TATE CONJECTURES FOR HILBERT MODULAR SURFACES

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Let k be a field that is finitely generated over its prime field

eg. k is an algebraic number field or $\mathbb{F}_p(T_1, \cdots, T_r)$.

Let $\overline{k}$ denote a separable algebraic closure of k.

Let X denote a smooth projective scheme over k.

Set $\overline{X} = X \times_k \overline{k}$.

Set for $\ell \neq p =$ char k,

$$H^i_\ell(X) = H^i(\overline{X}_{\acute{e}t}, \mathbb{Q}_\ell).$$

The Tate twist is defined as follows:

Define

$$\mathbb{Z}_\ell(1) \;=\; \lim\; \mu_{\ell^n}$$

with the inverse limit defined by the ℓ-th power map

$$\mu_{\ell^{n+1}} \;\hookrightarrow\; \mu_{\ell^n}$$

and with the natural G action on ℓ-power roots of unity.

It is $H_{2,\ell}(\mathbb{P}^1)$.

Set for $n > 0$

$$\mathbb{Z}_\ell(n) \;=\; \mathbb{Z}_\ell(1)^{\otimes n}$$

and for $n < 0$, set

$$\mathbb{Z}_\ell(n) \;=\; \mathrm{Hom}(\mathbb{Z}_\ell(-n), \mathbb{Z}_\ell).$$

Finally, set

$$\mathbb{Q}_\ell(n) \;=\; \mathbb{Z}_\ell(n) \;\otimes\; \mathbb{Q}_\ell.$$

For any $\mathbb{Q}_\ell$ vector space V_ℓ, define

$$V_\ell(n) \;=\; V \;\otimes\; \mathbb{Q}_\ell(n)$$

with the accompanying Galois action.

For $0 \leq j \leq \dim X$,

$$V_\ell^j(X) \;=\; H_\ell^{2j}(X)(j)$$

where the round brackets denote the Tate twist.

Let $Z^j(X,k)$ denote the $\mathbb{Q}$-span of algebraic subvarieties of codimension j on X which are defined over k.

Denote by $A_\ell^j(X,k)$ the image of $Z^j(X,k)$ under the ℓ-adic cycle class map.

The Galois group G acts on $V_\ell^j(X)$ and the first Tate conjecture is that the map

$$\mathbb{Q}_\ell \otimes Z^j(X,k) \longrightarrow V_\ell^j(X)^G$$

induced by the ℓ-adic cycle class map, is surjective.

Equivalently, one may ask that the map

$$\mathbb{Q}_\ell \otimes A_\ell^j(X,k) \longrightarrow V_\ell^j(X)^G$$

induced by inclusion, is an isomorphism.

Tate made this conjecture in 1965.

Let us denote this conjecture by $TC^j(X,k)$.

The Galois action on both sides is compatible so it suffices to prove the conjecture for large k.

It is conjectured that the kernel of the ℓ-adic cycle class map is independent of ℓ. That is, the notion of ℓ-adic homological equivalence is independent of ℓ.

This is known to be true in characteristic zero. Indeed, if $k \subset \mathbb{C}$, Then there is a comparison theorem that gives an isomorphism

$$H^*_B(X(\mathbb{C}), \mathbb{Q}(m)) \otimes \ \mathbb{Q}_\ell \ \simeq \ H^*_{et}(X \otimes \mathbb{C}, \mathbb{Q}_\ell(m))$$

between the Betti cohomology and the etale cohomology.

Here, $\mathbb{Q}(m)$ is the Hodge structure that is of rank 1, of pure bidegree $(-m,-m)$ and as a vector space is $(2\pi i)^m\mathbb{Q}$. It is the Betti analogue of the Tate twist.

The space

$$H^*_{et}(X\otimes\mathbb{C},\mathbb{Q}_\ell(m))$$

is isomorphic to

$$H^*_{et}(X\otimes\overline{k},\mathbb{Q}_\ell(m)).$$

Moreover, the cycle class map c_ℓ factors through c_B.

Consider the case $j = 1$, namely the surjectivity of

$$\mathbb{Q}_\ell \otimes \; Z^1(X,k) \;\longrightarrow\; V_\ell^1(X)^G.$$

For general X, this case is still open. In particular, an analogue of Lefschetz's $(1,1)$ theorem is not known.

Faltings proved the $j = 1$ case for Abelian varieties over number fields.

We shall discuss this problem in the context of Hilbert modular surfaces.

In general, studying the Tate conjecture for Shimura varieties is interesting because there is a natural supply of algebraic cycles (for example Shimura subvarieties) and there is a large algebra of correspondences (Hecke) that can be used to decompose the cohomology and relate it to automorphic forms.

