} (C^2 + p^2 + 1) \), which gives
\[
C^2 = -2p. \quad \text{(8.4)}
\]

We conclude that \( C \cdot D_1 = C^2 = -2p \) and \( C \cdot D_2 = p^2 + 1 \). Using that \( D_1 = p\ell_1 - \ell_2 \), \( D_2 = p\ell_2 - \ell_1 \), we solve for \( \ell_1, \ell_2 \) and get
\[
C \cdot \ell_1 = -1, \quad C \cdot \ell_2 = p. \quad \text{(8.5)}
\]

We conclude that if \( C \cdot (a_1 \ell_1 + a_2 \ell_2) > 0 \) then \( pa_1 > a_2 \). By symmetry, if \( C \) is a component of \( D_2 \) such that \( C \cdot (a_1 \ell_1 + a_2 \ell_2) > 0 \) then \( pa_2 > a_1 \).

Applying the Nakai-Moishezon criterion to \( a_1 \ell_1 + a_2 \ell_2 \), we conclude that if \( a_1 \ell_1 + a_2 \ell_2 \) is ample then \( pa_1 > a_2 > \frac{1}{p} a_1 \). We now claim that if \( pa_1 > a_2 > \frac{1}{p} a_1 \).
Lemma 8.2.1. Proof. Suppose that \( \alpha \) is a superspecial point on the moduli space \( \text{moduli space} \) following lemma holds for any totally real \( \text{field} \) structure can be defined over \( \text{field} \) the non-ordinary locus \( \text{locus} \) by the “Raynaud trick”, hence has positive intersection with \( \text{intersection} \) with \( b_1D_1 + b_2D_2 \).

8.2. The split case.

To conform with the notation of §7, we let \( V \) be the Satake compactification of \( \mathfrak{M} \), the moduli space with \( \mu_N \)-level structure, \( \mathcal{V}^* \) be a smooth toroidal compactification of \( \mathfrak{M} \), \( \pi : \mathcal{V}^* \to V \) the projection and \( \mathcal{V}^\circ = \mathfrak{M} \) the non-singular part of \( V \).

One knows that the non-ordinary locus consists of two divisors \( D_1 = W_{(1), \theta} \) and \( D_2 = W_{(\bar{1}), \theta} \) that intersect transversely; the intersection being the set of superspecial points. We also know that each \( D_i \) is a disjoint union of non-singular curves. See [BG]. However, we have very little information on the components of the \( D_i \). One can show that they are not Shimura curves. In the following, we obtain some information on the field of definition and genus of the components.

8.2.1. Fields of definition We examine the field of definition of the superspecial points and the non-ordinary locus, under some restriction on \( N \) and \( p \). The following lemma holds for any totally real field \( L \) of degree \( g > 1 \) and for any prime \( p \).

Lemma 8.2.1. Let \( N \geq 3 \) be an integer such that \( N|(p-1) \) or \( N|(p+1) \). Every superspecial point on the moduli space \( \mathfrak{M} \) of \( RM \) abelian varieties with \( \mu_N \)-level structure can be defined over \( \mathbb{F}_{p^2} \).

Proof. We use Honda-Tate theory for which [Wat] is a good reference. Consider the Weil numbers \( \pm p \) over \( \mathbb{F}_{p^2} \). There exist elliptic curves \( E_{\pm} \) over \( \mathbb{F}_{p^2} \) with that Weil number. The endomorphism ring of \( E_{\pm} \), at least after tensor with \( \mathbb{Q} \), is “the” quaternion algebra \( B_{p, \infty} \) over \( \mathbb{Q} \) ramified at \( p \) and \( \infty \). However, one easily sees that if \( f \in \text{End}_{\mathbb{F}_{p^2}}(E_{\pm}) \) and \( mf \in \text{End}_{\mathbb{F}_{p^2}}(E_{\pm}) \), for some non-zero integer \( m \), then \( f \in \text{End}_{\mathbb{F}_{p^2}}(E_{\pm}) \). It follows that \( \text{End}_{\mathbb{F}_{p^2}}(E) \) is a maximal order in \( B_{p, \infty} \).

The Frobenius endomorphism \( \pi : = \text{Fr}_{p^2} : E \to E \) is equal to \( \pm p \). It follows that \( E_{\pm}[N] \subseteq E_{\pm}(\mathbb{F}_{p^2}) \) iff \( N|(\pi-1) \) in \( \text{End}(E_{\pm}) \). But \( \pi = \pm p \) as an endomorphism and we conclude that \( E_{\pm}[N] \subseteq E_{\pm}(\mathbb{F}_{p^2}) \) iff \( N|(\pm p-1) \) as integers.

