Hilbert modular forms and cohomology

Shervin Shahrokhi Tehrani

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April 9th, 2011
Notations

- $F$ is totally real quadratic field extension.
- $\mathcal{O}_F$ is the ring of integers of $F$.
- $\alpha$ is a fractional ideal of $\mathcal{O}_F$.
- $\text{Cl}(F)$ is the ideal class group of $F$.
- $\text{Cl}(F)^+$ is the narrow ideal class group of $F$.
- $\mathbb{H}$ is upper half plane.
- $\mathbb{P}^1(F) = F \cup \{\infty\}$.
- $e(\omega) = e^{2\pi i \omega}$.
- If $M \subseteq F$ is a $\mathbb{Z}$-module of rank 2, then
  
  $$M^\vee = \{ \lambda \in F; \text{tr}(\mu \lambda) \in \mathbb{Z}, \forall \mu \in M \}$$

- The $\mathbb{A} = \mathbb{A}_\infty \mathbb{A}_f$ is adelic ring over $F$ where $\mathbb{A}_f$ is finite part of $\mathbb{A}$. 
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Let $F$ be a real quadratic field.

$$SL_2(F) \hookrightarrow SL_2(\mathbb{R}) \times SL_2(\mathbb{R}).$$

$SL_2(F)$ acts on $\mathbb{H} \times \mathbb{H}$ by

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( z_1, z_2 \right) = \left( \frac{az_1 + b}{cz_1 + d}, \frac{a'z_2 + b'}{c'z_2 + d'} \right)$$

**Definition**

$$\Gamma(\mathcal{O}_F \oplus a) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(F); a, d \in \mathcal{O}_F, b \in a^{-1}, c \in a \right\}$$
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**Definition**

The Hilbert full modular group is

$$\Gamma_F = \Gamma(\mathcal{O}_F \oplus \mathcal{O}_F) = \text{SL}_2(\mathcal{O}_F)$$

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Any subgroup of $\text{SL}_2(F)$ which is commensurable with $\Gamma_F$ is called an arithmetic subgroup.

Let $\Gamma$ be an arithmetic subgroup. It acts properly discontinuous on $\mathbb{H}^2$, i.e., if $W \subseteq \mathbb{H}^2$ is compact, then $\{\gamma \in \Gamma; \gamma W \cap W \neq \emptyset\}$ is finite.
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Modular Surfaces

Definition

The space

$$X'_\Gamma = \Gamma \backslash \mathbb{H}^2$$

is the modular surface.
Elliptic fixed points

The stabilizer of \( a \in \mathbb{H}^2 \)

\[ \Gamma_a = \{ \gamma \in \Gamma; \gamma a = a \} \]

is finite subgroup of \( \Gamma \).

**Definition**

\( a \) is called elliptic fixed point if

\[ \overline{\Gamma_a} = \Gamma_a / \{ \pm 1 \} \]

is not trivial.

**Proposition**

There are finite number of elliptic fixed points, and these are only singularities of \( X_\Gamma \).
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Cusp points

The $X'_{\Gamma}$ is not compact in general, therefore, there are points at infinity.

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\left( \frac{\alpha}{\beta} \right) = \frac{a\alpha + b\beta}{c\alpha + d\beta}.
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$\Gamma$-classes of $\mathbb{P}^1(F)$ are called cusp points of $X'_{\Gamma}$.
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$\Gamma$-classes of $\mathbb{P}^1(F)$ are called cusp points of $X'_\Gamma$. 
Cusp points

**Theorem**

The map

\[ \varphi : \Gamma_F \backslash \mathbb{P}^1(F) \to \text{Cl}(F) \]

\[ (\alpha : \beta) \mapsto \alpha \mathcal{O}_F + \beta \mathcal{O}_F \]

is bijective.

**Corollary**

The number of cusp points of \( X'_{\Gamma_F} \) is the class number of \( F \).
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The number of cusp points of \(X'_F\) is the class number of \(F\).
Let $G = R_{F/Q} \text{GL}_2(F)$ be reductive algebraic group over $\mathbb{Q}$. Therefore,

$$G(\mathbb{R}) = \text{GL}_2(\mathbb{R})^2$$

$$K_\infty = \text{SO}(2).\mathbb{R}_{>0} \times \text{SO}(2).\mathbb{R}_{>0}$$

The quotient $G(\mathbb{R})/K_\infty$ is homeomorphic with $\mathbb{H}^\pm \times \mathbb{H}^\pm$. 
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Let $K_f$ be compact open subgroup of $G_f = G(\mathbb{A}_f)$. Using Strong Approximation Theorem, we have

**Theorem**

*There is an identification*

$$G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K_f = \bigcup_{j=1}^{m} \Gamma_j \backslash \mathbb{H}^2$$

*with* $\Gamma_j = g_j G(\mathbb{R})^0 K_f g_j^{-1} \cap G(\mathbb{Q})$.  