We can try to produce algebraic cycles by day and use properties of automorphic forms to bound the dimension of Tate cycles by night. If we are lucky, we will find that the two are equal and that proves the Tate conjecture.

Let F be a real quadratic field and denote by $\mathcal{O}_F$ its ring of integers. The group $\mathsf{SL}_2(\mathcal{O}_F)$ embeds as a discrete subgroup of $\mathsf{SL}_2(\mathbb{R}) \times \mathsf{SL}_2(\mathbb{R})$ and thus, acts on two copies of the upper half plane $\mathfrak{h} \times \mathfrak{h}$ by fractional linear transformations. We also consider congruence subgroups Γ of $\mathsf{SL}_2(\mathcal{O}_F)$. The quotient

$$Y_\Gamma \;=\; \Gamma \backslash (\mathfrak{h} \times \mathfrak{h})$$

is an open Hilbert modular surface.

Let G denote the group $Res_{F/\mathbb{Q}}\mathsf{GL}_{2/F}$. It is an algebraic group defined over $\mathbb{Q}$ with the property that for any $\mathbb{Q}$-algebra $\mathcal{A}$, we have

$$G(\mathcal{A}) \;=\; \mathsf{GL}_2(\mathcal{A} \otimes_{\mathbb{Q}} F).$$

It is useful to work adelically: denote by $\mathbb{A} = \mathbb{A}_Q$ the ring of adeles over Q. Denote by $\mathbb{A}_f$ the subring of finite adeles.

From a Γ as above, we can define a compact open subgroup of $G(\mathbb{A}_f)$.

Given a compact open subgroup K_f of $G(\mathbb{A}_f)$ and K_∞ the maximal compact open subgroup of $G(\mathbb{R})$, set $K = K_f K_\infty$ and consider

$$Y_K \;=\; G(\mathbb{Q})\backslash G(\mathbb{A})/K.$$

In general, this is not connected, but it is defined over $\mathbb{Q}$. The connected components are permuted by $\mathsf{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in a way that is described by Shimura's reciprocity law.

In the case that $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$, the number of cusps is equal to the class number h_F of F.

Adding the cusps, gives a singular variety which we denote X_Γ or X_K. It is the Baily-Borel compactification and has the structure of a quasi-projective variety. It has a model defined over $\mathbb{Q}$.

Following Hirzebruch, there is a canonical smooth resolution Z_Γ or Z_K, also defined over $\mathbb{Q}$.

If K is sufficiently "small", this resolution introduces cycles at the cusps, which from the point of view of the Tate conjecture are spurious.

Using intersection cohomology, we can work with the Baily-Borel compactification X_Γ and thus avoid consideration of the cuspidal cycles.

Indeed, we have a Galois equivariant decomposition

$$H^2_\ell(\overline{Z}) \;=\; IH^2_\ell(\overline{X}) \;\oplus\; \oplus \mathbb{Q}_\ell(-1).$$

(From now on, we suppress the K or Γ from the notation, unless it is explicitly required.)

The algebra of Hecke correspondences $\mathbb{T}$ acts on X and on the cohomology. It decomposes the cohomology into components indexed by certain automorphic representations π of $\mathrm{GL}_{2/F}$:

$$IH^2_\ell(\overline{X}) \;=\; \oplus_\pi IH^2_\ell(\pi).$$

The notation $IH^2_\ell(\pi)$ simply signfies the π component.

This decomposition commutes with the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Inside $IH^2_\ell(\overline{X})$, there is a two-dimensional subspace coming from the "volume forms". These correspond to line bundles on X and the corresponding automorphic forms are Eisenstein series. The complement corresponds to the contribution from cusp forms.

We can restrict attention to the cuspidal part.

Denote by E_π the field of Fourier coefficients of π.

Each piece $IH^2_\ell(\pi)$ is 4 dimensional over $E_\pi \otimes \mathbb{Q}_\ell$.

Oda conjectures that there is an Abelian variety A_π defined over F of dimension $[E_\pi : \mathbb{Q}]$ such that

$$IH^2_\ell(\pi) \;=\; H^1_\ell(\overline{A_\pi}) \;\otimes\; H^1_\ell(\overline{A^\sigma_\pi})$$

where $1 \neq \sigma \in \mathrm{Gal}(F/\mathbb{Q})$.

This conjecture is known in the case π is a base change from $\mathbb{Q}$ (Oda) or if π has complex multiplication (M and Ramakrishnan).