Note that \( \text{End}(E_{\pm}^N) = \text{M}_g(\text{End}(E_{\pm})) \) is defined over \( \mathbb{F}_{p^2} \). It follows that any \( \mathcal{O}_L \) structure on \( E_{\pm}^N \) is defined over \( \mathbb{F}_{p^2} \). Note also that \( E_{\pm}^N \) has an obvious polarization defined over \( \mathbb{F}_{p^2} \) induced from the canonical identification of \( E \) with its dual, and hence (using that polarization to identify the polarizations with the symmetric positive elements of \( \text{End}(E_{\pm}^N) \)) every polarization of \( \text{End}(E_{\pm}^N) \) is defined over \( \mathbb{F}_{p^2} \).
To conclude the proof we notice that by a theorem of P. Deligne every superspecial abelian variety of dimension \( g > 1 \) is isomorphic over \( \mathbb{F}_p \) to \( E^g \) and, under our assumptions, \( \mu_N \cong \mathbb{Z}/N\mathbb{Z} \) as group schemes over \( \mathbb{F}_p^2 \).

\[ \square \]

**Corollary 8.2.2.** Every component of \( D_i \) is defined over \( \mathbb{F}_p^2 \).

**Proof.** It is enough to show that if \( C \) is a component of \( D_i \) then \( \sigma(C) = C \) if \( \sigma \in \text{Gal}(\mathbb{F}_p^2/\mathbb{F}_p) \).

We first note that \( D_i \) is defined over \( \mathbb{F}_p \). Let \( x \in C \) be a superspecial point (such exists, because \( D_i \setminus W_{(1,1)} \) is quasi-affine). It is a \( \mathbb{F}_p^2 \) rational point of \( V \) and hence \( \sigma(C) \) is also a component of \( D_i \) passing through \( x \). However, there is a unique such component passing through \( x \). We conclude that \( \sigma(C) = C \).

\[ \square \]

### 8.2.2. Calculation of intersection numbers

We shall make the following assumption regarding continuity of intersection numbers (cf. Equation (8.3), Remark 8.3.3).

**Assumption:** \( \ell_1^2 = 0, \quad \ell_2^2 = 0, \quad \ell_1 \ell_2 = \eta. \)

It follows that

\[ D_1^2 = 0, \quad D_2^2 = 0, \quad D_1 \cdot D_2 = (p - 1)^2 \eta. \]  

(8.6)

Therefore,

\[
0 = D_1^2 = \sum_{C \in D_1} C^2 \\
= \sum_{C \in D_1} (2g(C) - 2 - C \cdot K) \\
= \sum_{C \in D_1} (2g(C) - 2) - D_1 \cdot K \\
= \sum_{C \in D_1} (2g(C) - 2) - (p - 1)\ell_1 \cdot 2(\ell_1 + \ell_2) \\
= \sum_{C \in D_1} (2g(C) - 2) - 2(p - 1)\eta.
\]

That is,

\[
(p - 1)\eta = \sum_{C \in D_1} (g(C) - 1).
\]

(8.7)

This already shows that on average the genus of components of \( C \) should be greater than 1.

We can do slightly better. Assume that \( N \geq 3 \) and either \( N|(p - 1) \) or \( N|(p + 1) \). Let \( \{C_1, \ldots, C_r\} \) be the irreducible components of \( D_1 \). Let \( r_i \) be the number of
superspecial points on $C_i$. Let $g_i$ be the genus of $C_i$, and $G = \sum_{i=1}^{\ell} g_i$. Then $R := \sum_{i=1}^{\ell} r_i = (p-1)^2 \eta$ and, together with Equation (8.8), we get,

$$R = (p-1) \sum_{i=1}^{\ell} (g_i - 1) = (p-1)(G - \ell).$$

(8.9)

We have the estimate $r_i > 0$ (‘Raynaud’s trick’), but since $r_i = \deg \mathfrak{L}_p^{-1}|_{C_i}$ (existence of partial Hasse invariants and simplicity of their zeros) we actually have $r_i \geq p - 1$. Summing over the components, we get

$$R \geq \ell(p-1).$$

(8.10)

We obtain the following:

**Proposition 8.2.3.** Assume that $N \geq 3$ and $N|(p-1)$ or $N|(p+1)$. Then the average genus $g$ of the non-ordinary locus satisfies the inequality $g = G/\ell \geq 2$.

**Proposition 8.2.4.** The line bundle $\mathbb{L}_1^{n_1}\mathbb{L}_2^{n_2}$ is ample if and only if both $n_1$ and $n_2$ are positive.

The proof follows the same lines as for Theorem 8.1.1.

### 8.3. The ramified case.

Again, to conform with the notation of §7, we let $V$ be the Satake compactification of $\mathfrak{M}$, the moduli space with $\mu_N$-level structure, $V^*$ be a smooth toroidal compactification of $\mathfrak{M}$ (sic!), $\pi : V^* \rightarrow V$ the projection. Let $V^o = V \setminus V^s$. For every $S \subset W_{1,1}$, let $\mu_S = \pi^{-1}(S)$.

