Schoen's CM cycles: Part 1

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Overview

- Schoen, Chad. *Complex multiplication cycles and a conjecture of Beilinson and Bloch*. Trans. Amer. Math. Soc., vol 339. (1993)

Goal is to "subject [the Beilinson-Bloch conjecture] to a modest but needed test".

- Schoen constructs a projective 3-fold $\widetilde{W}$ fibred over $\mathbb{P}^1_{\mathbb{Q}}$.
- Then Beilinson-Bloch would say, for any number field F

$$rank\ CH^2(\widetilde{W}_F)_{\mathrm{hom}} = ord_{s=2} L_F \left(H^3(\widetilde{W}), s \right)$$

- For any $F/\mathbb{Q}$, let η_F be the generic point of $\mathbb{P}^1_F$, and define

$$CH^2(\widetilde{W}_F)_{CM} := Ker\left[CH^2(\widetilde{W}_F)_{hom} \to CH^2(\widetilde{W}_F \times \eta_F)\right];$$

essentially, cycles that are 'vertical'.

Theorem

Suppose $[F:\mathbb{Q}] \leq 2$, and the sign in the functional equation for $L_F\left(H^3(\widetilde{W}), s\right)$ is positive. Then

$$rank\ CH^2(\widetilde{W}_F)_{CM} = 0.$$

Theorem

Suppose $[F : \mathbb{Q}] \leq 2$*, and the sign in the functional equation for* $L_F\left(H^3(\widetilde{W}), s\right)$ *is positive. Then*

$$\text{rank } CH^2(\widetilde{W}_F)_{CM} = 0.$$

- Proof involves an analysis of Galois action on $CH^2(\widetilde{W}_{\overline{\mathbb{Q}}})_{CM}$, and determine which fields F (don't) give contributions. Then compare with the sign in functional equation.
- For real quadratic fields unramified at 3, condition on sign is satisfied. The values of the L-functions at $s = 2$ are given by Fourier coefficients of $c_d$of a weight $5/2$ modular form ($d = Disc(F)$), which can be explicitly computed. So for d such that $c_d \neq 0$, have

$$\text{rank } CH^2(\widetilde{W}_F)_{CM} = ord_{s=2} L_F\left(H^3(\widetilde{W}), s\right).$$

- For quadratic fields with negative sign in functional equation, Schoen also describes some numerical evidence consistent with BB:
- For $F = \mathbb{Q}(i)$, Schoen constructs infinitely many CM cycles.
- Expect lots of relations in $CH^2(\tilde{W}_F)_{CM}$.
- Compute images under Abel-Jacobi: Schoen computes about 50, and finds strong evidence their Abel-Jacobi images are all multiples of a single CM cycle.
- Remark: *rank* $CH^2(\widetilde{W_{\overline{\mathbb{Q}}}})_{CM} = \infty$ (Schoen 1986).

Our plan:

- Siddarth: Define $\tilde{W}$, and a discussion of $PSL_2(\mathbb{F}_3)$ action on $CH^2(\widetilde{W_{\mathbb{C}}})$.
- Zavosh: Define CM cycles + Galois action.
- Brian: L-functions, and calculations when the sign in functional equation is negative.

First view: By an equation

- Let $Y \subset \mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^2_{\mathbb{Q}}$ defined by

$$Y: \; u_0(x_0^3 + x_1^3 + x_2^3) + u_1(x_0 x_1 x_2) = 0.$$

- Denote projection onto first factor ($[u_0 : u_1]$-coordinate.) by $\pi : Y \to \mathbb{P}^1_{\mathbb{Q}}$
- Let $W = Y \times_\pi Y$. The singularities are ordinary double points (more on this), and $\widetilde{W}$ is the blowup along these points.

- $Y: \ u_0(x_0^3 + x_1^3 + x_2^3) + u_1(x_0x_1x_2) = 0$
- What do fibres of $Y \to \mathbb{P}^1$ look like?
- Fibres are non-singular (Desboves curves) *unless* either $u_0 = 0$ or $u_1^3 - 27u_0^3 = 0$.
- Take parameter $u = u_1/u_0$ on $\mathbb{P}^1$, and let

$$\dot{X} = \mathbb{P}^1 \setminus \{\infty, 3, 3\zeta, 3\zeta^2\} = Spec\,\mathbb{Q}[u, (u^3 - 27)^{-1}],$$

  where ζ is a primitive 3rd root of unity.
- Then the fibres of $\dot{\pi} : \dot{Y} \to \dot{X}$ are elliptic curves.

Second view: moduli space of elliptic curves

- Let $F : Sch/\mathbb{Q} \to Set$ be the moduli functor

$$F(S) = \{(E, s_1, \mathcal{C})\} / \simeq$$

  namely isom. classes of triples consisting of:

  1. E/S an elliptic curve over S
  2. s_1 a section of order 3.
  3. $\mathcal{C}$ a subgroup-scheme of E s.t. $\mathcal{C} \simeq \mu_3$, and disjoint from s_1.

