Notes on Shimura Varieties

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Notation

The derived group G^{der}, the adjoint group G^{ad} and the centre Z of a reductive algebraic group G fit into exact sequences:

$$1 \longrightarrow G^{\text{der}} \longrightarrow G \xrightarrow{\nu} T \longrightarrow 1$$

$$1 \longrightarrow Z \longrightarrow G \xrightarrow{\text{ad}} G^{\text{ad}} \longrightarrow 1\,.$$

The connected component of the identity of $G(\mathbb{R})$ with respect to the real topology is denoted $G(\mathbb{R})^+$.

We identify algebraic groups with the functors they define. For example, the restriction of scalars $\mathbb{S} = \text{Res}_{\mathbb{C}\backslash\mathbb{R}}\mathbb{G}_m$ is defined by

$$\mathbb{S} = (- \otimes_{\mathbb{R}} \mathbb{C})^{\times}\,.$$

Base-change is denoted by a subscript: $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$.

References

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Shimura Datum

Definition

A *Shimura datum* is a pair (G, X) consisting of a reductive algebraic group G over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class X of homomorphisms $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ satisfying, for every $h \in X$:

- (SV1) $Ad \circ h : \mathbb{S} \longrightarrow \mathrm{GL}(\mathrm{Lie}(G_{\mathbb{R}}))$ defines a Hodge structure on $\mathrm{Lie}(G_{\mathbb{R}})$ of type $\{(-1,1),(0,0),(1,-1)\}$;
- (SV2) $\mathrm{ad}\, h(i)$ is a Cartan involution on G^{ad};
- (SV3) G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial.

These axioms ensure that $X = G(\mathbb{R})/K_\infty$, where K_∞ is the stabilizer of some $h \in X$, is a finite disjoint union of hermitian symmetric domains.

The Weight Homomorphism

Recall, the inclusion $\mathbb{R} \subset \mathbb{C}$ defines a morphism

$$\mathbb{G}_{m/\mathbb{R}} = (- \otimes_{\mathbb{R}} \mathbb{R})^{\times} \longrightarrow \mathbb{S} = (- \otimes_{\mathbb{R}} \mathbb{C})^{\times} .$$

For every $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$, (SV1) implies that $h(\mathbb{R}^{\times}) \in Z(\mathbb{R})$ since $h(\mathbb{R}^{\times})$ acts (trivially) on $\mathrm{Lie}(G)_{\mathbb{C}}$ through the characters $z/\bar{z}, 1, \bar{z}/z$ and $\ker \mathrm{Ad} = Z$. Therefore, $h|_{\mathbb{G}_m} = h'|_{\mathbb{G}_m}$ for all $h, h' \in X$ since h and h' are conjugate.

Definition

The homomorphism $w_X = h^{-1}|_{\mathbb{G}_m} : \mathbb{G}_m \longrightarrow G_{\mathbb{R}}$, for any $h \in X$, is the *weight homomorphism* of the Shimura datum (G, X).

Theorem of Baily and Borel

Theorem

Let $D(\Gamma) = \Gamma \backslash D$ be the quotient of a hermitian symmetric domain by a torsion free arithmetic subgroup Γ of $Hol(D)^+$. Then $D(\Gamma)$ has a canonical realization as a zariski-open subset of a projective algebraic variety $D(\Gamma)^*$. In particular, it has a canonical structure as a quasi-projective complex algebraic variety.

Essentially, there are enough automorphic forms on $D(\Gamma)$ which allow us to embed it in projective space.

Double Coset Space of a Shimura Datum

For every compact open subgroup $K \subset G(\mathbb{A}_f)$, let

$$\mathrm{Sh}(G,X)_K = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K\,.$$

In fact

$$\mathrm{Sh}(G,X)_K = G(\mathbb{Q})_+\backslash X^+ \times G(\mathbb{A}_f)/K = \coprod_{g\in\mathscr{C}} \Gamma_g\backslash X^+$$

where

- $G(\mathbb{Q})_+ = ad^{-1}(G^{\mathrm{ad}}(\mathbb{R})^+) \cap G(\mathbb{Q})$
- $\mathscr{C}$ is a set of representatives of $G(\mathbb{Q})_+\backslash G(\mathbb{A}_f)/K$
- Γ_g is the image in $G^{\mathrm{ad}}(\mathbb{R})^+$ of $gKg^{-1} \cap G(\mathbb{Q})_+$
- X^+ is a connected component of X.

