ASSIGNMENT 2 - NUMBER THEORY, WINTER 2009

Submit by Monday, January 26, 16:00 (use the designated mailbox in Burnside Hall, 10^{th} **floor).**

Solve the following questions:

(6) Prove Kummer's Theorem: For $n \geq m$, $\mathrm{ord}_p\binom{n}{m}$ is the number of carry-over operations one needs to make in adding m and $n-m$, written to base p.

Calculate this way $\mathrm{ord}_3\binom{551}{182}$.

(7) Prove that $\Delta\binom{x}{k} = \binom{x}{k-1}$ for $k \geq 1$.

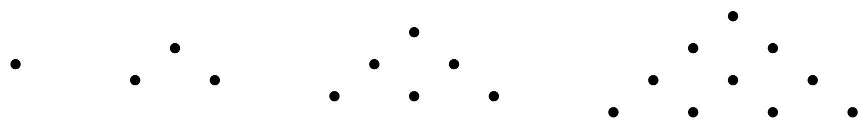
(8) Expand $(x+1)^3$ and $(x+1)^4$ as a sum of binomial functions. Use that to find formulas for sums of cubes and sums of fourths:

$$1^3 + 2^3 + \cdots + n^3, \qquad 1^4 + 2^4 + \cdots + n^4.$$

(9) The triangular numbers are the integers of the form $\frac{n(n+1)}{2}$ for $n = 0, 1, 2, 3, \ldots$. The first triangular numbers are

$$0, 1, 3, 6, 10, 15, 21, \ldots$$

They are so called because of the following diagrams in which the number of dots are triangular numbers:



Find a formula for the sum of the first n triangular numbers. The n-th pyramidal number is the sum of the first n triangular numbers. The series of pyramidal numbers thus starts as follows:

$$1, 4, 10, 20, 35, 56, \ldots$$

Find a formula for the sum of the first n pyramidal numbers.

(10) Give a formula for the following arithmetic functions (possibly in terms of the prime factorization of n).

$$\underline{1} * \underline{1}, \qquad \underline{1} * \underline{1} * \underline{1}, \qquad \underline{1} * \mathrm{Id}, \qquad \mathrm{Id} * \mathrm{Id},$$

and

$$\sum_{d|n} \mu(d)\tau(n/d), \qquad \sum_{d|n} \mu(d)\sigma(n/d), \qquad \sum_{d|n} \mu(d)\sigma(d).$$

(Explain why they are all multiplicative functions.)

The honors students need also to submit the following problem.

C. Fix an integer $k \geq 1$ and look at the series of k-powers: $1^k, 2^k, 3^k, \ldots$. Perform the series of differences: $2^k - 1^k, 3^k - 2^k, 4^k - 3^k, \ldots$. Repeat this so as to get k rows of differences. Prove that the k-row is: $k!, k!, k!, \ldots$.

Here is an example: take $k = 3$. The series we start with is of cubes: $1, 8, , 27, 64, 125, 216, \ldots$. The series of differences are

$$\begin{array}{ccccccccccc}
1 & & 8 & & 27 & & 64 & & 125 & & \ldots \\
& 7 & & 19 & & 37 & & 61 & & & \ldots \\
& & 12 & & 18 & & 24 & & & \ldots & \\
& & & 6 & & 6 & & 6 & & \ldots &
\end{array}$$