#### 8.3.1. The local structure of the moduli space. First we compute the local deformation theory at a point of $\mathfrak{M}$. It follows from Example 4.3.1 that the moduli space is regular at points of type $(0, n)$, $0 \leq n \leq 2$. By loc. cit. at a point of type $(0, 2)$, the universal deformation ring is $\mathbb{K}[c_0, c_1]$. Recall Remark 6.1.3. We may take $m = \infty$ and $c_3 = 1$ as in (6.2) so that the universal Frobenius is

$$F = \begin{pmatrix} 0 & T^2 \\ 1 & -c_0^2 - c_0 T \end{pmatrix}.$$

A deformation has type $(0, 1)$ if and only if it is not ordinary. This is equivalent to $TF^2 \equiv 0 \pmod{T^2}$. Equivalently,

$$\begin{pmatrix} 0 & 0 \\ 1 & -c_0^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & -c_0^2 \end{pmatrix} = 0 \pmod{T}.$$

This gives the condition $c_0 = 0$. We conclude that in the local deformation space the condition for deforming into $W_{(0,1)}$ is given by $c_0 = 0$ and it defines a smooth formal curve.
By Example 4.3.2, at a point of type (1, 1) the universal deformation ring $R$ is defined by
\[ k[a_0, b_0, c_0, d_0]/(a_0 + d_0, a_0 d_0 - b_0 c_0) \cong k[a_0, b_0, c_0]/(a_0^2 + b_0 c_0). \]
Hence, $\text{Spec}(R)$ is a cone. By (6.2) we may take $m = \infty$ and $c_3 = 1$ so that the universal Frobenius is given by
\[ F = \begin{pmatrix} -b_0 & T^2 + a_0 \\ T - a_0 & -c_0 \end{pmatrix}. \]
In order to have deformation of $(0, 1)$ we must have $TF^2 = 0 \pmod{T^2}$, which is equivalent to
\[ F = \begin{pmatrix} -b_0 & a_0 \\ -a_0 & -c_0 \end{pmatrix} \begin{pmatrix} -b_0 & a_0 \\ -a_0 & -c_0 \end{pmatrix} = 0. \]
This gives the system of equations modulo $p$:
\[ b_0^{p+1} - a_0^{p+1} = 0, \quad a_0 b_0^p + c_0 a_0^p = 0, \quad b_0 a_0^p + a_0 c_0^p = 0, \quad -a_0^{p+1} + c_0^{p+1} = 0. \]
If $b_0 = 0$ it follows that $a_0$ and $c_0 = 0$ are nilpotent. The associated reduced scheme is the point we started with. Inverting $b_0$, the second equation can be eliminated using $b_0(a_0 b_0 + c_0 a_0^p) = a_0 b_0^{p+1} - a_0^2 c_0 = a_0(b_0^{p+1} - a_0^{p+1})$. If $a_0 = 0$, the associated reduced scheme is the point we started with. Inverting $a_0$ we deduce from $a_0(b_0 a_0^p + a_0 c_0^p) = b_0(a_0^{p+1} - c_0^{p+1})$ and from the other relations that $b_0 a_0^p + a_0 c_0^p = 0$. Hence, on the complement of the point we are reduced to the equations
\[ a_0^2 + b_0 c_0 = 0, \quad b_0^{p+1} - a_0^{p+1} = 0, \quad -a_0^{p+1} + c_0^{p+1} = 0. \] (8.11)
We conclude that the non-ordinary locus consists of $p + 1$ branches given by $b_0 = \zeta a_0$ and $c_0 = \zeta^{-1} a_0$ for $\zeta$ a $p + 1$-st root of unity.

Finally, we compute the structure of $\pi_1: \mathcal{R} \rightarrow \mathcal{M}$. The morphism $\pi_1$ is projective [AG1, Lemma 8.4]. Outside $\pi_1^{-1}(W(1,1))$ it is one-to-one [AG1, Prop 8.7] and so is an isomorphism. Since $\mathcal{M}/W(1,1)$ is smooth, we conclude that $\pi_1^{-1}(\mathcal{M}/W(1,1))$ is smooth. Let $s \in W(1,1)$. Let $R := k[a_0, b_0, c_0]/(a_0^2 + b_0 c_0)$ be the completed local ring of $\mathcal{R}$ at $s$. Let $A \rightarrow \text{Spec}(R)$ be the universal abelian scheme over $R$. Using the theory of local models §4.3.2, we can find a $R \otimes_k k[T]/(T^p)$-basis $\alpha, \beta$ of $H^1_{dR}(A/R)$ such that the relative cotangent space $H^0(A, \Omega^1_{A/R})$ in $H^1_{dR}(A/R)$ is generated as $R \otimes_k k[T]/(T^p)$-module by $(T + a_0)\alpha + b_0 \beta$ and $c_0 \alpha + (T - a_0)\beta$. The scheme $\mathcal{R} \times_{\mathcal{M}} \text{Spec}(R)$ represents the Grassmanian of $R \otimes_k k[T]/(T^p)$-submodules of $H^0(A, \Omega^1_{A/R})$, free as $R$-modules and killed by $T$. Any such module is generated by an element $TX + TZ\beta$ which is zero in $H^1_{dR}(A/R)/H^0(A, \Omega^1_{A/R})$. Hence,
\[ \mathcal{R} \times_{\mathcal{M}} \text{Spec}(R) \cong \text{Proj} R[X, Z]/(a_0 X + c_0 Z, -b_0 X + a_0 Z), \] (8.12)

