The Work of Schoen on CM Cycles: Part II

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Fourth Toronto-Montreal Workshop in Number Theory, 2012

Recipe for a CM cycle

- Pick elliptic curve E with CM by order $\mathscr{O}_D \subset L$ quadratic
- E corresponds to some x in

$$CM_D := \{x \in \dot{X}(\bar{\mathbb{Q}}) : \mathrm{End}(\pi^{-1}(x)) = \mathscr{O}_D\}$$

- Embed $\mathscr{O}_D \subset L$ by action on the $1-$dim L-space $H^0(E, \Omega_{E/L})$
- Choose $\sqrt{-1}$, i.e. $\mathfrak{H} \subset \mathbb{C}$, then $\mathscr{O}_D = \mathbb{Z}[\nu]$, for some $\nu \in \mathfrak{H}$
- Let Γ_ν denote the graph of ν in $E \times E$

Definition (The CM cycle corresponding to E)

$$z_E = \begin{cases} \Gamma_\nu - \Gamma_{\bar{\nu}} & \text{if } D \text{ is odd} \\ (\Gamma_\nu - \Gamma_{\bar{\nu}})/2 & \text{if } D \text{ is even} \end{cases}$$

Other characterizations of the CM cycle

- We have a map

$$\phi : \mathrm{NS}(E \times E) \to \mathrm{End}(\mathrm{CH}_0(E)_{\deg 0}) = \mathrm{End}(E_L)$$

$$\Gamma \mapsto (c \mapsto \mathrm{pr}_{2*}(\Gamma.\mathrm{pr}_1{}^*(c)))$$

- $S := \mathrm{Span}\{E \times e, e \times E, \Delta\}$ goes to $\mathbb{Z}$ under ϕ
- E has $\mathrm{CM} \Leftrightarrow S^\perp \simeq \mathbb{Z}$
- z_E is the *positive* generator of $S^\perp$: $\phi(z_E) \in \mathrm{End}(E_L) \cap \mathfrak{H}$
- B. Gross: z_E is the unique element of $S^\perp$ that ϕ takes to the positive trace zero element of minimal norm

Null-homologous cycles on the threefold

- $\mathrm{CM}_D := \{x \in \dot{X}(\bar{\mathbb{Q}}) : \mathrm{End}(E_x) = \mathscr{O}_D\}$
- The inclusion $E_x \subset Y$ induces

$$\psi_x : \mathrm{NS}(E \times E) \to \mathrm{CH}_1(\tilde{W}_{\bar{\mathbb{Q}}})$$

- For $x \in \mathrm{CM}_D$, let

$$z_x = \psi_x(z_E) \in \mathrm{CH}_1(\tilde{W}_{\mathbb{Q}})$$

- The cycles z_x are null-homologous: there exist explicit cycles $\mathscr{E}_x$ in $\tilde{W}_{\bar{\mathbb{Q}}}$ such that $\partial \mathscr{E}_x = z_x$ [Sch86]

There is a *localization sequence* due to Bloch[Blo86][Blo94]:

$$\bigoplus_{x\in X(\bar{\mathbb{Q}})} CH^1(\tilde{W}_x)\otimes\mathbb{Q} \xrightarrow{\oplus\psi_x} \mathrm{CH}^2(\tilde{W}_{\bar{\mathbb{Q}}})\otimes\mathbb{Q} \to \mathrm{CH}^2(\tilde{W}_{\eta_{\bar{\mathbb{Q}}}})\otimes\mathbb{Q} \to 0.$$

Lemma

Complex multiplication cycles generate a finite index subgroup of

$$\mathrm{Ker}(\mathrm{CH}^2(\tilde{W}_{\bar{\mathbb{Q}}})_{\mathrm{hom}} \to \mathrm{CH}^2(W\times_X \eta_{\bar{\mathbb{Q}}}))$$

Key fact in proof an (easier) analogue of Drinfeld-Manin due to [Sch93]: Null-homologous 1-cycles on the singular fibres contribute a finite goup to $CH^2(\tilde{W}_{\bar{\mathbb{Q}}})_{\mathrm{hom}}$.

