Canonical Subgroups over Hilbert Modular Varieties

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Related work

In various settings:

Lubin, Katz, Abbes-Mokrane, Andreatta-Gasbarri, Kisin-Lai, Nevens, Rabinoff, Fargues, B. Conrad, Lau, Buzzard-Taylor (?), and perhaps others...
1 Introduction

2 Stratifications of the special fibers

3 Valuations on $X_{\text{rig}}, Y_{\text{rig}}$

4 Construction of the canonical subgroup
The Problem. Associate in a natural way to a $g$-dimensional abelian variety $A/R$, $[R : \mathbb{Z}_p] < \infty$, with real multiplication, an invariant isotropic subgroup $H$ such that $pH = \{0\}$, and $\#H = p^g$.

Motivation. Properties of the $U$-operator on overconvergent modular forms; used to prove classicality results for modular forms, study of $p$-adic families of modular forms, special values of $L$-functions, and modularity of Galois representations.

If $A$ has ordinary reduction: classical ✓
(There is a unique way to lift the kernel of Frobenius $\text{Fr}: \overline{A} \to \overline{A}^{(p)}$, where $\overline{A} = A \pmod{pR}$.)

Therefore, the problem is:

- Extend this to non-ordinary abelian varieties;
- Do it in families.
Reformulation

For appropriate moduli schemes $X$, $Y$ over $\mathbb{Z}_p$, $\mathcal{X}_{\text{rig}}$, $\mathcal{Y}_{\text{rig}}$ the associated (Raynaud) generic fibers, we have a diagram,

$$(A, H) \xrightarrow{\sim} \mathcal{Y}_{\text{rig}} \xrightarrow{\pi} \mathcal{X}_{\text{rig}},$$

and the section $s$ exists over the ordinary locus.

The Problem. Extend $s$ “as much as possible”.
The main theorem

Theorem (G. - Kassaei)

Let \( \tilde{h}_\beta \) be (Zariski local) lifts of the partial Hasse invariants. Let \( U \subset X_{\text{rig}} \) be

\[
U = \{ P : \nu(\tilde{h}_\beta(P)) + p\nu(\tilde{h}_{\sigma^{-1}\beta}(P)) < p, \ \forall \beta \in \mathbb{B} \}.
\]

There exists a section \( s^\dagger : U \to Y_{\text{rig}}, \) extending the section \( s \) on the ordinary locus.
What comes into the proof?

1. Stratifications of $\overline{X}$, $\overline{Y}$ (the special fibers).
2. Study of $\pi : \overline{Y} \rightarrow \overline{X}$ on completed local rings.
3. “Dissection” of $\mathfrak{Y}_{\text{rig}}$, the generic fiber of $Y$, using $g$ different valuations.

($g = [L : \mathbb{Q}]$, where $L$ is the totally real field acting.)

Remark. *The structure suggests strategy should be applicable to many Shimura varieties of PEL type.*
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Notation

- $L$ - totally real field, $[L : \mathbb{Q}] = g$.
- $p$ unramified in $L$.
  \[
  \mathbb{B} = \text{Hom}(L, \mathbb{Q}_p^{ur}) = \bigsqcup_{p \mid p} \mathbb{B}_p \circlearrowleft \sigma.
  \]
  ($\sigma = \text{Frobenius, lift of } x \mapsto x^p.$)

- For $S \subseteq \mathbb{B}$, let $S^c = \mathbb{B} \setminus S$ and
  \[
  \ell(S) = \{\sigma^{-1} \circ \beta : \beta \in S\}, \quad r(S) = \{\sigma \circ \beta : \beta \in S\}.
  \]

- $\kappa = \text{minimal field } \supseteq \mathcal{O}_L/p, \ \forall p \mid p$.

- $\mathcal{O}_L \otimes_{\mathbb{Z}} \mathcal{W}(\kappa) \cong \bigoplus_{\beta \in \mathbb{B}} \mathcal{W}(\kappa)_\beta$
  induces a decomposition of any $\mathcal{O}_L \otimes \mathcal{W}(\kappa)$-module.
Moduli Spaces

$X/W(\kappa)$ parameterizes $A = (A, \iota, \alpha, \lambda_A)/S$, where $S$ is a $W(\kappa)$-scheme and:

- $A \to S$ abelian scheme, of rel. dim' n $g$, $\iota : O_L \hookrightarrow \text{End}_S(A)$,
- $\alpha = \text{rigid } \Gamma_{00}(N)$-level structure.
- $\lambda_A : (a, a^+) \xrightarrow{\sim} (\mathcal{P}_A, \mathcal{P}_A^+)$ a polarization: $A \otimes a \cong A^t$,
- $\mathcal{P}_A = \text{Hom}_{O_L}(A, A^t)^{\text{sym}}$ with the positive cone of polarizations.