**Proposition 8.3.1.** The following hold:
- the singular points of $\mathcal{M}$ are the cusps and the points contained in $W(1,1)$;
Hilbert modular varieties of low dimension

- the variety \( \mathcal{M} \) is smooth over \( k \);

- \( \pi: \mathcal{M} \to \mathcal{M} \) is the blow-up along \( W_{(1,1)} \);

- for every \( s \in W_{(1,1)} \), the scheme \( \mu_s \) is a non-singular rational curve with self intersection \(-2\).

Proof. It follows from (8.12) that \( \mathcal{M} \) is a smooth variety. Let \( \overline{V^o} \) be the blow-up of \( \mathcal{M} \) at \( W_{(1,1)} \). Since \( W_{(1,1)} \) is reduced, we also get that the inverse image of \( W_{(1,1)} \) is a disjoint union of curves and, hence, is a divisor. By the universal property of blow-up we get a birational map \( \rho: \mathcal{M} \to \overline{V^o} \) compatible with the projections onto \( \mathcal{M} \). It is an isomorphism over \( \mathcal{M} \backslash W_{(1,1)} \). The completed local ring of \( \mathcal{M} \) at a point of \( W_{(1,1)} \) is isomorphic to \( R = k[a_0, b_0, c_0]/(a_0^2 + b_0 c_0) \). Since the blow-up is defined in terms of \textit{Proj} of the ideal defining \( W_{(1,1)} \) and \( W_{(1,1)} \) is reduced, the fibre product \( \overline{V^o} \times_\mathcal{M} \text{Spec}(R) \) coincides with the blow-up of \( \text{Spec}(R) \) at its closed point. In particular, the inverse image of the closed point of \( R \) in \( \overline{V^o} \times_\mathcal{M} \text{Spec}(R) \) is isomorphic to \( \mathbb{P}^1_k \) and has self intersection \(-2\). Using (8.12) one easily checks that the base change of \( \rho \) to the product of the completed local rings at the points of \( W_{(1,1)} \) is an isomorphism. Hence, \( \rho \) is an isomorphism. \( \Box \)

8.3.2. Calculation of some intersection numbers. Assume that \( p > 2 \) in this section. Let \( D \) be the divisor that is equal to the non-ordinary locus of \( V \). Let \( h \) be the total Hasse invariant, \( h \in \Gamma(V^o, \det E^{p-1}); \) it admits a square root \( \sqrt{h} \in \Gamma(V^o, \det E^{(p-1)/2}) \) - see [AG2]. We have \( (\sqrt{h}) = D \). It follows from the Kodaira-Spencer isomorphism that (initially on \( V^o \), but then on \( V \))

\[
K \sim \frac{4}{p-1} D. \tag{8.13}
\]

We know [BG] that the number of components of \( D \) is \( \eta = \frac{1}{2}(\zeta_L(-1))C(N) \), where \( C(N) \) is the degree of the level structure, and that the number of points of \( W_{(1,1)} \) is also \( \eta \). We also note that Proposition 8.3.1 implies that the variety \( V^o \) is suitable for calculating the intersections of divisors support on \( D \). The following calculations are done using the results and notations of §7. On the one hand,
\[ D^2 = (D^*)^2 \]
\[ = \left( \sum_{C \in D} \tilde{C} + \frac{p + 1}{2} \mu_{W(1,1)} \right)^2 \]
\[ = \left( \sum_{C \in D} \tilde{C} \right)^2 + (p + 1) \left( \sum_{C \in D} \tilde{C} \right) \cdot \mu_{W(1,1)} + \frac{(p + 1)^2}{4} \mu_{W(1,1)}^2 \]
\[ = \sum_{C \in D} \tilde{C}^2 + (p + 1) \sum_{u \in W(1,1)} \sum_{C \in D} \tilde{C} \cdot \mu_u + \frac{(p + 1)^2}{4} \sum_{u \in W(1,1)} \mu_u^2 \]
\[ = \sum_{C \in D} \tilde{C}^2 + \frac{(p + 1)^2}{2} \eta. \]