If the Abelian variety A_π exists, the Tate conjecture would follow (from the known case of divisors on Abelian varieties (due to Faltings)). In the CM case, the existence is deduced from the Tate conjecture.

Harder, Langlands, and Rapaport showed that as Gal($\overline{Q}/F$)-modules, there is a factorization

$$IH^2_\ell(\pi) \ = \ V_\ell(\pi) \ \otimes V_\ell(\pi^\sigma)$$

with each factor being 2-dimensional. This would follow from Oda's conjecture.

A key step in the proof of the factorization uses Tate's theorem, namely that

$$H^2(\mathrm{Gal}(\overline{\mathbb{Q}}/F), \overline{\mathbb{Q}}_\ell^\times) = 0.$$

HLR proved that if π does not have complex multiplication, then each of these factors are irreducible and remain irreducible as H-modules for any open subgroup H of Gal($\overline{Q}/F$).

Using this, HLR computed the dimension of the space $Ta(\pi)$ of Tate cycles in each $IH^2_\ell(\pi)$.

In particular, suppose that π does not have complex multiplication (that is, is not the automorphic induction of a representation of GL_1). Then,

$$Ta(\pi) \;=\; 0$$

if π is not the lift of a representation of $\mathrm{GL}_{2\mathbb{Q}}$. If it is a lift, then

$$\dim\, Ta(\pi) \;=\; 1.$$

If π does have complex multiplication, then

$$\dim\, Ta(\pi) \;=\; 2.$$

Some cycles can be constructed by embedding the modular curve into the Hilbert modular surface. We have

$$\mathsf{SL}_2(\mathbb{Z}) \hookrightarrow \mathsf{SL}_2(\mathcal{O}_F)$$

and

$$\mathfrak{h} \hookrightarrow \mathfrak{h} \times \mathfrak{h}$$

giving rise to

$$\mathsf{SL}_2(\mathbb{Z})\backslash\mathfrak{h} \hookrightarrow \mathsf{SL}_2(\mathcal{O}_F)\backslash(\mathfrak{h} \times \mathfrak{h}).$$

More generally, we can work with a congruence subgroup. The projection of these cycles to each π component produces a Tate class in each $IH^2(\pi)$ for which π is a lift.

Embedding the modular curve inside the Hilbert modular surface is the geometric analogue of base change.

In general, we do not know the geometric meaning of functorial lifts (or drops) of automorphic forms.

The Jacquet-Langlands correspondence between forms on GL(2) and a quaternion division algebra over Q is known to be geometric because of Faltings. However, there is no explicit construction of a cycle.

Later, we shall see that we do not know the geometric analogue of "simultaneous base change".

Does the Saito-Kurokawa lift have a geometric meaning? The Tate conjecture predicts that it should.

Returning to Hilbert modular surfaces, this leaves the question of the extra cycles in the complex multiplication case. This case was settled by Klingenberg and independently by M and Ramakrishnan.

Both papers appeared in Inventiones, volume 89, 1987.

We shall describe the latter approach.

It does not proceed by producing new cycles: it is still an open problem to find these cycles explicitly.

Rather, it shows that the Tate cycles match the Hodge cycles and appeals to Lefschetz.

π is of complex multiplication type if and only if there an imaginary quadratic extension M (say) of F and a quadratic character χ (say) of F that defines M, such that

$$\pi \ \simeq \ \pi \otimes \chi.$$

We consider the twisting correspondence R_χ introduced and studied in the elliptic case by Shimura.

In the case of other Shimura curves, it was studied by Hida.

It was used by Momose and Ribet in their proof of the Tate conjecture for Jacobians of modular curves.

In the case of curves, the R_χ can be used to produce cycles on the Jacobian.

We use it very differently.

On cohomology, R_χ has the effect of twisting each π by χ.

In particular, if $\pi \simeq \pi \otimes \chi$, it preserves the π component.

This operator also acts on the Betti and deRham cohomology. In particular, analyzing its effect on the deRham cohomology and the Hodge components, produces a period relation.

This period relation can be used to construct a $(1,1)$ cohomology class with rational periods. Such a class is rational, and so is a Hodge cycle.

More precisely, the operator R_χ (which is defined as a twisted linear combination of Hecke operators) acts on $IH^2(\pi)$. This is a 4 dimensional space over the field E_π of Fourier coefficients.

The Hodge structure has a one dimensional $(2,0)$ and $(0,2)$ piece and a two dimensional $(1,1)$ piece. These can be described explicitly as follows.

Inside the space of π, there is a classical Hilbert modular form $f(z,w)$ and the $(2,0)$ piece is generated by $\omega_f = f(z,w)dz \wedge dw$.

