Introduction to Borcherds Forms
Montreal-Toronto Workshop in Number Theory

September 3, 2010
Main Goal

Extend theta lift to construct (meromorphic) modular forms on Sh. var. associated to $O(p,2)$ with amazing properties (explicit divisors, product formulas, etc.)
Let

- $(V, q)$ - rational quadratic space of signature $(p, 2)$. Denote the associated bilinear form by $<,>$, i.e $q(x) = 1/2 < x, x >$
- $H = GSpin(V)$
- $\mathbb{D} = \{\text{oriented negative-definite planes in } V(\mathbb{R})\} = \text{Herm. symm. domain attached to } H$ (of complex dimension $p$)
Let

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For any \(K\) a c.o.s of \(H(\mathbb{A}_f)\), we have the Shimura variety

\[
X_K = H(\mathbb{Q}) \backslash (\mathcal{D} \times H(\mathbb{A}_f)/K)
\]

which has a (canonical) model defined over \(\mathbb{Q}\).
For a vector $x \in V(\mathbb{Q})$, $q(x) > 0$, let

- $D_x = \{ z \in D \mid z \perp x \} \subset D$
- $H_x = GSpin(x^\perp) \subset H$

For any $h \in H(\mathbb{A}_f)$, the map

\[ H_x(\mathbb{Q}) \backslash D_x \times H_x(\mathbb{A}_f)/(hKh^{-1} \cap H_x(\mathbb{A}_f)) \to X_K \]

\[ [z, g] \mapsto [z, gh] \]

defines a divisor on $X_K$, which we denote by $Z(x, h)$. 
We’ll want to take certain linear combinations of these divisors as well. Take

- $L$ - an even integral lattice in $V$
- $L^\vee = \{ x \in V(\mathbb{Q}) \mid < x, L > \subset \mathbb{Z} \}$ the dual lattice
- Fix a set of representatives $\{x_\mu\}$ of $L^\vee / L$
- From now on, we assume $K$ stabilizes all the adelic cosets $x_\mu + \hat{L}$, where $\hat{L} = L \otimes \hat{\mathbb{Z}}$, 

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For $m \in \mathbb{Q}$, $m > 0$, and $\mu \in L^\vee/L$, suppose there is an $x_0 \in V$ with $q(x_0) = m$. Can write

$$\left\{ x \in x_\mu + \widehat{L} \mid q(x) = m \right\} = \coprod_r K \xi_r^{-1} x_0$$

for some finite collection of elements $\xi_1, \ldots, \xi_n$ of $H(\mathbb{A}_f)$. 

Definition

Define

\[ Z(m, \mu) = \sum_r \chi_\mu(\xi_r^{-1} x_0) Z(x_0, \xi_r), \]

where \( \chi_\mu \) is the characteristic function of the adelic coset \( x_\mu + \hat{L} \), and \( Z(m, \mu) = 0 \) if there is no rational vector of length \( m \).
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Remarks:

1. Definition is independent of \( x_0 \) and choice of \( \xi_r \)'s
2. As \( K \) varies, get a compatible system of cycles, so can actually define the cycle \( Z(m, \mu) \) on full Shimura variety \( Sh(G, \mathbb{D}) \).
3. If \( X_K \) is connected, then

\[ Z(m, \mu) = \sum_x pr(D_x), \]

where the sum is over rational vectors of norm \( m \) in the coset \( x_\mu + L \), modulo the action of \( \Gamma = H(\mathbb{Q}) \cap K \), and \( pr : \mathbb{D}^+ \to \Gamma \backslash \mathbb{D}^+ \simeq X_K \) is the projection.
Example: for $N \in \mathbb{Z}$, let

- $V = \{ A \in M_2(\mathbb{Q}) | A = {}^t A \}$
- $q(A) = N \det(A)$, so signature is $(1, 2)$, and $H = GL_2$.
- $L = \left\{ \begin{pmatrix} c/N & b \\ b/2N & a \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\}$
- $L^\vee = \left\{ \begin{pmatrix} c/N & b/2N \\ b/2N & a \end{pmatrix} \right\}$, and $L^\vee/L \cong \mathbb{Z}/2N \mathbb{Z}$
- $K_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \mid c \in N \mathbb{Z}_p \right\}$, and $K = \prod K_p$
Example: for \( N \in \mathbb{Z} \), let

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- \( L^\vee = \left\{ \begin{pmatrix} c/N & b/2N \\ b/2N & a \end{pmatrix} \right\} \), and \( L^\vee / L \simeq \mathbb{Z} / 2N \mathbb{Z} \)
- \( K_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p) \mid c \in N \mathbb{Z}_p \right\} \), and \( K = \prod K_p \)