Shimura Varieties

Definition

The *Shimura variety* associated to the Shimura datum (G, X) is the inverse system

$$\mathrm{Sh}(G, X) = \varprojlim_{K} \mathrm{Sh}(G, X)_K \, .$$

The theorem of Baily and Borel implies that $\mathrm{Sh}(G, X)_K$ is an algebraic variety (when K is sufficiently small so that the arithmetic subgroups Γ_g of $G^{\mathrm{ad}}(\mathbb{R})^+$ are torsion free). Moreover, for $K \subset K'$, the natural map

$$\mathrm{Sh}(G, X)_K \to \mathrm{Sh}(G, X)_{K'}$$

is algebraic. A Shimura variety is actually an inverse limit of complex quasi-projective varieties.

Connected Components

Theorem

Let (G, X) be a Shimura datum with such that G^{der} is simply connected. Then, for $K \in G(\mathbb{A}_f)$ sufficiently small, there is an isomorphism

$$\pi_0(Sh(G,X)_K) \cong T(\mathbb{Q})^\dagger \backslash T(\mathbb{A}_f)/\nu(K)\,.$$

where

$$T(\mathbb{R})^\dagger = Im(Z(\mathbb{R}) \to T(\mathbb{R})) \text{ and } T(\mathbb{Q})^\dagger = T(\mathbb{Q}) \cap T(\mathbb{R})^\dagger\,.$$

The $G(\mathbb{A}_f)$-action on $\mathrm{Sh}(G, X)$

Each $g \in G(\mathbb{A}_f)$ defines a map, for every K, of algebraic varieties

$$g : \mathrm{Sh}(G, X)_K \longrightarrow \mathrm{Sh}(G, X)_{g^{-1}Kg} \; : \; [x, h] \mapsto [x, hg]$$

and so there is a right action of $G(\mathbb{A}_f)$ on $\mathrm{Sh}(G, X)$.

Furthermore, since there is an action of $G(\mathbb{A}_f)$ on the entire inverse system, we have an action on the ℓ-adic cohomology

$$H^i(\mathrm{Sh}(G, X), \overline{\mathbb{Q}}_\ell) := \varinjlim_K H^i_{\text{ét}}(\mathrm{Sh}(G, X)_K, \overline{\mathbb{Q}}_\ell) \, .$$

The variety $\mathrm{Sh}(G, X)$ has a model over a number field and so the cohomology has a Galois action as well. Thus the cohomology of Shimura varieties is a natural setting in which to compare automorphic and Galois representations as in the Langlands correspondence.

Constructing h's via $\mathbb{R}$-algebra homomorphisms

How can we write down a homomorphism of affine group schemes $h : \mathbb{S} \to G_{\mathbb{R}}$?

It's easy if $G = (- \otimes_{\mathbb{Q}} A)^\times$ for some algebra A over $\mathbb{Q}$ such that we can define a homomorphism of $\mathbb{R}$-algebras $h : \mathbb{C} \hookrightarrow A_{\mathbb{R}}$.

For such a G and h, we can define

$$h : \mathbb{S} = (- \otimes_{\mathbb{R}} \mathbb{C})^\times \to G_{\mathbb{R}} = (- \otimes_{\mathbb{R}} A_{\mathbb{R}})^\times .$$

The magic here is that we can define an h simply by choosing the image of i in $A_{\mathbb{R}}$. The $\mathbb{R}$-linearity of h and the functoriality of $(-)^\times$ take care of everything else.

Modular Curves

Consider $\mathrm{GL}_{2\,/\mathbb{R}} = (- \otimes_{\mathbb{R}} \mathrm{M}_2(\mathbb{R}))^{\times}$. Choose some $g \in \mathrm{M}_2(\mathbb{R})$ such that $g^2 = -1$ and define a homomorphism of $\mathbb{R}$-algebras

$$h : \mathbb{C} \longrightarrow \mathrm{M}_2(\mathbb{R}) : i \mapsto g\,.$$

Then h defines a morphism of algebraic groups

$$h : \mathbb{S} \longrightarrow \mathrm{GL}_{2\,/\mathbb{R}}\,.$$

To determine the stabilizer of h, we need only consider the centralizer of g. For $g = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, we have

$$\mathrm{Cent}_{\mathrm{GL}_2(\mathbb{R})}(g) = \mathbb{R}^{\times}\,\mathrm{SO}_2(\mathbb{R})$$

and the conjugacy class of h is $X = \mathbb{C} \backslash \mathbb{R}$.

