ASSIGNMENT 7 - NUMBER THEORY, WINTER 2009

Submit by Monday, March 16, 16:00.

Solve following questions:

(29) Develop $1/7, 2/7, \dots, 6/7$ into decimal expansions. What can you observe? Do the same for $1/19, 5/19, 6/19$ (you would need about 60 digits to be convinced of the pattern). Develop also $1/11, \dots, 10/11$ and $1/9, \dots, 8/9$. What can you observe?

Now calculate the order of 10 in $\mathbb{Z}/7\mathbb{Z}^\times, \mathbb{Z}/19\mathbb{Z}^\times, \mathbb{Z}/11\mathbb{Z}^\times, \mathbb{Z}/9\mathbb{Z}^\times$. (In $\mathbb{Z}/19\mathbb{Z}^\times$ it is 18). Make a conjecture relating the nature of 10 modulo n and the decimal expansions of $1/n, \dots, (n-1)/n$.

(30) Prove that $\sum_{n=0}^{\infty} \frac{1}{n!(n+1)!}$ is irrational in the same way we proved e is irrational.

(31) Construct $|\mathbb{R}|$ transcendental numbers by making variations on the construction in Liouville's theorem. You may use the statement that the cardinality of the set

$$\{(a_0, a_1, a_2, \dots) : a_i \in \{0, 1\}\}$$

is $|\mathbb{R}|$ (the cardinality of the real numbers).

(32) What is the continued fraction $[1, 2, 3, 1, 2, 3, 1, 2, 3, 1, 2, 3, \dots]$?

(33) Let $x = [c, a, b, a, b, a, b, \dots]$ where a and b are positive integers and c is an integer. Prove that x solves a quadratic equation. Which?

(34) Develop π into continued fraction in the form $[a_0, a_1, a_2, \dots]$, finding all a_i for $i = 0, 1, 2, 3, 4$. For which $0 \le i \le 4$ is p_i/q_i is an optimal approximation? Which approximations p_i/q_i are worth remembering in your opinion?

The honors students need to submit also 1 of the following problems.

(J) Prove the conjecture you have made in question (29).

(K) Develop into a continued fraction the numbers

$$\frac{e^1 - 1}{2}, \frac{e^{1/2} - 1}{2}, \frac{e^{1/3} - 1}{2}, \frac{e^{1/4} - 1}{2}, \dots$$

(Preferably, use some software for that.) Formulate a conjecture as to the continued fraction expansion of $\frac{e^{1/k}-1}{2}$, where k is a positive integer.

Can you think about a statement concerning the function $(e^{1/x} - 1)/2$ that, if true, will prove this conjecture.