Have \( X_K \simeq Y_0(N) = \Gamma_0(N) \backslash \mathbb{H} \) after identifying \( \mathbb{H} \simeq \mathbb{D}^+ \) via the map

\[
\tau \mapsto \text{span}_\mathbb{R} \left( \text{Re} \left( \begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix} \right), \text{Im} \left( \begin{pmatrix} \tau^2 & \tau \\ \tau & 1 \end{pmatrix} \right) \right)
\]
For $x = \begin{pmatrix} c/N & b/2N \\ b/2N & a \end{pmatrix} \in L^\vee$, have

$$\tau \in \mathbb{D}_x \iff Na\tau^2 + b\tau + c = 0.$$
For \( x = \begin{pmatrix} c/N & b/2N \\ b/2N & a \end{pmatrix} \in L^\vee \), have

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So for \( m \in \mathbb{Q}_{>0} \), and \( \mu \in \mathbb{Z}/2N\mathbb{Z} \), have

\[ Z(m, \mu) = \sum [\tau] \]

where the sum is over the images in \( Y_0(N) \) of \( \tau \in \mathbb{H} \) satisfying

\[ A\tau^2 + B\tau + C = 0 \]

with

\[ N | A, \quad B \equiv \mu \mod 2N, \quad B^2 - 4AC = 4Nm \]

ie. \( Z(m, \mu) = P_{4Nm, \mu} \) is a Heegner divisor in the terminology of Gross-Kohnen-Zagier.
Vector-valued modular forms

Want to extend the theta integral to a larger class of functions:

- Let $S(V(\mathbb{A}_f))$ be the space of ($\mathbb{C}$-valued) Schwartz functions on $V(\mathbb{A}_f)$, and let $S_L$ be the subspace spanned by the indicator functions $\chi_\mu$ of $x_\mu + \hat{L}$. 
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- There is an action of the metaplectic group $Mp_2(\mathbb{Z})$ on $S(V(\mathbb{A}_f))$, via the Weil representation, which restricts to a representation on $S_L$, denoted by $\rho_L$. 
For $k \in (1/2) \mathbb{Z}$, let $M_{k, \rho_L}$ be the space of holomorphic functions $F : \mathbb{H} \rightarrow S_L$ such that

1. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \varphi(\tau) \in Mp_2(\mathbb{Z})$, where $\varphi(\tau)^2 = c\tau + d$, have the transformation law

$$F \left( \frac{a\tau + b}{c\tau + d} \right) = \varphi(\tau)^{2k} \rho_L(\gamma) F(\tau)$$

2. $F$ is meromorphic at cusps
In terms of the basis elements $\chi_\mu$, have Fourier expansion at $\infty$:

$$F(\tau) = \sum_{\mu \in L^\vee / L} \sum_{m \in \mathbb{Q}} c_\mu(m) \, e(m\tau) \, \chi_\mu$$

where $c_\mu(m) = 0$ for $m << 0$ (meromorphic condition), $c_\mu(m) = 0$ for $m \notin q(x_\mu) + \mathbb{Z}$, and $e(x) = e^{2\pi ix}$.
The theta kernel

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Define the theta kernel (for $g \in G(\mathbb{A}), h \in H(\mathbb{A}), \phi \in S(V(\mathbb{A}))$)

$$\theta(g, h)(\phi) = \sum_{x \in V(\mathbb{Q})} \omega(g)\phi(h^{-1}x)$$
For $z \in \mathbb{D}$, define a positive definite quadratic form

$$q_z(x) = |q(pr_z(x))| + q(pr_z\perp(x))$$

where $pr_z(x)$ is the projection of $x$ onto $z$, and let

$$\phi_z^\infty(x) = e^{-\pi q_z(x)}, \quad \phi_z^\infty \in S(V(\mathbb{R}))$$
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Think of the theta kernel as a map

$$\theta : \mathbb{H} \times \mathbb{D} \times H(\mathbb{A}_f) \to S(V(\mathbb{A}_f))^*$$

by setting

$$\theta(\tau, z, h)(\phi) = v^{1-p/2} \theta(g_\tau, h)(\phi_z^\infty \otimes \phi)$$

$$= v^{1-p/2} \sum_{x \in V(\mathbb{Q})} \phi(h^{-1}x) \cdot (\omega(g_\tau)\phi_z^\infty)(x)$$
Restrict $\theta(\tau, z; h)$ to a functional in $(S_L)^*$. Then for fixed $z$ and $h$, the function

$$\theta(\cdot, z; h) : \mathbb{H} \to (S_L)^*$$

transforms like a modular form of weight $k = p/2 - 1$ for the dual representation $(\rho_L^*, S_L^*)$!
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transforms like a modular form of weight \( k = p/2 - 1 \) for the dual representation \((\rho_L^*, S_L^*)\)!

