Jacobi Forms

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History of Jacobi Forms

Notation

Let $e(x)$ denote $e^{2\pi ix}$ for $x \in \mathbb{C}$. Let $q = e(\tau)$ and $\zeta = e(z)$ where $\tau \in \mathcal{H}$ and $z \in \mathbb{C}$.

Jacobi forms are meant to be a natural generalization of Jacobi theta series.

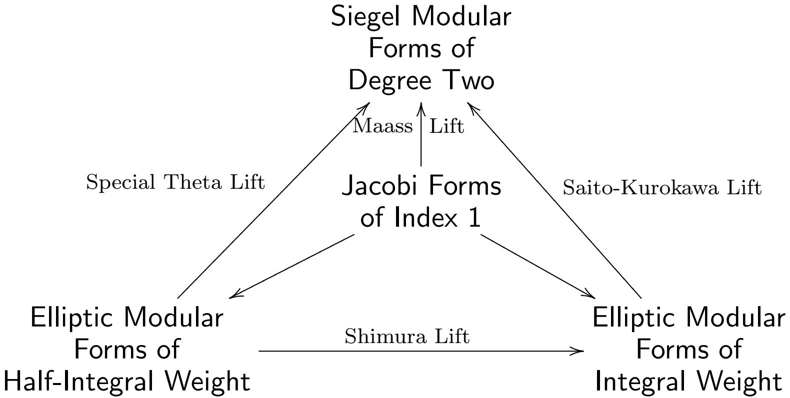
Definition

Let L be a lattice of rank $2k$ with a positive-definite quadratic form $Q(x)$ and bilinear form $B(x, y) = Q(x + y) - Q(x) - Q(y)$. Given a vector $y \in L$ we define the Jacobi theta series $\Theta_y(\tau, z)$ by

$$\Theta_y(\tau, z) = \sum_{x \in L} e((Q(x)\tau + B(x, y)z)).$$

Motivation

The main reference for this topic is *The Theory of Jacobi Forms* by Eichler and Zagier (1985). Their main interest in Jacobi forms was their relation to the Saito-Kurokawa lift.



Our interest in Jacobi forms will be in their connection with the Borcherds lift.

The Transformation Law

Jacobi forms are complex functions on $\mathcal{H} \times \mathbb{C}$ which are invariant under an action of the Jacobi group.

Definition

The *Jacobi group* is $SL_2(\mathbb{Z})^J = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^2$ where

$$[M, X][M', X'] = [MM', XM' + X'].$$

For a congruence subgroup Γ let $\Gamma^J = \Gamma \ltimes \mathbb{Z}^2$.

Notation

Given integers k and m, the slash operator is

$$(\phi|_{k,m}\gamma)(\tau, z) = (c\tau + d)^{-k} e\left(m\left(\frac{-c(z + \lambda\tau + \mu)^2}{c\tau + d} + \lambda^2\tau + 2\lambda z + \lambda\mu\right)\right) \cdot \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z + \lambda\tau + \mu}{c\tau + d}\right)$$

for $\gamma = [\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (\lambda, \mu)] \in SL_2(\mathbb{Z})^J$ This defines an action of the Jacobi group on complex function of $\mathcal{H} \times \mathbb{C}$.

Relationship to Modular Forms

Lemma

In the case $\gamma = \left[\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), (0,0)\right]$ we have

$$(\phi|_{k,m}\gamma)(\tau, z) = (c\tau + d)^{-k} e\left(\frac{-cmz^2}{c\tau + d}\right) \phi\left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}\right)$$

and in the case $\gamma = [\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right), (\lambda, \mu)]$ we find

$$(\phi|_{k,m}\gamma)(\tau, z) = e(\lambda^2 m\tau + 2\lambda mz)\, \phi(\tau, z + \lambda\tau + \mu).$$

Remark

In the case of $m = 0$ the previous slash operators reduce to the slash operator for elliptic modular forms.

A holomorphic function on $\mathcal{H} \times \mathbb{C}$ which is invariant under the action given above has a Fourier expansion.

Definition of Jacobi Forms

Definition

A *Jacobi form of weight k and index m for a congruence subgroup* Γ is a function $\phi : \mathcal{H} \times \mathbb{C} \to \mathbb{C}$ which

1. is holomorphic on $\mathcal{H} \times \mathbb{C}$,
2. satisfies $\phi|_{k,m}\gamma = \phi$ for all $\gamma \in \Gamma^J$ and
3. is holomorphic at each cusp $Mi\infty$ where $M \in \mathrm{SL}_2(\mathbb{Z})^J$, that is,
$$\phi|_{k,m}M = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn \geq r^2 h}} c_M\left(\tfrac{n}{h}, r\right) q^{n/h}\zeta^r$$
   where h is the width of the cusp $Mi\infty$ of Γ.

