Introduction to Borcherds Forms

Montreal-Toronto Workshop in Number Theory

September 3, 2010

Main Goal

Extend theta lift to construct (meromorphic) modular forms on Sh. var. associated to $O(p, 2)$ with amazing properties (explicit divisors, product formulas, etc.)

Let

- (V, q) - rational quadratic space of signature $(p, 2)$. Denote the associated bilinear form by $<,>$, i.e $q(x) = 1/2 < x, x >$
- $H = GSpin(V)$
- $\mathbb{D} = \{$oriented negative-definite planes in $V(\mathbb{R})\} =$ Herm. symm. domain attached to H (of complex dimension p)

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For any K a c.o.s of $H(\mathbb{A}_f)$, we have the Shimura variety

$$X_K = H(\mathbb{Q})\backslash (\mathbb{D} \times H(\mathbb{A}_f)/K)$$

which has a (canonical) model defined over $\mathbb{Q}$.

For a vector $x \in V(\mathbb{Q}), q(x) > 0$, let

- $\mathbb{D}_x = \{z \in \mathbb{D} \mid z \perp x\} \subset \mathbb{D}$
- $H_x = GSpin(x^\perp) \subset H$

For any $h \in H(\mathbb{A}_f)$, the map

$$\begin{aligned} H_x(\mathbb{Q}) \backslash \mathbb{D}_x \times H_x(\mathbb{A}_f)/(hKh^{-1} \cap H_x(\mathbb{A}_f)) \to &\quad X_K \\ [z, g] \quad \mapsto &\, [z, gh] \end{aligned}$$

defines a divisor on X_K, which we denote by $Z(x, h)$.

We'll want to take certain linear combinations of these divisors as well. Take

- L - an even integral lattice in V
- $L^\vee = \{x \in V(\mathbb{Q}) \mid \ < x, L > \subset \ \mathbb{Z}\}$ the dual lattice
- Fix a set of representatives $\{x_\mu\}$ of $L^\vee/L$
- From now on, we assume K stabilizes all the adelic cosets $x_\mu + \hat{L}$, where $\hat{L} = L \otimes \hat{\mathbb{Z}}$,

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For $m \in \mathbb{Q}$, $m > 0$, and $\mu \in L^\vee/L$, suppose there is an $x_0 \in V$ with $q(x_0) = m$. Can write

$$\left\{x \in x_\mu + \hat{L} \mid q(x) = m\right\} = \coprod_r K\xi_r^{-1}x_0$$

for some finite collection of elements $\xi_1, \ldots, \xi_n$ of $H(\mathbb{A}_f)$.

Definition

Define

$$Z(m,\mu) = \sum_r \chi_\mu(\xi_r^{-1} x_0) Z(x_0, \xi_r),$$

where χ_μ is the characteristic function of the adelic coset $x_\mu + \hat{L}$, and $Z(m,\mu) = 0$ if there is no rational vector of length m.

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Remarks:

1. Definition is independent of x_0 and choice of ξ_r's
2. As K varies, get a compatible system of cycles, so can actually define the cycle $Z(m,\mu)$ on full Shimura variety $Sh(G,\mathbb{D})$.
3. If X_K is connected, then

$$Z(m,\mu) = \sum_x pr(D_x),$$

where the sum is over rational vectors of norm m in the coset $x_\mu + L$, modulo the action of $\Gamma = H(\mathbb{Q}) \cap K$, and $pr : \mathbb{D}^+ \to \Gamma \backslash \mathbb{D}^+ \simeq X_K$ is the projection.

Example: for $N \in \mathbb{Z}$, let

- $V = \{A \in M_2(\mathbb{Q}) | A = {}^t A\}$
- $q(A) = N \det(A)$, so signature is $(1, 2)$, and $H = GL_2$.
- $L = \left\{ \begin{pmatrix} c/N & b \\ b & a \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$
- $L^\vee = \left\{ \begin{pmatrix} c/N & b/2N \\ b/2N & a \end{pmatrix} \right\}$, and $L^\vee / L \simeq \mathbb{Z} / 2N \mathbb{Z}$
- $K_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \mid c \in N \mathbb{Z}_p \right\}$, and $K = \prod K_p$