The other pieces of the Hodge structure can be obtained by applying certain non-holomorphic involutions.

By an explicit calculation $R(\chi)$ acts as the Gauss sum $g(\chi)$ on the $(2,0)$ and the $(0,2)$ pieces, and as $-g(\chi)$ on the $(1,1)$ piece.

On the other hand, $R(\chi)$ acts on the Betti cohomology as well. Choosing a rational basis, one gets four periods $c_{\pm,\pm}$ of ω_f.

The matrix formed from these periods gives the change of basis from the rational structure to the de Rham structure.

Another explicit calculation shows that the matrix of $R(\chi)$ with respect to the rational structure is an anti-diagonal matrix whose entries are

$$g(\chi)c_{++}/c_{--}, g(\chi)c_{+-}/c_{-+},$$

$$g(\chi)c_{-+}/c_{+-}, g(\chi)c_{--}/c_{++}.$$

This matrix has to have rational entries and so this implies a period relation.

Using this, one constructs two $(1,1)$ classes with rational periods, in other words Hodge cycles. This matches the dimension of the space of Tate cycles.

Thus the strategy is to match Tate cycles with Hodge cycles and this can be applied in greater generality.

Working with D. Ramakrishnan to do this matching of Tate cycles and Hodge cycles in the case of Quaternionic Shimura varieties.

In particular, we get a proof of the Tate conjecture for quaternionic Shimura surfaces over a real quadratic field.

It was proved by Lai if the quaternion algebra comes from $\mathbb{Q}$ since in this case, one has a supply of Hirzebruch-Zagier cycles and one can follow the method of HLR.

This method will not work in general and one has to rely on period relations.

Several other cases have been studied:

Unitary Shimura surfaces (Blasius, Rogawski)

Product of two Hilbert modular surfaces (M, D. Prasad)

Product of two unitary Shimura surfaces (Knightly)

The second conjecture of Tate asserts that the order of pole of the L-function $L(IH_\ell^2(\overline{X}), s)$ has a pole at $s = 2$ of order equal to the dimension of Tate cycles, i.e.

$$\dim\ (IH_\ell^2(\overline{X})(1))^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})}.$$

More generally, if the L-function is viewed over an extension M of $\mathbb{Q}$, then

$$\mathrm{ord}_{s=2}\ L(IH_\ell^2(\overline{X})_{/M}, s)\ =$$

$$-\dim\ (IH_\ell^2(\overline{X})(1))^{\mathrm{Gal}(\overline{\mathbb{Q}}/M)}.$$

Since the L-function is a product of L-functions $L(IH^2(\pi), s)$, the conjecture can be studied at this level.

Harder, Langlands and Rapaport showed that if π does not have complex multiplication, then the above conjecture holds for Abelian extensions M of $\mathbb{Q}$.

The case of π having complex multipication was proved by Klingenberg and independently by M and Ramakrishnan. In this case, one does not need to restrict M to be Abelian over $\mathbb{Q}$. In fact, one gets poles (in some cases) over dihedral extensions.

As an indication of our level of ignorance regarding cohomological conjectures about algebraic cycles, we give several examples of simple cohomology constructions whose analogues are not known for algebraic cycles. For lack of a better name, we call these "exotic cycles".

1) Complex multiplication cycles:

This is an old example (perhaps due to Harder?)

Let ψ denote an algebraic Hecke character of weight 1 for an imaginary quadratic field F (say). By class field theory, we may view it as a homomorphism

$$\rho_{\psi,\ell} : \mathrm{Gal}(\overline{\mathbb{Q}}/F) \longrightarrow \mathbb{Q}_\ell^\times$$

with the characteristic property that for almost all primes $\mathfrak{p}$ of F, the above homomorphism is

$$\mathrm{Frob}_\mathfrak{p} \mapsto \psi(\mathfrak{p}).$$

ψ defines a cusp form f (say) of weight 2 for $\mathrm{GL}_{2/\mathbb{Q}}$. Indeed, f has Fourier expansion

$$f(z) \;=\; \sum_{\mathfrak{a}} \psi(\mathfrak{a}) \exp 2\pi i \mathbb{N}(\mathfrak{a}) z$$

where the sum is over integral ideals $\mathfrak{a}$ of F.

If $\mathfrak{f}$ is the conductor of ψ, then f is a form for $\Gamma_0(N)$ where

$$N \;=\; d_F \mathbb{N}\mathfrak{f}$$

and d_F is the absolute value of the discriminant of F.