On the other hand,
\[ \sum_{C \in D} \tilde{C}^2 = \sum_{C \in D} (-2 - \tilde{C} \cdot K[V^*]) \quad \text{(adjunction on } V^*) \]
\[ = \sum_{C \in D} (-2 - \tilde{C} \cdot K^*) \quad \text{(Prop. 7.3.1 + Prop. 8.3.1)} \]
\[ = -2\eta - \sum_{C \in D} C^* \cdot K^* \quad \text{(Prop. 7.2.1)} \]
\[ = -2\eta - \sum_{C \in D} C \cdot K \quad \text{(Prop. 7.2.1)} \]
\[ = -2\eta - \frac{4}{p-1} D^2 \quad \text{(Equation (8.13))} \]
\[ = -2\eta - \frac{4}{p-1} D^2. \]

We conclude that \[ D^2 = -2\eta - \frac{4}{p-1} D^2 + \frac{(p + 1)^2}{2} \eta, \] which gives:

**Proposition 8.3.2.** The self intersection of \( D \) is given by
\[ D^2 = \frac{(p - 1)^2}{2} \eta. \]

**Remark 8.3.3.** Note that if we could argue by ‘continuity of intersection numbers’, we could write \( D = \frac{p-1}{2} (\ell_1 + \ell_2) \), whence \( D^2 = \frac{(p-1)^2}{2} \ell_1 \cdot \ell_2 = \frac{(p-1)^2}{2} \eta. \)

### 9. Hilbert modular threefolds

Let \( L \) be a totally real cubic field. In this section we study the local structure of the moduli variety \( \mathcal{M} \). Given the results for \( g = 2 \) and the unramified case, we may restrict our attention to the case when \( p = \mathfrak{p}^3 \) is maximally ramified. Assume that henceforth.
We recall from § 3.2 the strata and their hierarchy in terms of “being in the closure” as encoded in the following diagram

\[
\begin{array}{ccc}
(1, 2) & (0, 3) \\
\downarrow & \downarrow \scriptstyle g=3 \\
(1, 1) & (0, 2) \\
\downarrow & \downarrow \\
(0, 1) \\
\end{array}
\]

To begin with, it follows from Example 4.3.1 that the locus \(W_{(j, n)}^c\) for \(j = 0\) and \(n = 0, \ldots, 3\), or for \(j = 1\) and \(n = 1, 2\) (performing a similar computation), is formally smooth at points of type \((j; n)\) with \(n \geq n'\). Thus, we are interested in the structure of the strata \(W_{(0, 1)}^c\) at a point of type \((1, 1)\) and \((1, 2)\), and \(W_{(0, 2)}^c\) at a point of type \((1, 2)\).

### 9.1. Points of type \((1, 2)\).

In this case \(j = 1, i = 2\), and, since the point is superspecial, we may assume \(m = \infty\) and \(c_3 = 1\) in (6.2). The universal deformation space is of the form (cf. Example 4.3.3):

\[
\kappa[[a_0, a_1, b_0, c_0, c_1, d_0]]/(a_0 d_0 - b_0 c_0, a_0 + a_1 d_0 - b_0 c_1, a_1 + d_0) \\
\cong \kappa[[a_0, b_0, c_0, c_1, d_0]]/(a_0 d_0 - b_0 c_0, a_0 - d_0^2 - b_0 c_1).
\]

The results of §5.6 imply that the universal “mod p” Frobenius is given over this ring by

\[
F = \begin{pmatrix} -b_0^{\sigma} & T^2 + a_0^{\sigma} - d_0^{\sigma} T \\ T + d_0^{\sigma} & -c_0^{\sigma} - c_1^{\sigma} T \end{pmatrix}.
\]  

(9.1)

### 9.1.1. The non-ordinary locus \(W_{(0, 1)}^c\).

By Corollary 5.6.3, the condition that the deformation is non-ordinary is equivalent to the condition

\[
\begin{pmatrix} -b_0^{\sigma} & a_0^{\sigma} \\ d_0^{\sigma} & -c_0^{\sigma} \end{pmatrix} \begin{pmatrix} -b_0^{\sigma 2} & a_0^{\sigma 2} \\ d_0^{\sigma 2} & -c_0^{\sigma 2} \end{pmatrix} \equiv 0 \pmod{T}.
\]

This gives the following system of equations
We note that if any of the variables $a_0$, $b_0$, $c_0$, or $d_0$ is zero then so is a power of all the others. In this case, the associated reduced subscheme defines a smooth 1-dimensional deformation which coincides with the $j = 1$ locus, generically having invariants $(1,1)$. Else, to find the components of the non-ordinary locus, we may invert $a_0$, $b_0$, $c_0$, and $d_0$. Using (Eq5) one checks that

$$b_0 \cdot (Eq4) = d_0 \cdot (Eq2), \quad b_0 \cdot (Eq3) = d_0 \cdot (Eq1), \quad b_0^p \cdot (Eq2) = a_0^p \cdot (Eq1).$$