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Have $X_K \simeq Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$ after identifying $\mathbb{H} \simeq \mathbb{D}^+$ via the map

$$\tau \mapsto \mathrm{span}_{\mathbb{R}} \left(Re \begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix}, \ Im \begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix} \right)$$

For $x = \begin{pmatrix} c/N & b/2N \\ b/2N & a \end{pmatrix} \in L^\vee$, have

$$\tau \in \mathbb{D}_x \iff Na\tau^2 + b\tau + c = 0.$$

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So for $m \in \mathbb{Q}_{>0}$, and $\mu \in \mathbb{Z}/2N\mathbb{Z}$, have

$$Z(m,\mu) = \sum [\tau]$$

where the sum is over the images in $Y_0(N)$ of $\tau \in \mathbb{H}$ satisfying

$$A\tau^2 + B\tau + C = 0$$

with

$$N|A, \quad B \equiv \mu \mod 2N, \quad B^2 - 4AC = 4Nm$$

ie. $Z(m,\mu) = P_{4Nm,\mu}$ is a Heegner divisor in the terminology of Gross-Kohnen-Zagier.

Vector-valued modular forms

Want to extend the theta integral to a larger class of functions:

- Let $S(V(\mathbb{A}_f))$ be the space of ($\mathbb{C}$-valued) Schwartz functions on $V(\mathbb{A}_f)$, and let S_L be the subspace spanned by the indicator functions χ_μ of $x_\mu + \hat{L}$.

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- Let $S(V(\mathbb{A}_f))$ be the space of ($\mathbb{C}$-valued) Schwartz functions on $V(\mathbb{A}_f)$, and let S_L be the subspace spanned by the indicator functions χ_μ of $x_\mu + \hat{L}$.
- There is an action of the metaplectic group $Mp_2(\mathbb{Z})$ on $S(V(\mathbb{A}_f))$, via the Weil representation, which restricts to a representation on S_L, denoted by ρ_L.

Vector-valued modular forms

For $k \in (1/2)\mathbb{Z}$, let M_{k,ρ_L} be the space of holomorphic functions $F : \mathbb{H} \to S_L$ such that

- For $\gamma = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \varphi(\tau) \right) \in Mp_2(\mathbb{Z})$, where $\varphi(\tau)^2 = c\tau + d$, have the transformation law

$$F\left(\frac{a\tau + b}{c\tau + d}\right) = \varphi(\tau)^{2k} \, \rho_L(\gamma) F(\tau)$$

- F is meromorphic at cusps

In terms of the basis elements χ_μ, have Fourier expansion at ∞:

$$F(\tau) = \sum_{\mu \in L^\vee/L} \sum_{m \in \mathbb{Q}} c_\mu(m)\ e(m\tau)\ \chi_\mu$$

where $c_\mu(m) = 0$ for $m << 0$ (meromorphic condition), $c_\mu(m) = 0$ for $m \notin q(x_\mu) + \mathbb{Z}$, and $e(x) = e^{2\pi i x}$.

The theta kernel

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Define the theta kernel (for $g \in G(\mathbb{A}), h \in H(\mathbb{A}), \phi \in S(V(\mathbb{A}))$)

$$\theta(g,h)(\phi) = \sum_{x \in V(\mathbb{Q})} \omega(g)\phi(h^{-1}x)$$

- For $z \in \mathbb{D}$, define a positive definite quadratic form

$$q_z(x) = |q(pr_z(x))| + q(pr_{z^\perp}(x))$$

where $pr_z(x)$ is the projection of x onto z, and let

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- Think of the theta kernel as a map
$\theta : \mathbb{H} \times \mathbb{D} \times H(\mathbb{A}_f) \to S(V(\mathbb{A}_f))^*$ by setting

$$\begin{aligned} \theta(\tau, z, h)(\phi) &= v^{1-p/2}\ \theta(g_\tau, h)(\phi_z^\infty \otimes \phi) \\ &= v^{1-p/2} \sum_{x \in V(\mathbb{Q})} \phi(h^{-1}x) \cdot (\omega(g_\tau)\phi_z^\infty)(x) \end{aligned}$$

- Restrict $\theta(\tau, z; h)$ to a functional in $(S_L)^*$. Then for fixed z and h, the function

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- Hence, for $F \in M_{-k,\rho_L}$, the pairing

$$(F(\tau), \theta(\tau, z, h))$$

is $\Gamma = SL_2(\mathbb{Z})$-invariant.