Furthermore we say ϕ is a *Jacobi cusp form* if in addition to the conditions above ϕ vanishes at each cusp $Mi\infty$ for $M \in \mathrm{SL}_2(\mathbb{Z})^J$, that is, if
$$\phi|_{k,m}M = \sum_{\substack{n,r\in\mathbb{Z} \\ 4mn > r^2 h}} c_M\left(\tfrac{n}{h}, r\right) q^{n/h}\zeta^r.$$

Structural Theorem

Notation

Let $\mathrm{J}_{k,m}(\Gamma)$ denote the vector space of all Jacobi forms with weight k and index m on a congruence subgroup Γ. Let the subspace of cusp forms be denoted by $\mathrm{J}_{k,m}^{\mathrm{cusp}}(\Gamma)$. Furthermore let $M_k(\Gamma)$ denote the space of elliptic modular forms of weight k on the congruence subgroup Γ.

Theorem

Given a congruence subgroup Γ structurally $\bigoplus_{k,m} \mathrm{J}_{k,m}(\Gamma)$ forms a bigraded ring with each $\mathrm{J}_{k,m}(\Gamma)$ finite dimensional. Moreover $\mathrm{J}_{*,*}(\Gamma)$ is a module over $M_*(\Gamma)$.

Jacobi-Eisenstein Series for $\mathrm{SL}_2(\mathbb{Z})$

Notation

Let $\mathrm{SL}_2(\mathbb{Z})^J_\infty$ be the set elements $\gamma \in \mathrm{SL}_2(\mathbb{Z})^J$ which satisfy $1|_{k,m}\gamma = 1$; that is, let

$$\mathrm{SL}_2(\mathbb{Z})^J_\infty = \{[\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right), (0,\mu)] \mid \mu, n \in \mathbb{Z}\}.$$

Definition

Given integers $m \geq 0$ and $k \geq 4$ we define the *Jacobi-Eisenstein series* $E_{k,m}$ by

$$E_{k,m}(\tau, z) = \sum_{\gamma \in \mathrm{SL}_2(\mathbb{Z})^J_\infty \backslash \mathrm{SL}_2(\mathbb{Z})^J} 1|_{k,m}\gamma.$$

Theorem

The Jacobi-Eisenstein series $E_{k,m}$ is a Jacobi form on $\mathrm{SL}_2(\mathbb{Z})$.

Jacobi-Eisenstein Series for $\Gamma(N)$

Notation

Fix an integer $N > 0$ and $\bar{v} \in (\mathbb{Z}/N\mathbb{Z})^2$ where $\bar{v}$ has order N. Let $\delta = \left[\begin{pmatrix} a & b \\ c_v & d_v \end{pmatrix}, (\lambda, \mu)\right] \in \mathrm{SL}_2(\mathbb{Z})^J$ where $\overline{(c_v, d_v)} = \bar{v}$.

Definition

Given integers $m \geq 0$ and $k \geq 4$ we define the *Jacobi Eisenstein series* $E^{\bar{v}}_{k,m}$ by

$$E^{\bar{v}}_{k,m}(\tau, z) = \epsilon_N \sum_{\gamma \in (\mathrm{SL}_2(\mathbb{Z})^J_\infty \cap \Gamma(N)^J) \backslash \Gamma(N)^J \delta} 1|_{k,m}\gamma,$$

where $\epsilon_N = 1$ if $N > 2$ and $\epsilon_N = 1/2$ otherwise.

Theorem

The Jacobi-Eisenstein series $E^{\bar{v}}_{k,m}$ is a Jacobi form on $\Gamma(N)$.

Fourier Expansion of the Jacobi-Eisenstein Series

For $n' \in \mathbb{Z}/N\mathbb{Z}$ define

$$L_D^{n'}(s) = \sum_{\substack{n=1 \\ n\equiv n' \pmod{N}}}^{\infty} \left(\frac{D}{n}\right) n^{-s}$$

$$\zeta_+^{n'}(k,\mu) = \sum_{\substack{l=1 \\ l\equiv n' \pmod{N}}}^{\infty} \mu(l) l^{-s}.$$

Theorem

The Fourier expansion is of the form $E_{k,m}^{\bar{v}} = C + \sum_{\substack{q,r\in\mathbb{Z} \\ 4nm>r^2N}} c\left(\frac{n}{N}, r\right) q^{n/N}\zeta^r$

where

$$C = \begin{cases} \sum_{\lambda\in\mathbb{Z}} q^{\lambda^2 m}\zeta^{2m\lambda} & \text{if } v \equiv (0,1) \pmod{N} \\ 0 & \text{otherwise,} \end{cases}$$