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with $\Gamma_j = g_j G(\mathbb{R})^0 K_f g_j^{-1} \cap G(\mathbb{Q})$. 
If $K_0 = \prod_{\nu \in S_f} \text{GL}_2(\mathcal{O}_{\mathcal{F}_\nu})$, then

**Corollary**

$G(\mathbb{Q}) \backslash G(\mathbb{A}) / K_\infty K_0$ can be identified with $\bigcup_a \Gamma(\mathcal{O}_{\mathcal{F}} \oplus a) \backslash \mathbb{H}^2$, where $a$ runs over a complete set of representatives of $\text{Cl}(F)^+$. 
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Further properties

- There is fundamental domain for action of $\Gamma$ on $\mathbb{H}^2$ in terms of Siegel domains.
- The form $\omega = \omega_1 \wedge \omega_2$ where

  $$\omega_1 = \frac{1}{2\pi} \frac{dx_1 \wedge dy_1}{y_1^2}, \quad \omega_2 = \frac{1}{2\pi} \frac{dx_2 \wedge dy_2}{y_2^2}$$

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Compactification

We have

$$\mathbb{P}^1(F) \hookrightarrow \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}^1(\mathbb{R})$$

Let

$$(\mathbb{H}^2)^* = \mathbb{H}^2 \cup \mathbb{P}^1(F)$$

The group $\Gamma$ acts on $(\mathbb{H}^2)^*$. Let

$$X_\Gamma = \Gamma \backslash (\mathbb{H}^2)^*$$

then we have

**Theorem (Baily-Borel)**

On $(\mathbb{H}^2)^*$ there is unique topology such that the $\Gamma \backslash (\mathbb{H}^2)^*$ with quotient topology is a compact Hausdorff space. Moreover, there is a sheaf of functions $\mathcal{O}_{X_\Gamma}$ on $X_\Gamma$ such that $(X_\Gamma, \mathcal{O}_{X_\Gamma})$ is complex normal space.
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Remark: Using the line bundle of modular forms (in sufficiently large weights) on $X_{\Gamma}$, gives an embedding into projective space, therefore, $X_{\Gamma}$ is projective algebraic variety and $X'_{\Gamma}$ is quasi-projective.
There is smooth compactification of $X'_\Gamma$ using Toroidal Theory. Therefore, we can resolve the singularities at boundary of Baily-Borel compactification.

Also, by using the theory of Hironaka, we are able to resolve the singularities caused by Elliptic fixed points.

We are going to use the adelic version and fix following spaces:

- $X'_{K_f} = G(\mathbb{Q}) \backslash G(\mathbb{A}_\mathbb{Q})/K_f K_\infty$
- $X_{K_f}$ is its Baily-Borel compactification.
- $Z_{K_f}$ be the minimal resolution of the cusps.
- $Y_{K_f}$ be the minimal resolution of all singularities.
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Toroidal compactification and de singularization

- There is smooth compactification of $X'_{\Gamma}$ using **Toroidal Theory**. Therefore, we can resolve the singularities at boundary of **Baily-Borel** compactification.
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A holomorphic function \( f : \mathbb{H}^2 \to \mathbb{C} \) is called \textbf{Hilbert modular forms} of weight \( k = (k_1, k_2) \in \mathbb{Z}^2 \) on \( \Gamma \) if for all \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) one has

\[
 f(\gamma z) = (cz_1 + d)^{k_1} (c'z_2 + d')^{k_2} f(z).
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If \( k = k_1 = k_2 \) then \( k \) is called the weight of \( f \).
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If $f$ is Hilbert modular form then it has Fourier expansion at the cusp $\infty$ as:

There is $M \subset F$, $\mathbb{Z}$-module of rank 2, such that

$$f(z + \mu) = f(z) \quad \forall \mu \in M,$$

and

$$f(z) = \sum_{\nu \in M^\vee} a_{\nu} e(\text{tr}(\nu z)) \quad \text{where}$$

$$a_{\nu} = \frac{1}{\text{vol}(\mathbb{R}/M)} \int_{\mathbb{R}^2/M} f(z) e(-\text{tr}(\nu z)) \, dx_1 \, dx_2.$$

In contrast to the one dimensional case, we have
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Theorem (Koecher principle)

Let $f : \mathbb{H}^2 \to \mathbb{C}$ Hilbert modular form, then

$$a_\nu \neq 0 \quad \text{implies} \quad \nu = 0 \quad \text{or} \quad \nu \gg 0.$$ 

We denote the space of all Hilbert modular forms of weight $k$ by $M_k$. This has an interpretation as global section of line bundles over modular surface, and by using sheaf cohomology, $M_k$ is finite dimensional.