- As before, consider the θ-integral:

$$\Theta_F(z,h) = \int_{\Gamma\backslash\mathbb{H}} (F(\tau), \theta(\tau, z; h))\ v^{-2}\ du\ dv$$

This should be a function on X_K (ie. a function on $\mathbb{D} \times H(\mathbb{A}_f)$ invariant under $H(\mathbb{Q})$ and K), but...

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- Problem: doesn't converge!
- Solution:

$$\Theta_F^{\bullet}(z,h) = \int_{\Gamma\backslash\mathbb{H}}^{\bullet} (F(\tau), \theta(\tau, z; h))\ v^{-2}\ du\ dv$$

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Definition

Define the regularized integral $\Theta_F^\bullet(z, h)$ to be the constant term of the Laurent series expansion of $I(s)$ at $s = 0$.

Theorem (Borcherds (1998))

Given $F \in M_{1-p/2,\rho_L}$, *such that*

$$c_\mu(m) \in \mathbb{Z} \qquad \textit{for } m < 0$$

there is a function Ψ_F *on* $\mathbb{D} \times H(\mathbb{A}_f)$ *such that*

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2. $$div({\Psi_F}^2) = \sum_{\mu \in L^\vee/L} \sum_{m>0} c_\mu(-m)\, Z(m,\mu)$$
3. *In the neighbourhood of a cusp,* Ψ_F *has a* **product expansion**

$$\Psi_F(z,h) = C\, e(< z, \rho(W) >) \prod_{\mu,\xi} (1 - e(< z_f, \xi >))^{c_\mu(-q(\xi))\chi_\mu(h^{-1}\xi)}$$

Idea of Proof

Evaluate the regularized integral to get explicit expressions for the Fourier expansion of $\Theta_F^\bullet$, and $div(\Theta_F^\bullet)$, in terms of Fourier expansion of F. Then use a bit of complex analysis to show that there exists a function Ψ_F such that

$$2\log|\Psi_F(z,h)|^2 = -\Theta_F^\bullet(z,f) - c_0(0)\left(\log|y|^2 + \text{const.}\right)$$

Deduce rest of theorem from this.

Some of your favourite functions are Borcherds forms

Example: Let

$$F(\tau) = \sum_{n>0, n \text{ odd}} \sigma_1(n) q^n = \sum_{n>0, n \text{ odd}} \sum_{d|n} d q^n$$

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + \dots$$

$$\begin{aligned} f_0(\tau) &= F(\tau)\theta(\tau)\left(\theta(\tau)^4 - 2F(\tau)\right)\left(\theta(\tau)^4 - 16F(\tau)\right)E_6(4\tau)/\Delta(4\tau) \\ &\qquad + 56\theta(\tau) \\ &= q^{-3} \; - \; 248q + 26752q^4 + \dots \end{aligned}$$

The function f_0 is modular of weight $1/2$ for $\Gamma_0(4)$, and satisfies the "plus space" condition.

Let

$$J_d(\tau) = \prod_{\sigma} (j(\tau) - j(\sigma)) \in M_0(\Gamma),$$

where the product is taken over the set of imaginary quadratic numbers of discriminant $d < 0$, modulo $\Gamma = SL_2(Z)$.

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Borcherds shows how to explicitly find input functions G such that $\Psi_G = J_d$, by taking products of $\theta(\tau)$, $f_0(\tau)$ and $j(4\tau)$, in a way that one can read off a product formula for J_d in terms of the Fourier coefficients of these functions.