Hence, for \( F \in M_{-k, \rho_L} \), the pairing

\[
(F(\tau), \theta(\tau, z, h))
\]

is \( \Gamma = SL_2(\mathbb{Z}) \)-invariant.
As before, consider the $\theta$-integral:

$$\Theta_F(z, h) = \int_{\Gamma \backslash \mathbb{H}} (F(\tau), \theta(\tau, z; h)) \, v^{-2} \, du \, dv$$

This should be a function on $X_K$ (ie. a function on $\mathbb{D} \times H(\mathbb{A}_f)$ invariant under $H(\mathbb{Q})$ and $K$), but...
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- Problem: doesn’t converge!

- Solution:

$$\Theta^*_F(z, h) = \int_{\Gamma \backslash \mathbb{H}} (F(\tau), \theta(\tau, z; h)) \, v^{-2} \, du \, dv$$

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- Fact: For $\Re(s)$ sufficiently large,

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defines a holomorphic function in $s$.

- $I(s)$ can be meromorphically continued to all of $\mathbb{C}$. 
Define regularized integral by introducing a complex parameter $s$.

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**Definition**

Define the regularized integral $\Theta_F^\bullet(z, h)$ to be the constant term of the Laurent series expansion of $I(s)$ at $s = 0$. 
Theorem (Borcherds (1998))

Given $F \in M_{1-p/2, \rho_L}$, such that

$$c_\mu(m) \in \mathbb{Z} \quad \text{for } m < 0$$

there is a function $\Psi_F$ on $\mathbb{D} \times H(\mathbb{A}_f)$ such that

1. $\Psi_F$ is a meromorphic modular form of weight $c_0(0)/2$ (with respect to a multiplier system of finite order)
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1. \[ \Psi_F \text{ is a meromorphic modular form of weight } c_0(0)/2 \text{ (with respect to a multiplier system of finite order)} \]

2. \[ \text{div}(\Psi_F^2) = \sum_{\mu \in \mathbb{L}^\vee/L} \sum_{m > 0} c_\mu(-m) Z(m, \mu) \]
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2. $\text{div}(\Psi_F^2) = \sum_{\mu \in L^\vee / L} \sum_{m > 0} c_\mu(-m) \ Z(m, \mu)$

3. In the neighbourhood of a cusp, $\Psi_F$ has a product expansion

$$\Psi_F(z, h) = C \ e(\langle z, \rho(W) \rangle) \prod_{\mu, \xi} (1 - e(\langle z_f, \xi \rangle))^{c_\mu(-q(\xi))} \chi_\mu(h^{-1} \xi)$$
Idea of Proof

Evaluate the regularized integral to get explicit expressions for the Fourier expansion of $\Theta^*_{F}$, and $\text{div}(\Theta^*_{F})$, in terms of Fourier expansion of $F$. Then use a bit of complex analysis to show that there exists a function $\Psi_{F}$ such that

$$2 \log |\Psi_{F}(z, h)|^2 = -\Theta^*_{F}(z, f) - c_0(0) \left( \log |y|^2 + \text{const.} \right)$$

Deduce rest of theorem from this.
Some of your favourite functions are Borcherds forms

Example: Let

\[ F(\tau) = \sum_{n > 0, n \text{ odd}} \sigma_1(n)q^n = \sum_{n > 0, n \text{ odd}} \sum_{d | n} dq^n \]

\[ \theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2q + 2q^4 + \ldots \]

\[ f_0(\tau) = F(\tau)\theta(\tau) \left( \theta(\tau)^4 - 2F(\tau) \right) \left( \theta(\tau)^4 - 16F(\tau) \right) \frac{E_6(4\tau)}{\Delta(4\tau)} + 56\theta(\tau) \]

\[ = q^{-3} - 248q + 26752q^4 + \ldots \]

The function \( f_0 \) is modular of weight 1/2 for \( \Gamma_0(4) \), and satisfies the "plus space" condition.
Let

\[ J_d(\tau) = \prod_{\sigma} (j(\tau) - j(\sigma)) \in M_0(\Gamma), \]

where the product is taken over the set of imaginary quadratic numbers of discriminant \( d < 0 \), modulo \( \Gamma = SL_2(\mathbb{Z}) \).
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Borcherds shows how to explicitly find input functions \( G \) such that \( \Psi_G = J_d \), by taking products of \( \theta(\tau), f_0(\tau) \) and \( j(4\tau) \), in a way that one can read off a product formula for \( J_d \) in terms of the Fourier coefficients of these functions.
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eg. for \( d = -3 \), take \( 3f_0 \). Product formula gives

\[ j(\tau) = \Psi_G(\tau) = q^{-1} \prod_{n>0} (1-q^n)^{c(n^2)} = q^{-1}(1-q)^{744}(1-q^2)^{80256} \ldots \]