Definition

A Hilbert modular form is called cusp form if it vanishes at all cusps of $\Gamma$. $S_k$ is the space of all cusp forms of weight $k$. 
Hilbert modular forms

**Theorem (Koecher principle)**

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Non-vanishing conditions

**Theorem**

Let $f$ be Hilbert modular form of weight $k = (k_1, k_2)$ for $\Gamma$. The $f$ vanishes identically unless $k_1, k_2$ are both positive or $k_1 = k_2 = 0$. In latter case $f$ is constant.

**Corollary**

If $\pi : Z_{K_f} \longrightarrow X_{K_f}$ is the natural map, then any holomorphic 1-form on $Z_{K_f}$ vanishes identically, i.e

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Let $\omega$ be 1-form, and $\eta$ be pullback on regular points of $X_{kf}$. We have

$$\eta = f_1(z)dz_1 + f_2(z)dz_2$$

where $f_1, f_2$ are Hilbert modular form of weight $(2, 0)$, and $(0, 2)$. Using last theorem, we can say $\eta$ vanishes. Therefore, $\omega$ is zero.

Remark: Using Hodge theory we can show $H^1(Z_{kf}, \mathbb{C})$ vanishes. This means that the interesting part of cohomology of Hilbert modular surfaces is in degree 2.
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We are looking at $H^2(Z_{\Gamma}, \mathbb{Q})$. Using Poincaré duality, we have a non-degenerate pairing

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Let $E_{\sigma}$ be the subspace of $H_2(Z_{\Gamma}, \mathbb{Q})$ generated by the classes of the curves $S_{\sigma}$ in the resolving of cusp $\sigma$. We have the decomposition

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Mixed Hodge Structure

$X'_\Gamma$ is quasi-projective algebraic variety, therefore by Deligne Hodge Theory, there is mixed Hodge structure as:

- There is decreasing filtration $\{F_p\}_{p \in \mathbb{Z}}$ on $H^i(X'_\Gamma, \mathbb{Q}) \otimes \mathbb{C}$,
- There is an increasing weight filtration $\{W_k\}_{k \in \mathbb{Z}}$ on $H^2(X'_\Gamma, \mathbb{Q})$ as
  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = 0$ for $k \leq 1$
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  - $W_k H^2(X'_\Gamma, \mathbb{Q}) = H^2(X'_\Gamma, \mathbb{Q})$ for $k \geq 4$

- **Remark**: There exists a pure hodge structure of weight $k$ on $W_k H^2(X'_\Gamma, \mathbb{Q}) / W_{k-1} H^2(X'_\Gamma, \mathbb{Q})$. 
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Recall

\[ j^* H^2(Z_\Gamma, \mathbb{Q}) = j^* \left( \bigoplus_{\sigma} E_\sigma^\vee \right) \oplus \pi^* H^2(X_\Gamma, \mathbb{Q}) \].

Since \( j^* E_\sigma^\vee = 0 \) for all cusps,

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Let \( f = (\ldots, f_j, \ldots) \) be Hilbert modular form of weight 2. This defines a 2-form \( \omega_f \) by

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\omega_f = (2\pi i)^2 f_j(z) \, dz_1 \wedge dz_2 \quad \text{on} \quad \Gamma_j \backslash \mathbb{H}^2.
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**Lemma**

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F^2 \mathbb{H}^2(X_{\Gamma}, \mathbb{C}) = \{ \omega_f : f \in S_2 \}
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Let \( \varepsilon_1 = (\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, 1), \varepsilon_2 = (1, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}) \) acts on \( \mathbb{H}_+ \times \mathbb{H}_+ \) by

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\varepsilon_1 : (z_1, z_2) \mapsto (\overline{z}_1, z_2), \quad \varepsilon_1 : (z_1, z_2) \mapsto (z_1, \overline{z}_2)
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We define

\[ \eta_f = \varepsilon_2^* \omega_f \quad \eta_f' = \varepsilon_1^* \omega_f \]

**Theorem**

\( H^2(X_\Gamma, \mathbb{C}) \) is direct sum of

1. it is \((2, 0)\)-component \( \{ \omega_f : f \in \mathcal{S}_2 \} \)
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3. it is \((1, 1)\)-component \( \{ \eta_f + \eta'_g : f, g \in \mathcal{S}_2 \} \oplus W \), where \( W \) is the space generated by the forms \( \omega_1, \omega_2 \) on all components of \( X_\Gamma \).
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Cuspidal Cohomology

Definition

$H^2_{cusp}(X_\Gamma, \mathbb{C})$ is the orthogonal complement of $W$ in $H^2(X_\Gamma, \mathbb{C})$. 
Hecke ring

Let

\[ B = G(\mathbb{A}_\mathbb{Q}) \cap \left( G(\mathbb{R})^0 \times \prod_{\nu \in S_f} GL_2(O_{\mathcal{F}_\nu}) \right) \]