Modular Curves (continued)

Let $N = \prod_p p^{n_p}$ and define the compact open subgroup

$$K(N) = \prod_{p|N} (1 + p^{n_p} \mathrm{M}_2(\mathbb{Z}_p)) \prod_{p \nmid N} \mathrm{GL}_2(\mathbb{Z}_p) .$$

The modular curve $Y_1(N)$ is a connected component of $\mathrm{Sh}(\mathrm{GL}_2, X)_{K(N)}$.

Further, the group of connected components is

$$\pi_0(\mathrm{Sh}(\mathrm{GL}_2, X)_{K(N)}) = \mathbb{Q}^\times \backslash \mathbb{A}_f^\times / \det(K(N)) = (\mathbb{Z}/N\mathbb{Z})^\times .$$

Quaternionic Shimura Curves

Let B be an indefinite quaternion algebra over $\mathbb{Q}$ (ie. $B_{\mathbb{R}} = \mathrm{M}_2(\mathbb{R})$) and let $G = (- \otimes_{\mathbb{Q}} B)^{\times}$.

The previous construction applies and although the hermitian symmetric domain is the same as for GL_2, the Shimura varieties are very different since the quotients are by arithmetic subgroups of $B^{\times}$ (which are quite different than those of $\mathrm{GL}_2(\mathbb{R})$).

Symplectic Groups

Let (V, ψ) be a symplectic space (ie. V is a vector space over $\mathbb{Q}$ equipped with an alternating bilinear form ψ). Define the algebraic group $\mathrm{GSp}(V, \psi)$ over $\mathbb{Q}$ by

$$\mathrm{GSp}(V, \psi)(R) = \left\{ g \in \mathrm{GL}(V_R) : \begin{array}{c} \psi(gu, gv) = \nu(g)\psi(u, v) \\ \text{with } \nu(g) \in R^\times \text{ for all } u, v \in V_R \end{array} \right\}$$

for all $\mathbb{Q}$-algebras R.

Let $J \in \mathrm{End}(V_{\mathbb{R}})$ such that $J^2 = -1$, $\psi(Ju, Jv) = \psi(u, v)$ for all $u, v \in V_{\mathbb{R}}$ and $\psi(u, Jv)$ is positive definite.

Define a map of $\mathbb{R}$-algebras

$$h : \mathbb{C} \longrightarrow \mathrm{End}(V_{\mathbb{R}}) : i \mapsto J$$

and note that $\psi((a + Jb)u, (a + Jb)v) = (a^2 + b^2)\psi(u, v)$ for all $a, b \in \mathbb{R}$.

Symplectic Groups (continued)

The map of $\mathbb{R}$-algebras defines an algebraic map

$$h : \mathbb{S} \longrightarrow \mathrm{GSp}(V, \psi)_{\mathbb{R}}$$

and a Shimura datum $(\mathrm{GSp}(V, \psi), X)$.

Each conjugate of J by an element of $\mathrm{GSpin}(V, \psi)(\mathbb{R})$ defines a complex structure on $V_{\mathbb{R}}$. In fact,

$$X = \left\{ \begin{array}{c} \text{complex structures } J \text{ on } V_{\mathbb{R}} \text{ such that } \psi(Ju, Jv) = \psi(u, v) \\ \text{and } \psi(u, Jv) \text{ is either positive or negative definite} \end{array} \right\}$$

which is a hermitian symmetric domain of complex dimension $g(g+1)/2$ where $2g$ is the dimension of V.

In fact, a choice of a symplectic basis of V implies $X^+ = \mathscr{H}_g$ where

$$\mathscr{H}_g = \{Z \in \mathrm{M}_g(\mathbb{C}) : Z^t = Z \text{ and } \mathrm{im}(Z) > 0\}$$

is the Siegel space of genus g.

Siegel Modular Varieties

Definition

The *Siegel modular variety attached to* (V, ψ) is the Shimura variety $\mathrm{Sh}(\mathrm{GSp}(V, \psi), X)$.