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Definition

A point $x \in \mathrm{CM}_D$, corresponding to $(E, s_1, \mathscr{H})$ is *Heegner* if both $\langle s_1 \rangle$ and $\mathscr{H}$ are $\mathscr{O}_D$-modules

- If E is Heegner, $E[3] \simeq \mathscr{O}_D/3 \Rightarrow 3$ splits in $\mathscr{O}_D$
- $(E, s_1, \mathscr{H})$ has CM by $\mathscr{O}_D$, and 3 splits in $\mathscr{O}_D \Rightarrow 2$ Heegner points in its $PSL_2(\mathbb{F}_3)$ orbit
- For $\tau \in \mathfrak{H}$, if $a\tau^2 + b\tau + c = 0$, $gcd(a, b, c) = 1$, $a > 0$,
$$\tau \text{ is Heegner} \Leftrightarrow 3|a \text{ and } 3|c$$

Fields of Definition

Given a point $x \in \mathrm{CM}_D$, set $k_D = \mathbb{Q}(\sqrt{D})$, and let

$$H_D = k_D(j(\mathscr{O}_D)) \text{ be the ring class field of } \mathscr{O}_D$$

$$H = \begin{cases} H_D & \text{if } x \text{ is Heegner} \\ H_{9D} & \text{otherwise} \end{cases}$$

Proposition

Both x and z_x are defined over H. Moreover, H is the smallest field of definition of z_x.

- Above is due to an accident: $(\mathbb{Z}/N\mathbb{Z})^* \simeq \{\pm 1\}$ when $N = 3$.
- $\forall T \in SL_2(\mathbb{F}_3)$, $T_* z_x = z_{T_x}$ (level str. doesn't affect cycle)
- $\forall \sigma \in \mathrm{Gal}(H_{9D}/\mathbb{Q})$, $\sigma(z_x) = \epsilon_D(\sigma) z_{\sigma(x)}$, where ϵ_D is the quadratic character of $k_D/\mathbb{Q}$

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Galois Action on CM cycles: Passage to a quotient

- For each discriminant $D < 0$, let

$$Z(D) = \text{ subgroup of } \bigoplus_{x \in \mathrm{CM}_D} NS(p^{-1}(x)) \text{ generated by CM cycles}$$

- Goal: describe action of $\mathrm{Gal}(H_{9D}/\mathbb{Q})$ on image of $Z(D)$ in $CH^2(\tilde{W}_{\mathbb{C}})_{\text{hom}} \otimes \mathbb{Q}$
- $A = PSL_2(\mathbb{F}_3)$ acts on Z(D).
- $A_2 = Syl_2(A) \lhd A$ acts trivially on $J_1(\tilde{W})$ and $CH^2(\tilde{W}_{\mathbb{C}})_{\text{hom}} \otimes \mathbb{Q}$.
- If $U \in A$ has order 3, $1 + U + U^2$ kills the same two spaces.
- Let

$$\mathscr{Z}(D) := (Z(D)_{A_2})/(1 + U + U^2)$$

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- $\mathscr{G} = SL_2(\mathbb{F}_3) \rtimes \mathrm{Gal}(H_{9D}/\mathbb{Q}) \subset \mathrm{Aut}(\tilde{W}_{H_{9D}})$ acts on $Z(D)$
- $(A/A_2) \rtimes \mathrm{Gal}(H_{9D}/\mathbb{Q})$ acts on $Z(D)_{A_2}$
- $Gal(H_{9D}/\mathbb{Q})$ acts on $\mathscr{Z}(D)$

Reduced to studying $\mathscr{Z}(D)$, an explicit Galois representation

- Aim: decompose $\mathscr{Z}(D) \otimes \mathbb{C}$ into irreducibles
- Will generalize Heegner points
- Need points stable under A_2 action