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eg. for $d = -3$, take $3f_0$. Product formula gives

$$j(\tau) = \Psi_G(\tau) = q^{-1} \prod_{n>0} (1-q^n)^{c(n^2)} = q^{-1}(1-q)^{744}(1-q^2)^{80256} \cdots$$

(see Borcherds' 1995 Inventiones paper)

Extending Gross-Kohnen-Zagier

Theorem (Borcherds (1999))

Let $\mathcal{M}_k$ be the bundle of modular forms of weight k on X_K, and c_1 be the first Chern class map $c_1 : Pic(X_K) \to CH^1(X_K)$. The formal series

$$A(\tau) = c_1(\mathcal{M}_{-1})\chi_0^* + \sum_{\mu \in L^\vee/L} \sum_{m>0} Z(m,\mu)\, q^m\, \chi_\mu^*$$

is a holomorphic modular form of weight $1 + p/2$ for ρ_L^* with coefficients in $CH^1(X_K)_{\mathbb{Q}}$ (ie. for any linear functional $\lambda \in CH^1(X_K)_{\mathbb{Q}}^*$, the formal sum

$$\lambda\,(c_1(\mathcal{M}_{-1}))\,\chi_0^* + \sum \lambda(Z(m,\mu))q^m\chi_\mu$$

is the q-expansion of a hol. mod. form.)

Proof

There is a pairing

$$M_{-k,\rho_L} \times \left\{\text{appropriate subspace of } S_L{}^*[[q^{1/h}]]\right\} \to \mathbb{C}$$

$$\left(\sum c_\mu(m)q^m\chi_\mu, \sum_{\mu,n\geq 0} b_\mu(n)q^n\chi_\mu^*\right) \mapsto \sum_\mu \sum_{m\leq 0} c_\mu(m)b_\mu(-m)$$

Under this pairing,

$$(M_{-k,\rho_L})^\perp = S_{2+k,\rho_L^*} = \text{ hol. mod. forms} \qquad \text{(Serre duality)}$$

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For any $f = \sum c_\mu(m) q^m \chi_\mu$, the Borcherds form Ψ_f^2 is a rational section of $\mathcal{M}_{c_0(0)}$ (up to torsion in Pic), hence

$$\sum_{m<0} c_\mu(m) Z(-m,\mu) = div(\Psi_f{}^2) = c_0(0) c_1(\mathcal{M}_1) \in CH^1(X_K)$$

Proof

Hence for any $\lambda \in CH^1(X)^*_{\mathbb{Q}}$, and $f = \sum c_\mu(m) q^m \chi_\mu \in M_{-k,\rho_L}$, have

$$\begin{aligned}(f, \lambda(A)) &= \lambda\left(c_0(0)c_1(\mathcal{M}_{-1}) + \sum c_\mu(m) Z(-m, \mu)\right)\\ &= \lambda\left(c_1(\mathcal{M}_{-c_0(0)}) + c_1(\mathcal{M}_{c_0(0)})\right) = \lambda(0) = 0\end{aligned}$$

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\end{aligned}
$$

Thus $\lambda(A)$ is a hol. mod. form, as required.

Version 2: Let $Heeg(X_K)$ be the free abelian group generated by symbols $Z(m,\mu)$ and $Z(0,0)$. Let $PHeeg(X_K)$ be the subgroup generated by elements

$$cZ(0,0) + div(\Psi_f^2),$$

where Ψ_f is a Borcherds form on X_K of weight $c/2$ for some input function f. Let $HeegDiv = Heeg(X_K)/PHeeg(X_K)$.

Theorem (Borcherds, generalizing GKZ)

The formal power series

$$A(\tau) = Z(0,0)\chi_0^* + \sum_{\mu, m>0} Z(m,\mu)q^m\chi_\mu^*$$

is a holomorphic modular form of weight $1 + p/2$ *for* ρ_L^* *with coefficients in HeegDiv (ie. for any linear functional* $\lambda \in$ *HeegDiv***, the formal sum*

$$\lambda(Z(0,0))\chi_0^* + \sum \lambda(Z(m,\mu))q^m\chi_\mu$$

is the q-expansion of a hol. mod. form.)

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Under this pairing,

$$(M_{-k,\rho_L})^\perp = S_{2+k,\rho_L^*} = \text{ hol. mod. forms} \qquad \text{(Serre duality)}$$

Now for any $f = \sum c_\mu(m)q^m\chi_\mu \in M_{-k,\rho_L}$, and $\lambda \in HeegDiv^*$, have

$$\begin{aligned}(f, \lambda(A)) &= c_0(0)\lambda(Z(0,0)) + \sum_\mu \sum_{m<0} c_\mu(m)\lambda(Z(-m,\mu)) \\ &= \lambda\left(c_0(0)Z(0,0) + div(\Psi_f^2)\right) = 0\end{aligned}$$