$Y/W(\kappa)$ parameterizes $(A, H)$ such that:

- $H$ is killed by $p$, degree $p^g$, $O_L$-invariant, isotropic

Equivalently,

$$(f : A \to B),$$

such that $\text{deg}(f) = p^g, \text{Ker}(f) \subseteq A[p], f^*\mathcal{P}_B = p\mathcal{P}_A$.

Atkin-Lehner: $w(f : A \to B) = (f^t : B \to A)$, $f \circ f^t = p$
Invariants for $\overline{X}$

For $A/k$, $k \supseteq \kappa$ perfect. Let

$$\alpha_A = \text{Ker}(\text{Fr}_A) \cap \text{Ker}(\text{Ver}_A).$$

Decomposition of the Dieudonné modules:

$$\mathcal{D}(A[p]) = \bigoplus_{\beta \in \mathcal{B}} k^2 \supseteq \mathcal{D}(\alpha_A) = \bigoplus_{\beta \in \mathcal{B}} \mathcal{D}(\alpha_A)_\beta$$

0 or 1 dim’l

The type of $A$ is

$$\tau(A) = \{\beta \in \mathcal{B} : \mathcal{D}(\alpha_A)_\beta \neq \{0\}\}.$$  

Define strata of $\overline{X}$,

$$W_\tau \leftrightarrow \{A : \tau(A) = \tau\} \quad \text{(locally closed),}$$

$$Z_\tau \leftrightarrow \{A : \tau(A) \supseteq \tau\} \quad \text{(closed).}$$
Theorems (G. - Oort)

1. \( \overline{W}_\tau = Z_\tau = \bigsqcup_{\tau' \supseteq \tau} W_{\tau'} \). So \( \{ W_\tau : \tau \subseteq \mathbb{B} \} \) is a stratification of the moduli space \( \overline{X} \) by \( 2^g \) strata.

2. \( W_\tau \) is non-singular, quasi-affine of dimension \( g - \# \tau \).

3. \( \exists h_\beta \), a Hilbert modular form of weight \( p \cdot \sigma^{-1} \circ \beta - \beta \), such that \( (h_\beta) = Z_\beta \).
   (In classical terms: weight \((0, \ldots, 0, p, -1, 0 \ldots, 0)\).)

4. \( \hat{O}_{\overline{X},P} \cong k[[t_\beta : \beta \in \mathbb{B}]] \) and if \( h_\beta(P) = 0 \) then we may identify \( h_\beta \) with \( t_\beta \).

5. The kernel of the \( q \)-expansion map on the graded ring of Hilbert modular forms modulo \( p \) is the ideal \( \langle h_\beta - 1 : \beta \in \mathbb{B} \rangle \).
Given $A \xleftarrow{f} B$, we have

$$\bigoplus_{\beta} \text{Lie}(A)_{\beta} \xrightarrow{\bigoplus_{\beta} \text{Lie}(f)_{\beta}} \bigoplus_{\beta} \text{Lie}(B)_{\beta}.$$ 

Define

- $\varphi = \varphi(f) = \{\beta \in \mathbb{B} : \text{Lie}(f)_{\sigma^{-1} \circ \beta} = 0\}$,
- $\eta = \eta(f) = \{\beta \in \mathbb{B} : \text{Lie}(f^t)_{\beta} = 0\}$,
- $I = \mathcal{L}(\varphi) \cap \eta$ (the “critical indices”).

Properties:

1. $(\varphi \triangle \eta)^c \supseteq \tau(A) \supseteq \varphi \cap \eta$.
2. $\eta \supseteq \mathcal{L}(\varphi)^c$. 
A pair \((\varphi, \eta)\) (for \(\varphi, \eta \subseteq \mathbb{B}\)) is admissible if

\[
\eta \supseteq \ell(\varphi)^c.
\]

Exist 3\(^g\) such pairs.

Define strata in \(\overline{Y}\):

\[
\begin{align*}
W_{\varphi,\eta} &\leftrightarrow \{ (f : A \to B) : \varphi(f) = \varphi, \eta(f) = \eta \} \quad \text{(loc. closed)}, \\
Z_{\varphi,\eta} &\leftrightarrow \{ (f : A \to B) : \varphi(f) \supseteq \varphi, \eta(f) \supseteq \eta \} \quad \text{(closed)}.
\end{align*}
\]
Theorems

1. \( \overline{W_{\varphi,\eta}} = Z_{\varphi,\eta} = \bigsqcup_{(\varphi',\eta') \geq (\varphi,\eta)} W_{\varphi',\eta'} \) and so \( \{ W_{\varphi,\eta} \} \) is a stratification of \( \overline{Y} \) with \( 3^g \) strata.