Furthermore suppose $m = 1$ and $(c_v, N) = 1$ then

$$c\left(\frac{n}{N}, r\right) = \epsilon_N \frac{(-1)^{\frac{k}{2}} \pi^{k-\frac{1}{2}} (4m\frac{n}{N} - r^2)^{k-\frac{3}{2}}}{N 2^{k-2} m^{k-1}} A \sum_{\substack{j \pmod{N} \\ \gcd(j,N)=1}} \left(\zeta_+^j(2k,\mu)\, L_{Nr^2-4mn}^{c_v j^{(-2)}}(k)\right).$$

Definition

Fix integers $m \geq 0$ and $k \geq 4$. Let b be the largest positive number such that $m = ab^2$ for some a. For an integer s define the *Jacobi-Eisenstein series* $E^{\bar{v}}_{k,m,s}$ by

$$E^{\bar{v}}_{k,m,s}(\tau, z) = \epsilon_N \sum_{\gamma \in (\mathrm{SL}_2(\mathbb{Z})^J_\infty \cap \Gamma(N)^J)\backslash\Gamma(N)^J\delta} \left(1|_{k,m}\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\frac{s}{b}, 0\right)\right]\right)|_{k,m}\gamma$$

$$= \epsilon_N \sum_{\gamma \in (\mathrm{SL}_2(\mathbb{Z})^J_\infty \cap \Gamma(N)^J)\backslash\Gamma(N)^J\delta} \left(q^{as^2}\zeta^{2abs}\right)|_{k,m}\gamma.$$

Theorem

The Jacobi-Eisenstein series $E^{\bar{v}}_{k,m,s}$ is a Jacobi form on $\Gamma(N)$.

Definition

Fix integers $m \geq 0$ and $k \geq 4$. Let the Jacobi-Eisenstein space $\mathrm{E}_{k,m}(\Gamma(N))$ be the span of the Jacobi-Eisenstein series $E^{\bar{v}}_{k,m,s}$.

Basis of Jacobi-Eisenstein Space

Theorem

If $\bar{v}M \not\equiv \overline{(0,1)} \pmod{N}$ then $E^{\bar{v}}_{k,m,s}$ has no constant term at $Mi\infty$, otherwise $E^{\bar{v}}_{k,m,s}$ has constant term at $Mi\infty$ which are:

1. $\sum_{r \equiv 2abs \pmod{2m}} \left(q^{\frac{r^2}{4m}} \zeta^r + (-1)^{-k} q^{\frac{r^2}{4m}} \zeta^{-r} \right)$ if $N \leq 2$ and
2. $\sum_{r \equiv 2abs \pmod{2m}} (\pm 1)^{-k} q^{\frac{r^2}{4m}} \zeta^{\pm r}$ if $N > 2$.

Corollary

If K is the number of cusps the dimension of $\mathrm{E}_{k,m}(\Gamma(N))$

1. is $K\left(\lfloor \frac{b}{2} \rfloor + 1\right)$ if $N \leq 2$ and k is even,
2. is $K \lfloor \frac{b-1}{2} \rfloor$ if $N \leq 2$ and k is odd, and
3. is $K\,(b+1)$ if $N > 2$.

Corollary

$$\mathrm{J}_{k,m}(\Gamma(N)) = \mathrm{E}_{k,m}(\Gamma(N)) \oplus \mathrm{J}^{\mathrm{cusp}}_{k,m}(\Gamma(N))$$

Number of Coefficients which Determine a Modular Form

Let e_a denote the ramification index at a (if a is a cusp $\frac{1}{e_a} = 0$).

Theorem

For a fixed congruence subgroup Γ let g denote the genus $X(\Gamma)$. For a non-zero modular form $f \in M_k(\Gamma)$ we have the following formula

$$\deg(\mathrm{div}(f)) = k(g-1) + \frac{k}{2} \sum_{a \in \Gamma \backslash \mathcal{H}^*} \left(1 - \frac{1}{e_a}\right).$$

Theorem

The number of coefficients $N_{k,\Gamma}$ needed to determine a modular form of weight k on a congruence subgroup Γ satisfies

$$N_{k,\Gamma} \leq k(g-1) + \frac{k}{2} \sum_{a \in \Gamma \backslash \mathcal{H}^*} \left(1 - \frac{1}{e_a}\right).$$

Number of Coefficients which Determine a Jacobi Form

Theorem

Given a congruence subgroup Γ there exists an injection

$$\mathcal{D} = \left(\bigoplus_{\nu=0}^{2m} \mathcal{D}_\nu \right) : \mathrm{J}_{k,m}(\Gamma) \to \mathrm{M}_k(\Gamma) \oplus \mathrm{S}_{k+1}(\Gamma) \oplus \ldots \oplus \mathrm{S}_{k+2m}(\Gamma).$$

where the coefficient $c(n)$ of the modular form $\mathcal{D}_{2\nu}(\phi)$ can be defined in terms of the coefficients $c(n,r)$ (where $4mn \geq r^2 h$) from the Jacobi form ϕ.