Thus, we may consider only the three equations (Eq1), (Eq5), (Eq6). Substituting using $a_0 = b_0c_1 + d_0^2$ we reduce to the equations

$$b_0^{p+1} + b_0c_1d_0^p + d_0^{p+2} = 0, \quad b_0c_1d_0 + d_0^3 - b_0c_0 = 0$$

in the ring $k[b_0, c_0, c_1, d_0][b_0^{-1}, c_0^{-1}, c_1^{-1}, d_0^{-1}]$. Multiply the second equation by $d_0^{p-1}$ and subtract from the first equation to reduce to the equations

$$b_0^p + c_0d_0^{p-1} = 0, \quad d_0^3 - b_0c_0 + b_0c_1d_0 = 0.$$

In order to compute the components of the non-ordinary locus through the given point, one proceeds as in the proof of 6.3.4 and computes the minimal prime ideals of $k[b_0, c_0, c_1, d_0][b_0^{-1}, c_0^{-1}, c_1^{-1}, d_0^{-1}]$ associated to the ideal defined by the equations

$$b_0^p + c_0d_0^{p-1} = 0, \quad d_0^{p+1} + c_0^{p+1} - c_0c_1^p d_0^p = 0.$$

As in loc. cit. one concludes that those prime ideals are in one to one correspondence with the minimal prime ideals associated to 0 in $R_1 := k[c_1, d_0][c_0]/(c_0^{p+1} - c_0c_1^p d_0^p + d_0^{p+1})$ not containing $d_0$. Since the polynomial $c_0^{p+1} + d_0^{p+1}$ in the variable $c_0$ is irreducible over $R_1/(c_1) = k[d_0]$, one concludes that $R_1$ is a domain.

We conclude that the non-ordinary locus is locally irreducible at points of type $(1,2)$. One can also calculate that the tangent space at a point of type $(1,2)$ to the deformation space into non-ordinary abelian varieties (given by (Eq1)-(Eq6)) is three dimensional and conclude that every point of type $(1,2)$ is a singular point of $W^c_{(0,1)}$.

**9.1.2. The locus $W^c_{(0,2)}$.** We next consider the problem of deforming a point of type $(1,2)$ into the $(0,2)$ locus. The condition that the $a$-number is at least 2 is
equivalent to the condition $TF^2 \equiv 0 \pmod{T^3}$, where $F$ is given by

$$F = \begin{pmatrix} -b_0^s & T^2 + a_0^s - d_0^s T \\ T + d_0^s & -c_0^s - c_1^s T \end{pmatrix}.$$ 

This is equivalent to the following matrix being congruent to $0$ modulo $T^2$:

$$\begin{pmatrix} -b_0^s & a_0^s - d_0^s T \\ T + d_0^s & -c_0^s - c_1^s T \end{pmatrix} \begin{pmatrix} -b_0^{s^2} & a_0^{s^2} - d_0^{s^2} T \\ T + d_0^{s^2} & -c_0^{s^2} - c_1^{s^2} T \end{pmatrix}.$$ 

This provides the following equations:

(Eq1) $a_0 d_0 - b_0 c_0 = 0$

(Eq2) $a_0 - d_0^2 - b_0 c_1 = 0$

(Eq3) $b_0^{p+1} + a_0 d_0^p = 0$

(Eq4) $a_0 - d_0^{p+1} = 0$

(Eq5) $d_0 b_0^p + c_0 d_0^p = 0$

(Eq6) $b_0^p + c_0 + c_1 d_0^p = 0$

(Eq7) $b_0 a_0^p + a_0 c_0^p = 0$

(Eq8) $b_0 d_0^p - a_0 c_0^p + d_0 c_0^p = 0$

(Eq9) $d_0 a_0^p + c_0^{p+1} = 0$

(Eq10) $a_0^p - d_0^{p+1} + c_1 c_0^p + c_0 c_1^p = 0$

We now substitute using (Eq4) $a_0 = d_0^{p+1}$ and obtain the following equations in the variables $b_0, c_0, c_1,$ and $d_0$:

(Eq1) $d_0^{p+2} - b_0 c_0 = 0$

(Eq2) $d_0^{p+1} - d_0^2 - b_0 c_1 = 0$

(Eq3) $b_0^{p+1} + d_0^{2p+1} = 0$

(Eq5) $d_0 b_0^p + c_0 d_0^p = 0$

(Eq7) $b_0 c_0 + c_1 d_0^p = 0$

(Eq8) $b_0 d_0^p - d_0^{p+1} c_1 + d_0 c_0^p = 0$

(Eq9) $d_0 d_0^{p+1} + c_0^{p+1} = 0$

(Eq10) $d_0^{p+1} - d_0^{p+1} + c_1 c_0^p + c_0 c_1^p = 0$

We distinguish two cases:

**Case 1:** $d_0 = 0$. 
This implies that a power of $b_0$ and of $c_0$ is zero. The associated reduced subscheme is the smooth curve given by $c_1$, which is the $(1, 1)$ curve already noticed above.