- The extra data is called a *weak level-3-structure* on E.

- Claim: F is representable by $\dot{X} = Spec\,\mathbb{Q}[u, (u^3 - 27)^{-1}]$.
- Reason: Given $(E, s_1, \mathcal{C})$ over $S = Spec(A)$ for a $\mathbb{Q}$-algebra A.
- Schoen shows that there exists two unique rational functions t_1, t_2 on E, such that

$$\mu := \frac{1 + t_1^3 + t_2^3}{t_1 t_2} \in A$$

- Note this means $1 + t_1^3 + t_2^3 = \mu t_1 t_2$ on E.
- Then

$$\mathbb{Q}[u, (u^3 - 27)^{-1}] \to A, \qquad u \mapsto \mu$$

is the isomorphism $F \to Hom(\cdot, \dot{X})$.

- Recall $\dot{Y} \to \dot{X}$; set $s_1 = [-1 : 0 : 1]$, and $s_2 = [0 : -\zeta : \zeta^2]$ and $\mathcal{C} = \langle s_2 \rangle$. Then

$$(\dot{Y}, s_1, \mathcal{C})$$

is the *universal object* for F.

Third view: uniformizations by modular functions

- Want to describe complex points as an arithmetic quotient (as in the case of full level structure)
- Let

$$\Gamma(3) = \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \ mod \ 3 \right\},$$

and $\widetilde{\Gamma}(3)$ be the semidirect product $\Gamma(3) \ltimes \mathbb{Z}^2$, with multiplication

$$\left(\gamma_1, \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} \right) \cdot \left(\gamma_2, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right) = \left(\gamma_1 \gamma_2, \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix} + \gamma_1 \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right)$$

- $\widetilde{\Gamma}(3)$ acts freely on $\mathfrak{H} \times \mathbb{C}$:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) \cdot (\tau, z) = \left(\frac{a\tau + b}{c\tau + d}, \frac{z + \alpha\tau + \beta}{c\tau + d} \right)$$

- Schoen constructs (very) explicit meromorphic functions $t_1, t_2 : \mathfrak{H} \times \mathbb{C} \to \mathbb{C}$ that are $\widetilde{\Gamma}(3)$ invariant and such that

$$\mu(\tau, z) := \frac{1 + t_1^3 + t_2^3}{t_1 t_2}$$

is independent of z, and $\mu^3 \neq 27$, so descends to

$$\mu(\tau) : \Gamma(3) \backslash \mathfrak{H} \to \mathbb{C} \setminus \{3, 3\zeta, 3\zeta^2\}$$

- So get a parametrization

$$\begin{array}{ccc} \widetilde{\Gamma}(3)\backslash\mathfrak{h} \times \mathbb{C} & \xrightarrow{([1:\mu],[1:t_1:t_2])} & \dot{Y}^{an} \\ \downarrow & & \downarrow \\ \Gamma(3)\backslash\mathfrak{h} & \xrightarrow[{[1:\mu]}]{} & \mathbb{P}^1(\mathbb{C}) \setminus \{\infty, 3, 3\zeta, 3\zeta^2\} \end{array}$$

- Also have sections $s_0(\tau) = (\tau, 0), s_1(\tau) = (\tau, 1/3), s_2(\tau) = (\tau, \tau/3)$.
- Fibres are elliptic curves with weak level-3-structure.

- Fact: $\widetilde{\Gamma}(3)\backslash\mathfrak{h}\times\mathbb{C}$ has compactification B such that the diagram extends to

$$\begin{array}{ccc} B & \longrightarrow & Y^{an} \\ \downarrow & & \downarrow \\ \Gamma(3)\backslash\mathfrak{h}^* & \longrightarrow & \mathbb{P}^1(\mathbb{C}) \end{array}$$

$\mathfrak{h}^* = \mathfrak{h}\cup\{cusps\}$.

- Also have

$$\dot{W}^{an} = \dot{Y}^{an} \times_{\dot{X}} \dot{Y}^{an} \simeq \widetilde{\Gamma}(3)_2 \backslash (\mathfrak{h} \times \mathbb{C}^2)$$

where $\widetilde{\Gamma}(3)_2 = \Gamma(3) \ltimes (\mathbb{Z}^2 \times \mathbb{Z}^2)$, and

$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \right] \cdot (\tau, z_1, z_2)$$
$$= \left(\frac{a\tau + b}{c\tau + d}, \frac{z_1 + \alpha_1\tau + \beta_1}{c\tau + d}, \frac{z_2 + \alpha_2\tau + \beta_2}{c\tau + d} \right)$$

- Let
$$\eta(\tau) = e^{2\pi i \tau/24} \prod_{n>0} \left(1 - e^{2\pi i n\tau}\right).$$
- Then
$$\omega := \frac{2\pi i}{3}\eta^8(\tau) d\tau dz_1 dz_2$$
is $\widetilde{\Gamma}(3)_2$ invariant, and in fact (!) extends to a holomorphic form on $\widetilde{W}^{an}$.
- Fact: $h^{2,1} = 0$, $h^{3,0} = 1$ (Schoen 1986).
- In particular ω is a basis for $H^{3,0}(\widetilde{W}^{an})$.