2. \( W_{\varphi,\eta} \) and \( Z_{\varphi,\eta} \) are non-singular, equi-dimensional of dimension \( 2g - (\# \varphi + \# \eta) \).

3. There are \( 2^g \) maximal strata, given by \( Z_{\varphi,\ell(\varphi)^c}, \varphi \subseteq \mathbb{B} \).

There are \( 2^r \) horizontal components, where \( r = \#\{p | p\} \).

Two of which are

\[ \overline{Y}_F = Z_{\mathbb{B},\emptyset} \leftrightarrow (A, \text{Ker}(\text{Fr}_A)) \]

\[ \overline{Y}_V = Z_{\emptyset,\mathbb{B}} \leftrightarrow (A, \text{Ker}(\text{Ver}_A)). \]

4. \( w(Z_{\varphi,\eta}) = Z_{r(\eta),\ell(\varphi)}. \)

5. \( \pi(Z_{\varphi,\eta}) = Z_{\varphi} \cap \eta. \)

6. If \( C \subseteq Z_{\varphi,\eta} \) is an irreducible component then

\[ C \cap \overline{Y}_F \cap \overline{Y}_V \neq \emptyset. \]
Let $\overline{Q} \in \overline{Y}$ be a closed $k$-point, then (H. Stamm, using methods of Th. Zink)

$$\hat{O}_{\overline{Y}, \overline{Q}} \cong \frac{W(k)[[\{x_\beta : \beta \in I\}, \{y_\beta : \beta \in I\}, \{z_\beta : \beta \in I^c\}]}{\langle\{x_\beta y_\beta - p : \beta \in I\}\rangle}.$$ 

Moreover, the variables can be chosen so that: If

$$\varphi \supseteq \varphi' \supseteq \varphi - r(I), \quad \eta \supseteq \eta' \supseteq \eta - I,$$

and $(\varphi', \eta')$ is admissible, write

$$\varphi' = \varphi - J, \quad \eta' = \eta - K,$$

then $Z_{\varphi', \eta'}$ is described in $\hat{O}_{\overline{Y}, \overline{Q}}$ by the ideal

$$\langle\{x_\beta : \beta \in I - K\}, \{y_\beta : \beta \in I - \ell(J)\}\rangle.$$ 

Moreover, if $\overline{Q} \in Z_{\varphi', \eta'}$ then $(\varphi', \eta')$ are as above.
Canonical Subgroups

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Key Lemma

Let $\beta \in \varphi \cap \eta \subseteq \tau$, $\pi(Q) = P$, 

$$\pi^* : \hat{\mathcal{O}}_{X,P} \rightarrow \hat{\mathcal{O}}_{Y,Q}.$$ 

Then:

$$\pi^*(t_\beta) = \begin{cases} 
ux_\beta + vy_{\sigma^{-1} \circ \beta} & \sigma \circ \beta \in \varphi, \ \sigma^{-1} \circ \beta \in \eta, \\
ux_\beta & \sigma \circ \beta \in \varphi, \ \sigma^{-1} \circ \beta \notin \eta, \\
vy_{\sigma^{-1} \circ \beta} & \sigma \circ \beta \notin \varphi, \ \sigma^{-1} \circ \beta \in \eta, \\
0 & \sigma \circ \beta \notin \varphi, \ \sigma^{-1} \circ \beta \notin \eta,
\end{cases}$$

where $u, v$ are units.
Ideas coming into the proof

- Study the situation on components of $\text{Spf}(\hat{\mathcal{O}}_{Y/Q})$; they correspond to strata $Z_{\varphi',\eta'}$ passing through $\overline{Q}$.
- Gain data on $\pi^*(t_\beta)$; roughly, $\pi^*(t_\beta) = u x_\beta^M + v y_\sigma^{N-1} \circ \beta$.
- Globalize so as to be able to study these expressions on components of $Z_{\varphi',\eta'}$ but at other points than $\overline{Q}$.
- Reduce to computation at a “special superspecial point” (using that any component $C$ of any strata $Z_{\varphi,\eta}$ intersects $\overline{Y_F} \cap \overline{Y_V}$).
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\( \mathcal{X}_{\text{rig}}, \mathcal{Y}_{\text{rig}} \) are the rigid spaces associated to \( X^{\wedge \mathcal{X}}, Y^{\wedge \mathcal{Y}} \). Given \( P \in \mathcal{X}_{\text{rig}} \), get \( \overline{P} = \text{sp}(P) \in \overline{X} \). The variables \( t_{\beta} \in \hat{O}_{\mathcal{X}, \overline{P}} \) are functions on \( \text{sp}^{-1} \overline{P}) = \text{residue “disc” about } P \). Let

\[
\nu(P) = (\nu_{\beta}(P)), \quad \nu_{\beta}(P) = \begin{cases} 
\nu(t_{\beta}(P)) & \beta \in \tau(\overline{P}), \\
0 & \text{else}.
\end{cases}
\]