**Case 2:** we invert $d_0$.

We now multiply each equation by a suitable power of $d_0$ so that to substitute expressions of the form $c_0d_0^{ \alpha}$ by $-d_0^{ \alpha+1}d_0$ (using (Eq5)). We remark that the elimination of $c_0$ was justified by (Eq6). We arrive at the following system of equations in $b_0$, $c_1$, and $d_0$:

\[
\begin{align*}
(Eq1') & \quad d_0^{2p+1} + b_0^{p+1} = 0 \\
(Eq2') & \quad d_0^{p+1} - d_0^2 - b_0c_1 = 0 \\
(Eq6') & \quad b_0^\alpha d_0^{p-1} - b_0^p + c_1 d_0^{p-1} = 0 \\
(Eq7') & \quad b_0 d_0^{2p-2p-1} - b_0^{p+1} = 0 \\
(Eq8') & \quad -b_0 d_0^{p-1} + d_0^p c_1 + b_0^p = 0 \\
(Eq9') & \quad d_0^{2p^2+p} - b_0^{2p^2} = 0 \\
(Eq10') & \quad d_0^{2p^2} - b_0^{2p+1} - b_0^2 c_1 - d_0^{2p-2p+1}b_0^{p+1} = 0 
\end{align*}
\]

Note that (Eq1') implies that $b_0 \neq 0$ and implies (Eq7') and (Eq9'). We may therefore consider only the system

\[
\begin{align*}
(Eq1') & \quad d_0^{2p+1} + b_0^{p+1} = 0 \\
(Eq2') & \quad d_0^{p+1} - d_0^2 - b_0c_1 = 0 \\
(Eq6') & \quad b_0^\alpha d_0^{p-1} - b_0^p + c_1 d_0^{p-1} = 0 \\
(Eq8') & \quad -b_0 d_0^{p-1} + d_0^p c_1 + b_0^p = 0 \\
(Eq10') & \quad d_0^{2p^2} - b_0^{2p+1} - b_0^2 c_1 - d_0^{2p-2p+1}b_0^{p+1} = 0 
\end{align*}
\]

We now show that (Eq1') and (Eq2') imply (Eq6') and (Eq8'), (Eq10'). Indeed, multiplying (Eq6') by $b_0$ we get

\[
b_0(Eq6') = d_0^{2p+1} + (-d_0^2 + d_0^{p+1})d_0^{2p-1} - d_0^{2p+1}d_0^{p-1} = 0.
\]

Multiplying (Eq8') by $b_0^p$, we get

\[
b_0^p(Eq8') = d_0^{2p+1}d_0^{p-1} + (-d_0^2 + d_0^{p+1})p d_0^p + (-d_0^{2p+1})p = 0.
\]

Finally,

\[
(Eq10') = d_0^{2p^2} - d_0^{2p+1} - (-d_0^{2p+1})p(-d_0^2 + d_0^{p+1}) - d_0^{2p-2p+1}(-d_0^2 + d_0^{p+1})p = 0.
\]

Hence, we are left with the system of equations
Recall that these equations are taken in a ring where $d_0$ is invertible, viz. in the ring $k[b_0, c_1, d_0]\langle d_0^{-1} \rangle$. If $I$ is the ideal generated by the equations (Eq1'), (Eq2') then the ring $k[b_0, c_1, d_0]\langle d_0^{-1} \rangle/I$ is equal to the ring

$$k[b_0, c_1, d_0]/(b_0^{-1}, d_0^{p+1} + b_0^{p+1}, d_0^{-1} - d_0^2 - b_0c_1).$$

Hence, we can eliminate $c_1$, putting $c_1 = d_0^2(d_0^{-1} - 1)b_0^{-1}$ (note that $c_1^{p+1} = -d_0(d_0^{-1} - 1)^{p+1}$, justifying the substitution) and conclude that the $(0,2)$-locus is given locally at a point $(1, 2)$ by the irreducible equation

$$d_0^{2p+1} + b_0^{p+1} = 0$$

in the ring $k[b_0, d_0]$ and hence is irreducible there.

### 9.2. Points of type $(1,1)$.

In this case $j = n = 1$ and $i = 1$. Hence, we may assume that $c_3 = 1$ in (6.2). The universal deformation space of $[\mathcal{A}_0]$ is defined by the ring

$$R := k[a_0, a_1, b_0, c_0, c_1, d_0]/(a_0d_0 - b_0c_0, a_1d_0 + a_0 - b_0c_1, a_1 + d_0).$$