Let $N_{k,\Gamma}$ denote the number of coefficients needed to determine a modular form of weight k on a congruence subgroup Γ. Let

$$N' = (k+2m)(g-1) + \frac{k+2m}{2} \sum_{a \in \Gamma \backslash \mathcal{H}^*} \left(1 - \frac{1}{e_a} \right).$$

Then $\max(N_{k,\Gamma}, \ldots, N_{k+2m,\Gamma}) \leq N'$.

Theorem

A Jacobi form of weight k and index m on Γ is determined by the coefficients $c(n,r)$ where $n \leq N'$ and $-m \leq r < m$.

Structural Results for Jacobi Forms for $SL_2(\mathbb{Z})$

Theorem

The ring $J_{2*,*}(SL_2(\mathbb{Z}))$ is contained in $M_*(SL_2(\mathbb{Z}))\left[E_{4,1}, E_{6,1}, \frac{1}{\Delta}\right]$. Furthermore $J_{*,*}(SL_2(\mathbb{Z}))$ is free as a module over $M_*(SL_2(\mathbb{Z}))$.

Notation

We introduce the functions

$$\phi_{10,1} = \frac{1}{144}(E_6 E_{4,1} - E_4 E_{6,1})$$
$$\phi_{12,1} = \frac{1}{144}(E_4^2 E_{4,1} - E_6 E_{6,1}).$$

Theorem

The Jacobi forms $E_{4,1}$ and $E_{6,1}$ form a basis for $J_{*,1}$ over M_* .
The Jacobi forms $\phi_{10,1}$ and $\phi_{12,1}$ form a basis for $J_{*,1}^{cusp}$ over M_* .

The Borcherds Lift

Let $\tilde{J}_{\kappa,t}(SL_2(\mathbb{Z}))$ denote the space of weak Jacobi forms of weight κ and index t on $SL_2(\mathbb{Z})$ and let $\rho_L : Mp_2(\mathbb{Z}) \to \mathbb{C}[L'/L]$ denote the Weil representation. Then

$$J_{\kappa,t}(SL_2(\mathbb{Z})) \cong M_{\kappa-1/2,\mathbb{Z},Q(x)=tx^2,\rho^*} \tag{1}$$

the space of vector-valued modular forms on the lattice $\mathbb{Z}$ with quadratic form $Q(x) = tx^2$ of weight $\kappa - 1/2$ with respect to ρ^*. This space is relevant to the obstruction space.

More directly

$$\tilde{J}_{\kappa,t}(SL_2(\mathbb{Z})) \cong M^!_{\kappa-1/2,\mathbb{Z},Q(x)=-tx^2,\rho} \tag{2}$$

the space of nearly holomorphic vector-valued modular forms on the lattice $\mathbb{Z}$ with quadratic form $Q(x) = -tx^2$ of weight $\kappa - 1/2$ with respect to ρ.

It is the functions in $M^!_{\kappa-1/2,\mathbb{Z},Q(x)=-tx^2,\rho}$ that can be lifted directly via the Borcherds lift.

An Example of the Borcherds Lift on a Jacobi Form

Let

$$\phi_{0,1}(\tau, z) = \frac{\phi_{12,1}(\tau, z)}{\Delta(\tau)} \in \tilde{J}_{0,1}(\mathrm{SL}_2(\mathbb{Z})).$$

By Equation 2

$$\tilde{J}_{0,1}(\mathrm{SL}_2(\mathbb{Z})) \cong \mathrm{M}^!_{\kappa-1/2,\mathbb{Z},Q(x)=-tx^2,\rho}.$$

The discriminant groups for the lattices D and $D \oplus H \oplus H$ where D is $\mathbb{Z}$ with $Q(x) = -tx^2$ are isomorphic. Thus

$$\tilde{J}_{0,1}(\mathrm{SL}_2(\mathbb{Z})) \cong \mathrm{M}^!_{\kappa-1/2,D\oplus H\oplus H,\rho}$$

and hence $\phi_{0,1}$ can be lifted through the Borcherds lift to a modular form on the Siegel upper-half plane $\mathrm{Sp}_4(\mathbb{Z})$.

Explicitly calculating the principal part of $\phi_{0,1}$ by Borcherds theorem gives us the divisor of the lift. This divisor corresponds to the Humbert surface of discriminant $|4m|$ which is also the divisor of a known function $\Delta_0^{(2)}$.

These two functions differ by a constant. Thus using the Borcherds product formula we get a product expansion for $\Delta_0^{(2)}$.