ASSIGNMENT 3 - NUMBER THEORY, WINTER 2009

Submit by Monday, February 2, 16:00 (use the designated mailbox in Burnside Hall, 10^{th} floor).

Solve the following questions:

(11) Let R be the set of all functions $f : \mathbb{N}^+ \to \mathbb{C}$, namely R is the set of arithmetic functions. Prove that R is a commutative ring under the following operations of addition and multiplication:

$$(f+g)(n) = f(n) + g(n),$$
$$(f * g)(n) = \sum_{d|n} f(d)g(n/d).$$

Its zero is the function that is identically zero and its identity is the function e defined in class.

(12) Find all integers n such that:

(a) $\varphi(n) = n/2$;
(b) $\varphi(n) = \varphi(2n)$;
(c) $\varphi(n) = 4$;
(d) $\varphi(n) = 12$.

If you have installed PARI on your laptop, or have access to some other mathematical software, it wouldn't hurt to experiment. For example, in PARI you might want to run

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for(n=1, 100, if(eulerphi(n)==n/2,print(n),))
for(n=1, 100, if(eulerphi(n)==eulerphi(2*n),print(n),))
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(13) For each of the following statements give a proof or a counterexample:

(a) If $(m, n) = 1$ then $(\varphi(m), \varphi(n)) = 1$;
(b) If n is composite[1] then $(n, \varphi(n)) > 1$;
(c) If the same primes divide n and m then $n\varphi(m) = m\varphi(n)$.

The honors students need also to submit the following problems.

D. Prove that a multiplicative function $f \in R$ is a unit.

One says that a function $f \in R$ is completely multiplicative if f is multiplicative and in addition $f(p^\alpha) = f(p)^\alpha$ for every prime p and $\alpha \in \mathbb{N}^+$. Equivalently, $f(mn) = f(m)f(n)$ for any m, n. Prove that the inverse, relative to Dirichlet multiplication, of a completely multiplicative function f is given by

$$f^{-1}(n) = \mu(n)f(n).$$

Give an example of a completely multiplicative function f such that f^{-1} is not completely multiplicative. (To put this in perspective: one can prove that the inverse of a multiplicative function is multiplicative.)

E. Find $\limsup_n \frac{\varphi(n)}{n}$ and $\liminf_n \frac{\varphi(n)}{n}$.

You may use the statement (to be proved later) $\sum_p \frac{1}{p} = \infty$, where the sum is over all primes.

[1] That means $n > 1$ and is not prime.