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Special points

Definition

1. If $D \equiv 1 (\mathrm{mod}\ 3)$, we say a CM point $(E, s_1, \mathscr{H})$ is *special* if either both $\langle s_1 \rangle$ and $\mathscr{H}$ are $\mathscr{O}_D$-modules, or neither is. This is equivalent to being in an A_2-orbit of a Heegner point.
2. If $D \equiv -1 (\mathrm{mod}\ 3)$ and $D < -4$, we say a CM point $(E, s_1, \mathscr{H})$ is *special* if the unique element of $Gal(H_{9D}/H_D) \simeq \mathbb{Z}/4\mathbb{Z}$ that has order 2 acts on it by switching $\langle s_1 \rangle$ and $\mathscr{H}$

Some basic properties, for $D = \pm 1 (\mathrm{mod}\ 3)$, $D < -4$:

- Special points stable under $\mathrm{Gal}(H_{9D}/\mathbb{Q})$-action
- A_2 acts simply transitively on $SL_2(\mathbb{F}_3)$-orbits $\mathbf{o}$
- $Z_{sp}(\mathbf{o}) \otimes \mathbb{C} = \mathrm{span}\langle \mathbf{o} \rangle_{\mathbb{C}} \Rightarrow Z_{sp}(\mathbf{o})_{A_2} \otimes \mathbb{C} \simeq \mathbb{C}$

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Description of the Galois representation $\mathscr{X}(D)$

- For any disc $D < 0$,
  $\mathrm{Gal}(H_D/\mathbb{Q}) \simeq$ a generalized dihedral group :

$$N \rtimes C_2, N \text{ abelian }, C_2 = \langle x : x^2 = 1 \rangle, \forall n \in N, xnx = n^{-1}$$

- $\forall \chi : \mathrm{Gal}(H_D/\mathbb{Q}) \longrightarrow \mathbb{C}, \chi^2 = 1$
- $\pi \in \mathrm{Rep}(G)^{\mathrm{irr}}, \dim \pi \neq 1 \Leftrightarrow \pi \simeq \mathrm{Ind}_N^G(\varsigma), \varsigma : N \to \mathbb{C}^*, \varsigma^2 \neq 1$

Definition (Ad-hoc, based on root number of $L_k(\tilde{W}, s)$)

$$\textit{type of } k := \delta(k) := \begin{cases} -1 & k \text{ imaginary and unramified at } 3 \\ -1 & k \text{ real and ramified at } 3 \\ 1 & \text{otherwise} \end{cases}$$

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Decomposition of $\mathscr{Z}(D) \otimes \mathbb{C}$

Proposition

1. *If* $D \not\equiv 0 (mod\ 3)$*, then*
   - $\mathrm{Gal}(H_{9D}/H_D(\mu_3))$ *acts trivially on* $\mathscr{Z}(D)$
   - *Every irreducible two dimensional representation of* $\mathrm{Gal}(H_D(\mu_3)/\mathbb{Q})$ *occurs in* $\mathscr{Z}(D) \otimes \mathbb{C}$ *with multiplicity one*
   - *A non-trivial character* χ *occurs if and only if* $\delta(\mathrm{Fix}(\ker \chi)) = -1$*, in which case it has multiplicity 1*
   - *The trivial representation does not occur*
2. *If* $D \equiv 0 (mod\ 3)$*, then*
   - *Irreducible representations that factors through* $\mathrm{Gal}(H_D/\mathbb{Q})$ *do not occur*
   - *All others occur with multiplicity one.*

Consequences

Corollary

1. *Let Q_0 be denote the compositum of all quadratic fields of type 1. Then CM cycles make no contribution to*

$$(\mathrm{CH}^2(\tilde{W}_{\mathbb{Q}}) \otimes_{\mathbb{Q}})^{\mathrm{Gal}(\bar{\mathbb{Q}}/Q_0)}$$