The action of $SL_2(\mathbb{F}_3)$

- Suppose F is a number field containing ζ, a fixed 3rd root of unity
- Let $(E, s_1, \mathcal{C})$ over F.
- The Weil pairing is an alternating pairing

$$\langle \cdot, \cdot \rangle : E[3] \times E[3] \to \mu_3$$

  so we can pick out a generator $s_2 \in \mathcal{C}$ such that $\langle s_1, s_2 \rangle = \zeta$
- Hence have natural action of $SL_2(\mathbb{F}_3) \simeq Sp(E[3])$ on $\dot{X}_F$ and $\dot{Y}_F$ by permutations of $E[3]$ preserving $\langle \cdot, \cdot \rangle$.

- Via explicit equations, action extends to $Y_F \to X_F$, and then to $\tilde{W}_F$ diagonally.
- On $\dot{Y}^{an}$: for $\overline{\gamma} \in SL_2(\mathbb{F}_3)$, choose $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ s.t. $\gamma \equiv \overline{\gamma}$ mod 3. Then the action becomes

$$\overline{\gamma} \cdot [\tau, z] = \left(\frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right),$$

which is well defined on $\widetilde{\Gamma}(3) \backslash \mathfrak{h} \times \mathbb{C}$.

- We have a quotient map $SL_2(\mathbb{F}_3) \to \mathbb{Z}/3\mathbb{Z}$
- Let $M = \mathbb{Q}^2$ with an action of $\mathbb{Z}/3\mathbb{Z}$ such that the generator acts via the matrix $\left(\begin{smallmatrix} & -1 \\ 1 & -1 \end{smallmatrix}\right)$. As a $\mathbb{Q}[SL_2(\mathbb{F}_3)]$ module, M is simple.

Theorem

As a $\mathbb{Q}[SL_2(\mathbb{F}_3)]$ module, $CH^2(\widetilde{W_{\mathbb{C}}})_{hom} \otimes \mathbb{Q}$ is isomorphic to a direct sum of copies of M.

Sketch of proof:

- Step 1: Show $-Id$ acts trivially, so descends to a rep. of $PSL_2(\mathbb{F}_3)$.
- Step 2: Show for any $U \in PSL_2(\mathbb{F}_3)$ of order 3, $1 + U + U^2$ annihilates $CH^2(\widetilde{W_{\mathbb{C}}})_{hom} \otimes \mathbb{Q}$.
- There are 3 simple $\mathbb{Q}[PSL_2(\mathbb{F}_3)]$ modules: trivial, Steinberg , and M. Of these, only M has property from Step 2.

For Step 2:

- First show that for any $U \in SL_2(\mathbb{F}_3)$ of order 3, the quotient $V := W_{\mathbb{C}}/\langle U \rangle$ is a smooth variety with a model over $\mathbb{Q}$, such that $H^3(V, \mathbb{Q}) = 0$:

Assume $U = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$, as all order 3 subgroups are conjugate. Schoen shows V is rational by providing an explicit equation.
There exists a cover $W_0 \to \tilde{W}$ s.t U acts without fixed points on W_0, $V = W_0/\langle U \rangle$, and

$$H^3(W_0^{an}, \mathbb{Q}) = H^3(\tilde{W}^{an}, \mathbb{Q})$$

Then $H^3(V, \mathbb{Q}) = H^3(\tilde{W}^{an}, \mathbb{Q})^U$ is the U-invariant subspace.
But $H^3(\tilde{W}^{an}) \otimes \mathbb{C} = span\{\omega, \overline{\omega}\}$, where

$$\omega = \frac{2\pi i}{3} \eta^8(\tau) d\tau_1 dz_1 dz_2.$$

Since $\eta(\tau) = e^{2\pi i \tau/24} \prod_{n>0} \left(1 - e^{2\pi i n \tau}\right)$, we have

$$\eta^8(\tau + 1) = \zeta \eta^8(\tau), \qquad \zeta = e^{2\pi i/3},$$

so

$$\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix} \cdot \omega = \zeta \omega,$$

so no U-invariants!
Therefore $H^3(V, \mathbb{Q}) = 0$.

- By Bloch (?) and Murre: for V a nonsingular rational projective 3-fold

$$H^3(V, \mathbb{Q}) = 0 \implies CH^2(V)_{hom} \otimes \mathbb{Q} = 0$$

- Let $\rho : \tilde{W}_{\mathbb{C}} \to V$ be the projection
- For $z \in CH^2(\tilde{W}_{\mathbb{C}})_{hom} \otimes \mathbb{Q}$,

$$(1 + U + U^2)z = \rho^* \rho_*(z) = 0,$$

as required.