The Siegel modular varieties are important because they are moduli spaces of abelian varieties with extra structure.
In particular, for every $K \subset G(\mathbb{A}_f)$, consider the set $\mathscr{M}_K$ of triples $(A, \pm s, \eta K)$ where

- A is an abelian variety over $\mathbb{C}$;
- s is an alternating form on $H_1(A, \mathbb{Q})$ such that s or $-s$ is a polarization on $H_1(A, \mathbb{Q})$;
- η is an isomorphism $V(\mathbb{A}_f) \to V_f(A)$ sending ψ to a multiple of s by an element of $\mathbb{A}_f^\times$.

Theorem

For every compact open subgroup $K \subset G(\mathbb{A}_f)$, there is a bijection

$$Sh(GSp(V, \psi), X)_K = \mathscr{M}_K / \cong \ .$$

Orthogonal Groups

We want to study Shimura varieties defined by orthogonal groups however these are not simply connected. The spin groups are the simply connected covers of orthogonal groups.

Let (V, q) be a rational quadratic space of signature $(p, 2)$. Denote the associated bilinear form by $\langle\,,\rangle$ so that $q(x) = \frac{1}{2}\langle x, x\rangle$. The Clifford algebra of (V, q) is

$$\mathrm{Cl}(V, q) = \bigoplus_{n\geq 0} V^{\otimes n}/(v \otimes v - q(v)\cdot 1; v \in V)$$

and has a $\mathbb{Z}/2$-grading, $\mathrm{Cl}(V, q) = \mathrm{Cl}^0(V, q) \oplus \mathrm{Cl}^1(V, q)$, according to the parity of the tensors. Note $V \hookrightarrow \mathrm{Cl}^1(V, q)$.

There is an anti-involution on $\mathrm{Cl}(V, q)$ defined by

$$(v_1 \otimes \cdots \otimes v_r)^* = (-1)^r v_r \otimes \cdots \otimes v_1$$

This defines a norm, called the spinor norm, on $\mathrm{Cl}(V, q)$, $\mathrm{Nm}(x) = x^* x$.

Orthogonal Groups (continued)

Define the algebraic group $GSpin(V, q)$ over $\mathbb{Q}$ by

$$GSpin(V, q)(R) = \{x \in \mathrm{Cl}^0(V, q)) \otimes R : \mathrm{Nm}(x) \in R^\times \text{ and } xVx^* \subset V\}$$

for any $\mathbb{Q}$-algebra R.

There is an natural map

$$\rho : \mathrm{GSpin}(V, q) \longrightarrow \mathrm{GO}(V, q) : x \mapsto (v \mapsto \alpha(x) \cdot v \cdot x^*)$$

where $\alpha(v_1 \otimes \cdots \otimes v_r) = (-1)^r v_1 \otimes \cdots \otimes v_r$. (It is an easy exercise to see that $\rho(x)$ preserves q up to a scalar). The point is that the derived group $\mathrm{Spin}(V, q)$ is simply connected.

Orthogonal Groups (continued)

The condition on the signature of (V, q) implies that there are two orthogonal vectors $e_1, e_2 \in V_{\mathbb{R}}$ with $q(e_1) = q(e_2) = -1$.

Let $j = e_1 e_2 \in \mathrm{Cl}^0(V, q)_{\mathbb{R}}$ and we see that $j^2 = -1$, $j^* = -j$ and $a + jb \in \mathrm{GSpin}(V, q)$ for all $a, b \in \mathbb{R}$. Notice that the order matters: $e_1 e_2 \neq e_2 e_1$ since $\mathrm{Cl}(V, q)$ is not commutative.

Finally, we have our homomorphism of $\mathbb{R}$-algebras with (anti-)involution

$$h : \mathbb{C} \to \mathrm{Cl}^0(V, q)_{\mathbb{R}} : i \mapsto j$$

which defines an algebraic map

$$h : \mathbb{S} \to \mathrm{GSpin}(V, q)_{\mathbb{R}} .$$

Orthogonal Groups (continued)

The elements $e_1, e_2 \in V_{\mathbb{R}}$ (*and the order of their indices*) define an *oriented* negative-definite 2-plane in $V_{\mathbb{R}}$. Elements of $\mathrm{GSpin}(V, q)(\mathbb{R})$ act transitively on

$$X = \{\text{oriented negative-definite 2-planes in } V_{\mathbb{R}}\}$$

which is a hermitian symmetric domain of complex dimension p.