2. *If $x \in CM_{D\,sp}$ then $z_x \in (CH^2(\tilde{W}_{\bar{\mathbb{Q}}}) \otimes \mathbb{Q})^{\mathrm{Gal}(\bar{\mathbb{Q}}/H_D)}$*

- $1 \Leftarrow \forall D < 0$, $(\mathscr{L}(D) \otimes \mathbb{Q})^{\mathrm{Gal}(H_{9D}/H_{9D} \cap Q_0)} = 0$.
- $2 \Rightarrow x \in CM_{D\,sp}$, z_x is defined over $H \subsetneq H_{9D}$ (cf. Heegner)
- $1 \Rightarrow F$ real quad. type 1, $\mathrm{CH}^2(\tilde{W}_F)_{\mathrm{CM}} \otimes \mathbb{Q} = 0$, so:

$$CH^2(\tilde{W}_F)_{\mathrm{CM}} \leq ord_{s=2} L_F(H^3(\tilde{W}), s)$$

An Experiment

- Much more interesting:
  - F is quadratic imaginary, and type -1
  - $ord_{s=2} L_{F_D}(H^3(\tilde{W}), s) = 1$
- Idea for experiment: Compute Abel-Jacobi image of various cycles over such an F, check if they live on a line

How to Cook Many Cycles over an Interesting Field

1. Pick and fix an imaginary quadratic discriminant $D_i \not\equiv 0 (\text{mod } 3)$.
2. Pick any real quadratic discriminant $D_r \not\equiv 0 (\text{mod } 3)$
3. $D = D_r D_i$ is another quadratic imaginary discriminant, and $k_{D_i} \subset H_D$
4. Let $\epsilon_{D_i} : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \pm 1$ be quadratic character of k_{D_i}
5. The proposition predicts $(\mathscr{Z}(D) \otimes \mathbb{Q})^{\epsilon_{D_i}}$ has dimension 1.
6. Find a non-zero element, perhaps $Tr_{H_{9D}/k_{D_i}}(z_x)$ for some $x \in CM_D$.
7. Pick another D_r, repeat.

Abel Jacobi and the cycle class map

- These cycles are explicit: one can try to compute their Abel-Jacobi image
- Schoen did so using BASIC in 1993 and found some relations
- Difficult problem: describe the image of

$$cl : \mathscr{Z}(D) \to \mathrm{CH}^2(\tilde{W}_{\mathbb{C}})_{\mathsf{hom}} \otimes \mathbb{Q}$$

- Schoen suggested investigating the case when Abel-Jacobi image is zero. He found one such case: $d = 172$ $(= D_f, D_i = -4)$.
- SAGE in a few seconds computes the next few: 172, 524, 1292, 1564, 1793, 3016, 4169, 4648

Real discriminants with $ord_{s=2} L_F(H^3(\tilde{W}), s) = 1$

- From a 2006 paper of M. Watkins[Wat08]:

172	524	1292	1564	1793	3016	4169	4648
6508	9149	9452	9560	10636	11137	12040	13784
14284	15713	17485	17884	22841	22909	22936	25729
27065	27628	29165	30392	34220	35749	38636	40108
41756	44221	47260	51512	54385	57548	58933	58936
58984	59836	59996	62353	64268	70253	74305	77320
77672	78572	84616	86609	86812	87013	92057	95861
96556	97237	99817					

Further Work

- Bloch's localization sequence: relations come from Higher Chow groups

$$\ldots \to \mathrm{CH}^2(\tilde{W}, 1) \to \mathrm{CH}^2(\tilde{W}_\eta, 1) \xrightarrow{\partial} \bigoplus_{x \in X} \mathrm{CH}^1(\tilde{W}_x) \to \ldots$$

- A. Besser[Bes95] has repeated Schoen's argument for r-fold product of $A \to X_\Gamma$ universal abelian surface A over a modular surface.
- R. Sreekantan [Sre01] has computed relations between CM classes on Shimura curves, using elements of $CH^2(X, 1)$ constructed by A. Collino [Col97]

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