We can induce the representation $\rho_{\psi,\ell}$ to $\mathbb{Q}$ and we get a 2-dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

The ℓ-adic representation associated to f is the above induced representation.

f contributes a subspace $H^1(f)$ to the cohomology $H^1(X)$ of the modular curve $X = X_0(N)$.

We can consider ψ^2. This is a Hecke character of weight 2 and corresponds to a cusp form g (say) of weight 3.

g contributes a subspace $H^2(g)$ to the cohomology $H^2(S)$ of the universal elliptic curve over X.

We have a natural map defined over $\mathbb{Q}$:

$$H^1(f) \otimes H^1(f) \longrightarrow H^2(g)$$

and hence also a map

$$H^1(X) \otimes H^1(X) \longrightarrow H^2(S)$$

and thus a Tate class in

$$H^2(X \times X) \otimes H^2(S) \subseteq H^4(X \times X \times S).$$

We don't know whether this is algebraic.

The squaring operation on Hecke characters is not understood algebraically.

2) Base change cycles

Let F be a real quadratic field. Let X_K be as above.

The non-trivial cohomology of X_K occurs in dimension 2. We have

$$IH^2(X_K) \;=\; \oplus_{\Pi}\, IH^2(\Pi).$$

Here, the sum is over certain automorphic representations Π of $\mathrm{GL}_2(F_{\mathbb{A}})$.

One way of producing Π that appear in the above sum is to begin with a π that appears in the cohomology of the modular curve.

Starting with such a π, one can consider $\Pi = \pi|_F$ (the base change of π to F).

Then, one shows that

$$H^2(\Pi) \simeq H^1(\pi) \otimes H^1(\pi)$$

as $\mathrm{Gal}(\overline{\mathbb{Q}}/F)$-modules.

Now, consider two such real quadratic fields F_1 and F_2 (say). We have two corresponding surfaces X_1 and X_2 (suppresing the compact subgroups for now).

Let π be as above and denote by Π_1 and Π_2 the base change of π to F_1 and F_2 respectively.

Then, as $\mathrm{Gal}(\overline{\mathbb{Q}}/F_1F_2)$-modules, we have

$$H^2(\Pi_1) \otimes H^2(\Pi_2) \;\simeq\; H^1(\pi)^{\otimes 4}.$$

If π is a "non-CM" form, as a Galois module, we may view $V = H^1(\pi)$ as the standard two dimensional representation of GL_2.

It is easy to see that in this case, the space of invariants in $V^{\otimes 4}$ is two dimensional. One invariant comes from

$$\wedge^2 V \otimes \wedge^2 V$$

and the other is a summand of

$$Sym^2 V \otimes Sym^2 V.$$

The first invariant is easily accounted for as coming from a product of a cycle on X_1 and on X_2. But the second invariant is a new cycle.

This construction (which can be made quite generally) is due to KM and D. Prasad.

3) Weil cycles:

Let A be an Abelian variety defined over k and let E be a subfield of End$(A)\otimes\mathbb{Q}$ in which $1 \in E$ acts as the identity endomorphism. Suppose that E acts k-rationally. Let

$$V_\ell \ = \ H^1_\ell(A)$$

and suppose that

$$r \ = \ \mathrm{dim}_{E\otimes\mathbb{Q}_\ell} V_\ell.$$

Consider

$$W_E(V_\ell) \ = \ \wedge^r_{E\otimes\mathbb{Q}_\ell} V_\ell.$$

This is a one dimensional $E \otimes \mathbb{Q}_\ell$- module.

The action of Gal($\overline{k}/k$) is through the determinant and apriori, this takes values in $(E \otimes \mathbb{Q}_\ell)^\times$.

The space $W_E(V_\ell)$ consists of Tate classes if and only if the determinant takes values in $\mathbb{Q}_\ell^\times$.

This is the ℓ-adic analogue of Weil's construction of exceptional Hodge classes.

Tate's conjecture on L-functions raises a compatibility question:

We may consider extension K/F and the base change $X_{/K}$ of X to K.

Tate's conjecture says that

$$\text{ord}_{s=j+1}\ L_{2j}(X_{/K}, s)\ =\ -\text{dim}\ A_\ell^j(X, K)$$

where the right hand side denotes the $\mathbb{Q}$-vector space spanned by codimension j-cycles modulo homological equivalence, that have a representative defined over K.

Since the right hand side stabilizes for K sufficiently large, compatibility would require the following to be true:

$$\sup_K\ \text{ord}_{s=j+1}\ L_{2j}(X_{/K}, s)\ \ll 1.$$

This is not known. It might be approachable in the function field case.