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Definition

Let \( H_K \) be the algebra over \( \mathbb{Q} \) generated by

\[ T(m) = \sum_b RbR \quad \text{where} \quad \det(b)O_{\mathcal{F}} = m. \]

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The action of Hecke operators on Modular Forms

**Note:** There is a version of action of Hecke operators on Modular forms, we are going to use this fact that $S_k$, i.e. the space of cusp forms has basis of of eigenforms for all Hecke operators and **Multiplicity one principle**, i.e. two eigenforms with same eigenvalue are multiple of each other.
The action of Hecke operators on Cohomology

We have

$$\mathbb{H}^2(X_{K_0}, \mathbb{Q}) = \mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q}) \oplus (\mathbb{Q}(-1))^{h^+}.$$  

where $\mathbb{Q}(-1)$ is the rational hodge structure of type $(1, 1)$ of $(2\pi i)\mathbb{Q}$.

Recall that $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$ has Hodge decomposition where each term is isomorphic with a space of cusp forms.

Therefore, $H_K$ acts on $\mathbb{H}^2_{cusp}(X_{\Gamma}, \mathbb{C})$ which is compatible with action on modular forms.
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The action of Hecke operators on $\mathbb{H}^2(X_{\Gamma}, \mathbb{C})$

**Theorem**

For any $T \in H_K$ we can attach $T^*$, an endomorphism of $\mathbb{H}^2(X_{K_0}, \mathbb{Q})$ such that preserves the Hodge decomposition, and

$$\int_{T^*c} \omega = \int_c T^*\omega \quad \forall c \in H_2(X'_{K_0}, \mathbb{Q}), \quad \omega \in H^2(X', \mathbb{Q})$$

where $T^*$ is its dual endomorphism on $\mathbb{H}_2(X_{K_0}, \mathbb{Q})$. Also,

$$\langle \omega_1, T^*\omega_2 \rangle = \langle T^*\omega_1, \omega_2 \rangle$$
Decomposition of $H_K$

**Proposition**

$H_K$ is a semi-simple finite dimensional algebra over $\mathbb{Q}$, and $S_2$ is an $H_K \otimes \mathbb{C}$-module of rank one. Moreover,

$$H_K = \bigoplus k_i$$

where $k_i$ is finite field extension of $\mathbb{Q}$.

We can choose set of primitive idempotents $\{e_1, e_2, \ldots, e_n\}$ such that $k_i = e_i H_K$. 
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Embedding of $k_i$

If $f$ is normalized eigenform $f$ in $e_i S_2$, then there is embedding

$$\sigma_i : k_i \longrightarrow \mathbb{C}$$

$$t(f) = \sigma_i(t)f$$

therefore, $k_i \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{\sigma} \mathbb{C}$. 

Using above embedding

$$e_i S_2 = \bigoplus_{\sigma} \mathbb{C}f^\sigma$$

where the $f^\sigma$ are the normalized eigenforms with $t(f^\sigma) = \sigma(t)f^\sigma$.

- **Remark:** $f^\sigma$ is called companion of $f$. 

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Decomposition of $\mathbb{H}^2_{cusp}(X_{K_0}, \mathbb{Q})$

Let $F = \{f_1, \ldots, f_n\}$ be a set of normalized eigenforms, one for each $k_i$ (i.e. $f_i \in e_i S_2$).

If $f \in e_i S_2$ the we shall write $e_i = e_f$ and $k_i = k_f$.

We let

$$H^2(M_f, \mathbb{Q}) := e_f \mathbb{H}^2_{cusp}(X, \mathbb{Q}).$$

**Theorem**

There is a decomposition of polarized Hodge structure on $\mathbb{H}^2_{cusp}(X, \mathbb{Q})$ as

$$\mathbb{H}^2_{cusp}(X, \mathbb{Q}) = \bigoplus_{f \in F} H^2(M_f, \mathbb{Q}).$$
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Further Decomposition of $H^2(M_f, \mathbb{Q})$

Let consider $\varepsilon_1, \varepsilon_2$ as involutions on $\mathbb{H}^2(X, \mathbb{Q})$.

Because the actions of $\varepsilon_1, \varepsilon_2$ commutes with Hecke operators, therefore,

$$H^2(M_f, \mathbb{Q}) = \bigoplus_{s, s' \in \{+, -\}} H^2(M_f, \mathbb{Q})_{ss'}$$

where $\varepsilon_1, \varepsilon_2$ act on $H^2(M_f, \mathbb{Q})_{ss'}$ as $s.Id$ and $s'.Id$ respectively.

**Proposition**

For every normalized eigenform $f \in S_2$ we have

$$\text{rank}_{k_f} H^2(M_f, \mathbb{Q})_{ss'} = 1.$$
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