(see Borcherds’ 1995 Inventiones paper)
Extending Gross-Kohnen-Zagier

**Theorem (Borcherds (1999))**

Let $\mathcal{M}_k$ be the bundle of modular forms of weight $k$ on $X_K$, and $c_1$ be the first Chern class map $c_1 : \text{Pic}(X_K) \to CH^1(X_K)$. The formal series

$$A(\tau) = c_1(\mathcal{M}_{-1})\chi_0^* + \sum_{\mu \in L^\vee/L} \sum_{m > 0} Z(m, \mu) q^m \chi_\mu^*$$

is a holomorphic modular form of weight $1 + p/2$ for $\rho_L^*$ with coefficients in $CH^1(X_K)_Q$ (i.e. for any linear functional $\lambda \in CH^1(X_K)_Q^*$, the formal sum

$$\lambda \left( c_1(\mathcal{M}_{-1}) \right) \chi_0^* + \sum \lambda(Z(m, \mu)) q^m \chi_\mu$$

is the $q$-expansion of a holomorphic modular form.)
Proof

There is a pairing

$$M_{-k, \rho_L} \times \left\{ \text{appropriate subspace of } S_L^*[q^{1/h}] \right\} \to \mathbb{C}$$

$$\left( \sum c_\mu(m)q^m\chi_\mu, \sum_{\mu,n \geq 0} b_\mu(n)q^n\chi_\mu^* \right) \mapsto \sum_\mu \sum_{m \leq 0} c_\mu(m)b_\mu(-m)$$

Under this pairing,

$$(M_{-k, \rho_L})^\perp = S_{2+k, \rho_L^*} = \text{hol. mod. forms} \quad \text{(Serre duality)}$$
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For any \( f = \sum c_\mu(m)q^m \chi_\mu \), the Borcherds form \( \Psi_f^2 \) is a rational section of \( \mathcal{M}_{c_0(0)} \) (up to torsion in \( Pic \)), hence

\[ \sum_{m<0} c_\mu(m)Z(-m, \mu) = \text{div}(\Psi_f^2) = c_0(0)c_1(M_1) \in CH^1(X_K) \]
Proof

Hence for any $\lambda \in CH^1(X)_\mathbb{Q}^*$, and $f = \sum c_\mu(m)q^m \chi_\mu \in M_{-k, \rho_L}$, have

$$(f, \lambda(A)) = \lambda \left( c_0(0)c_1(M_{-1}) + \sum c_\mu(m)Z(-m, \mu) \right)$$

$$= \lambda \left( c_1(M_{-c_0(0)}) + c_1(M_{c_0(0)}) \right) = \lambda(0) = 0$$
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Thus $\lambda(A)$ is a hol. mod. form, as required.
Version 2: Let $Heeg(X_K)$ be the free abelian group generated by symbols $Z(m, \mu)$ and $Z(0, 0)$. Let $PHeeg(X_K)$ be the subgroup generated by elements

$$cZ(0, 0) + div(\Psi_f^2),$$

where $\Psi_f$ is a Borcherds form on $X_K$ of weight $c/2$ for some input function $f$. Let $HeegDiv = Heeg(X_K)/PHeeg(X_K)$. 
Theorem (Borcherds, generalizing GKZ)

The formal power series

\[ A(\tau) = Z(0, 0) \chi_0^* + \sum_{\mu, m > 0} Z(m, \mu) q^m \chi_\mu^* \]

is a holomorphic modular form of weight $1 + p/2$ for $\rho_L^*$ with coefficients in HeegDiv (ie. for any linear functional $\lambda \in \text{HeegDiv}^*$, the formal sum

\[ \lambda(Z(0, 0)) \chi_0^* + \sum \lambda(Z(m, \mu)) q^m \chi_\mu \]

is the q-expansion of a hol. mod. form. )
Proof

There is a pairing

\[ M_{-k,\rho_L} \times \left\{ \text{appropriate subspace of } S_L^*[[q^{1/h}]] \right\} \to \mathbb{C} \]

\[
\left( \sum_{\mu,n \geq 0} c_\mu(m) q^m \chi_\mu, \sum_{\mu,n \geq 0} b_\mu(n) q^n \chi^*_\mu \right) \mapsto \sum_{\mu} \sum_{m \leq 0} c_\mu(m) b_\mu(-m)
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Under this pairing,

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Now for any \( f = \sum c_\mu(m) q^m \chi_\mu \in M_{-k, \rho_L} \), and \( \lambda \in \text{HeegDiv}^* \), have

\[
(f, \lambda(A)) = c_0(0) \lambda(Z(0, 0)) + \sum_{\mu} \sum_{m < 0} c_\mu(m) \lambda(Z(-m, \mu))
\]

\[
= \lambda \left( c_0(0) Z(0, 0) + \text{div}(\Psi_f^2) \right) = 0
\]