The matrix $M$ of Frobenius $F$ of the universal display is defined by

$$
\begin{pmatrix}
T - b_0^p + d_0^p & T^2 + a^p - c^p \\
T + d_0^p & -c^p
\end{pmatrix}
$$

with $a := a_0 + a_1T$ and $c := c_0 + c_1T$. The deformations in the non-ordinary locus, i.e., inside $W'_{(0,1)}$, are defined by the condition that $T^2F^2 = 0$ modulo $T$. This is equivalent to require that $M \cdot M^t = 0 \mod T$, i.e., to the vanishing of

$$
\begin{pmatrix}
-b_0^p + d_0^p & a_0^p - c_0^p \\
d_0^p & -c_0^p
\end{pmatrix}
\begin{pmatrix}
-b_0^2 + d_0^2 & a_0^2 - c_0^2 \\
a_0^2 & -c_0^2
\end{pmatrix}
$$

which is equal to

$$
\begin{pmatrix}
(b_0^2 + d_0^2 - a_0^2) + a_0^2 c_0^2 - c_0^2 d_0^2 & -b_0^2 + 2b_0^2 d_0 + a_0^2 d_0^2 - c_0^2 d_0^2 - a_0^2 c_0^2 + c_0^2 d_0^2 + a_0^2 c_0^2 + c_0^2 d_0^2 \\
-b_0^2 + a_0^2 c_0^2 & a_0^2 c_0^2 - c_0^2 d_0^2 + c_0^2 d_0^2 + a_0^2 c_0^2 + c_0^2 d_0^2
\end{pmatrix}
$$

Hence, we get the following seven equations in the variables $a_0$, $a_1$, $b_0$, $c_0$, $c_1$ and $d_0$:
Case 1: Assume $d_0 = 0$. Then, a power of $b_0$ is 0 from (Eq1), a power of $c_0$ is 0 from (Eq4), $a_0 = 0$ from (Eq6) and $a_1$ is 0 from (Eq7). The only free variable left is $c_1$. Hence, the reduced subscheme defined by $d_0 = 0$ is 1-dimensional and coincides with universal deformation space inside the locus $W_{(1,1)}^c$, as already known.

Case 2: Let us invert $d_0 = 0$. Then

- $d_0^3(Eq1) = (-b_0^p + b_0 d_0^p)(Eq2) + d_0^{2p}(Eq5)$;
- $c_0^2(Eq2) = -d_0^2(Eq4) + d_0^p(Eq5)$;
- $d_0^{2p}(Eq3 - Eq4) = -b_0^p d_0^p(Eq5)p^2 - d_0^p c_0^2(Eq5)p + b_0^p d_0^p(Eq2)$.

Hence, using $a_1 = -d_0$, $a_0 = d_0^3 + b_0 c_1$, (Eq5) and $d_0^{-p}(Eq2)$, the system of equations (Eq1)–(Eq7) becomes equivalent to the system of equations

- $d_0^3 + b_0 c_1 d_0 - b_0 c_0 = 0$;
- $-b_0^p + d_0^p - c_0 d_0^{p-1} = 0$;

in $k[b_0, c_0, c_1, d_0][d_0^{-1}]$.

It follows from 6.3.5 that the nilradical of the ideal defined by these equations has exactly two minimal prime ideals. Hence, the locus $W_{(1,1)}^c$ is not analytically irreducible at the points of $W_{(1,1)}$. Studying the tangent space it is easily seen that $W_{(1,1)}^c$ is singular in $W_{(0,1)}^c$.

9.3. Summary.

We now come to some conclusions concerning the global structure of moduli space $\mathfrak{M}$ for $L$ cubic totally real field and $p$ maximally ramified in $L$.

Let $\mathfrak{M}$ be any component of $\mathfrak{M}$. By Proposition 6.3.3 the non-ordinary locus is irreducible. The locus $W_{(1,1)}^c = W_{(1,1)} \cup W_{(1,2)}$ is irreducible and non-singular, by loc. cit. and (3.1). The locus $W_{(0,2)}^c = W_{(0,2)} \cup W_{(1,2)} \cup W_{(0,3)}$ is a union of Moret-Bailly families, each component is singular only at the unique point (cf. [AG1,
Prop. 6.8]) of $W_{(1,2)}$ lying on it. The components of the locus $W_{(0,2)}^c$ are disjoint, because intersection points can only be of type $(1,2)$, and by § 9.1.2 the locus is locally irreducible there.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{hilbert_modular_threefold_maximally_ramified_case}
\caption{Hilbert modular threefold - maximally ramified case.}
\end{figure}

One can prove that the $W_{(0,1)}^c$ locus and the $W_{(1,1)}^c$ locus are irreducible in each component of the moduli space in a different way. In fact, a similar use of the correspondences $\pi_1 \pi_2^{-1}$, $\pi_2 \pi_1^{-1}$, shows that one is irreducible if an only if the other is. We know by Theorem 6.2.3 that $\mathcal{T}_2 = W_{(1,1)}^c \cup W_{(0,2)}^c$ is connected, we know that each component of $W_{(0,2)}^c$ meets $W_{(1,1)}^c$ at a unique point, and we know that the locus $S_1 = W_{(1,1)} \cup W_{(1,2)}$ is non-singular. The implies that there is a unique component of $W_{(1,1)}$ in every component of $\mathcal{M}$.

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\section*{References}


Hilbert modular varieties of low dimension


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