\( (\nu(x) = \min(\text{val}(x), 1).) \) For \( Q \in \mathcal{Y}_{\text{rig}}, \overline{Q} = \text{sp}(Q) \in \overline{Y} \), let

\[
\nu(Q) = (\nu_{\beta}(Q)), \quad \nu_{\beta}(Q) = \begin{cases} 
1 & \beta \in \eta(\overline{Q}) - I(\overline{Q}), \\
\nu(x_{\beta}(Q)) & \beta \in I(\overline{Q}), \\
0 & \beta \notin I(\overline{Q}).
\end{cases}
\]

\( \nu(P), \nu(Q) \) belong to the valuation cube \( \Theta = [0, 1]^{B} \).

\( \nu(Q) + \nu(wQ) = 1 \) (easy!).
The Cube Theorem

Parameterize the “open faces” of $\Theta$ by $a = (a_\beta)$, $a_\beta \in \{0, 1, *\}$. For such $a$ define

$$\varphi(a) = \{ \beta \in \mathbb{B} : a_\beta \neq 0 \}, \quad \eta(a) = \{ \beta \in \mathbb{B} : a_{\sigma^{-1}\beta} \neq 1 \}.$$

There is a 1:1 order-reversing correspondence

$$\{\text{open faces of } \Theta\} \leftrightarrow \{\text{strata } W_{\varphi, \eta}\}.$$

- $\nu(Q) \in \mathcal{F}_a \iff \overline{Q} \in W_{\varphi(a), \eta(a)}$.
- $\nu(Q) \in \text{Star}(\mathcal{F}_a) \iff \overline{Q} \in Z_{\varphi(a), \eta(a)}$.

The open faces of $\Theta$ produce a “dissection” of $Y_{\text{rig}}$: $\mathcal{F}_a \leftrightarrow \{Q : \nu(Q) \in \mathcal{F}_a\}$. 
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The canonical subgroup theorem

\[ \mathcal{U} := \{ P \in \mathcal{X}_{\text{rig}} : \nu_{\beta}(P) + p\nu_{\sigma^{-1} \circ \beta}(P) < p, \forall \beta \in \mathcal{B} \} \]

\[ \mathcal{V} := \{ Q \in \mathcal{Y}_{\text{rig}} : \nu_{\beta}(Q) + p\nu_{\sigma^{-1} \circ \beta}(Q) < p, \forall \beta \in \mathcal{B} \} \]

\[ \lambda_{\beta}(Q) \]

Theorem

\[ \pi(\mathcal{V}) = \mathcal{U} \text{ and there is a section } \]

\[ s^\dagger : \mathcal{U} \to \mathcal{V}, \]

extending the canonical section on the ordinary locus.
Ideas in the proof

- Define for $p|p$,
  \[ \mathcal{V}_p := \{ Q : \lambda_\beta(Q) < p, \forall \beta \in \mathbb{B}_p \} \]
  \[ \mathcal{W}_p := \{ Q : \lambda_\beta(Q) > p, \forall \beta \in \mathbb{B}_p \} \]

We first show that:
- $\mathcal{U}$ is admissible.
- $\pi^{-1}(\mathcal{U}) = \mathcal{V} \bigsqcup \mathcal{W}$, admissible disjoint union, where
  \[
  \mathcal{W} = \bigcup_{\emptyset \neq S \subseteq \{ p | p \}} \left[ \bigcap_{p \in S} \mathcal{W}_p \cap \bigcap_{p \not\in S} \mathcal{V}_p \right].
  \]

Uses the notion of tubular neighborhoods and our strata on $\overline{Y}$.
- $\pi|_{\pi^{-1}(\mathcal{U})}$ is finite-flat.
- The connected components of $\mathcal{V}$ are in bijection with those of $\mathcal{U}$.
- We calculate that $\pi|_\mathcal{V}$ has degree 1 by restricting to $\text{sp}^{-1}(W_{\mathcal{B},\emptyset}) \subseteq \text{sp}^{-1}(\overline{Y}_F)$. 
Further properties

- We can determine what happens under

\[ P \mapsto s^\dagger(P) \mapsto w \circ s^\dagger(P) \mapsto \pi \circ w \circ s^\dagger(P), \]

and so can iterate the construction to construct higher level canonical subgroups (\( \subseteq A[p^n] \)).

- Can prove functorial behavior relative to morphisms between Hilbert modular varieties. In particular, (i) decent to \( \mathbb{Q}_p \) of the canonical subgroup, and (ii) prove \( \mathcal{U} \) is maximal (in a suitable sense) for the construction of the canonical subgroup.