Submit by Monday, January 26, 16:00 (use the designated mailbox in Burnside Hall, 10th floor).

Solve the following questions:

(6) Prove Kummer’s Theorem: For $n \geq m$, $\text{ord}_p (\binom{n}{m})$ is the number of carry-over operations one needs to make in adding $m$ and $n - m$, written to base $p$. Calculate this way $\text{ord}_3 (\binom{551}{182})$.

(7) Prove that $\Delta (\binom{x}{k}) = \binom{x-k}{k-1}$ for $k \geq 1$.

(8) Expand $(x + 1)^3$ and $(x + 1)^4$ as a sum of binomial functions. Use that to find formulas for sums of cubes and sums of fourths:

\[1^3 + 2^3 + \cdots + n^3, \quad 1^4 + 2^4 + \cdots + n^4.\]

(9) The triangular numbers are the integers of the form $\frac{n(n+1)}{2}$ for $n = 0, 1, 2, 3, \ldots$. The first triangular numbers are

\[0, 1, 3, 6, 10, 15, 21, \ldots\]

They are so called because of the following diagrams in which the number of dots are triangular numbers:

\[\begin{array}{cccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\end{array}\]

Find a formula for the sum of the first $n$ triangular numbers. The $n$-th pyramidal number is the sum of the first $n$ triangular numbers. The series of pyramidal numbers thus starts as follows:

\[1, 4, 10, 20, 35, 56, \ldots\]

Find a formula for the sum of the first $n$ pyramidal numbers.

(10) Give a formula for the following arithmetic functions (possibly in terms of the prime factorization of $n$).

\[1 * 1, \quad 1 * 1 * 1, \quad 1 * \text{Id}, \quad \text{Id} * \text{Id},\]

and

\[\sum_{d|n} \mu(d) \tau(n/d), \quad \sum_{d|n} \mu(d) \sigma(n/d), \quad \sum_{d|n} \mu(d) \sigma(d).\]

(Explain why they are all multiplicative functions.)

The honors students need also to submit the following problem.

C. Fix an integer $k \geq 1$ and look at the series of $k$-powers: $1^k, 2^k, 3^k, \ldots$. Perform the series of differences: $2^k - 1^k, 3^k - 2^k, 4^k - 3^k, \ldots$. Repeat this so as to get $k$ rows of differences. Prove that the $k$-row is:

\[k!, k!, k!, \ldots\]

Here is an example: take $k = 3$. The series we start with is of cubes: 1, 8, 27, 64, 125, 216, \ldots. The series of differences are:

\[
\begin{array}{cccccccc}
1 & 8 & 27 & 64 & 125 & \ldots \\
7 & 19 & 37 & 61 & \ldots \\
12 & 18 & 24 & \ldots \\
6 & 6 & 6 & \ldots \\
\end{array}\]