PRASANNA 2012/04/29

X smooth projective $/k$

$CH^j(X) \supseteq CH^j(X)_0 \supseteq CH^j(X)_{alg}$; $CH^j(X)_{\mathbb{Q}}$, $CH^j(X)_{0,\mathbb{Q}}$, $CH^j(X)_{alg,\mathbb{Q}}$

alg. eq. $\Rightarrow$ hom. eq. $\Rightarrow$ num. eq.

(i) If $j=1$, (divisors)

hom eq. $\Leftrightarrow$ alg. eq. (LEFTSCHETZ (1,1)-thm.)

$\overset{\text{even}}{(\text{integrally})}$

$\downarrow$ $k=\mathbb{C}$

(ii) GRIFFITHS ('69)

X = generic hypersurface of deg 5 in $\mathbb{P}^4$

$\dim X = 3$, $Gr^2(X)$ is nontorsion

(iii) CLEMENS, for ~~some~~ the example as (ii), $Gr^2(X) \otimes \mathbb{Q}$ is not f.g.

(iv) CERESA, C = generic curve of genus g

$X = Jac(C)$, $\Delta = C - [-1]^* C \in CH^{g-1}(X)_0$ is non-torsion.

Q. Can you examples over a number field?

BRUNO-HARRIS: $C: x^4+y^4=z^4$ (Fermat quartic)

Δ = Ceresa cycle $\subseteq CH^2(X)_0$; $X = \mathrm{Jac}(C)$

showed Δ is not zero in $Gr^2(X)$.

studying $AJ_{\mathbb{C}}: CH^2(X)_0 \longrightarrow (H^{3,0}(X) \oplus H^{2,1}(X))^\vee / \mathrm{Im}\, H_3(X,\mathbb{Z})$

$$\searrow \quad \frac{H^{3,0}(X)^\vee}{\mathrm{Im}(H_3(X,\mathbb{Z}))}$$

using $X \sim E \times E \times E'$

with $\underset{\substack{\updownarrow\\ \psi}}{E}, \underset{\substack{\updownarrow\\ \psi'}}{E'}$ e.c. with CM by $\mathcal{O}_K$, $K = \mathbb{Q}(\sqrt{-1})$

$$H^3(X) = \underset{\substack{\downarrow\\(3,0)+(0,3)}}{M_{\psi^2\psi'}} \oplus \Big(\underset{\substack{\downarrow\\(2,1)+(1,2)}}{\underline{\quad M' \quad}}\Big)$$

$\Delta \longmapsto$ get

$$\frac{2\int_0^1\int_0^x \frac{dt}{\sqrt{1-t^4}} \frac{dx}{(1-x^4)^{3/4}}}{\int_0^1 \frac{dt}{\sqrt{1-t^4}} \cdot \int_0^1 \frac{dx}{(1-x^4)^{3/4}}} = 1.24\ldots$$

$\uparrow$

so $\neq 0$

(if would show $\notin \mathbb{Q}$,
knew is in fact non-torsion in Gr)

BLOCH ~~get~~ showed non-torsionness using

$$\begin{tikzcd}
AJ_p: CH^2(X)_0 \arrow[r] & H^1(K, H^3(X,\mathbb{Q}_p)(2)) \arrow[d] \\
& H^1(K, M_\chi(2)) \arrow[d] \\
v \mid \ell,\ \ell \neq p & H^1(K_v, M_\chi(2))
\end{tikzcd}$$

Conjecture (BEILINSON - BLOCH)

$K = \#$ field , $X_{/K}$ smooth projective

(i) $CH^j(X)_0$ is a f.g. ab. gp.

(ii) $\operatorname{rk} CH^j(X)_0 = \operatorname{ord}_{s=j} L(H^{2j-1}(\bar{X}), s)$

$= \operatorname{ord}_{s=0} L(H^{2j}(\bar{X})(j), s)$

have

$$\begin{tikzcd} & J^j(X) \\ AJ : CH^j(X)_0 \arrow[ur] \arrow[dr] & \\ & H^1(K, H^{2j-1}(\bar{X}, \mathbb{Q}_p)(j)) \end{tikzcd}$$

Back in $X = \operatorname{Jac}(C)$:

$$\begin{tikzcd} CH^2(X)_0 \arrow[r, squiggly] & H^3(X)(2) \\ \cup & \cup \; M \\ CH^2(X)_{alg} \arrow[r, squiggly] & M' \\ \cup & \cup \\ (0) & (0) \end{tikzcd}$$

In this case, $L(X, 2) = 0$, but $L(M', s)$ is non-zero at the ~~center~~ central point.

A ~~[illegible]~~ refined version of the B-B conjecture involving :

Coniveau filtrations

$$N^i H^*(X) := \varinjlim_{\substack{Y \subset X_{\bar{k}} \\ \text{closed subscheme} \\ \text{of codim} \geq i}} \ker\left(H^*(X) \to H^*(X - Y) \right) = \varinjlim_{Y} \operatorname{Im}\left(H^*_{|Y|}(X) \to H^*(X) \right)$$

$$N^i CH^j(X)_{\mathbb{Q}} := \{ \Upsilon \in CH^j(X)_{\mathbb{Q}} \text{ s.t. } \exists\ Y \hookrightarrow X_{\bar{k}} \text{ of codim} \geq i$$

$$\text{\& cycle } c \text{ rep. } \Upsilon \text{ s.t. } \text{Supp}(c) \subseteq Y \ \& \ C \text{ is hom. triv. on } Y \}$$

$$\begin{array}{l}
N^0 CH^j(X)_{\mathbb{Q}} = CH^j(X)_{0,\mathbb{Q}} \\
\quad \rotatebox{90}{$\subseteq$} \\
\quad \vdots \\
\quad \rotatebox{90}{$\subseteq$} \\
N^{j-1} CH^j(X)_{\mathbb{Q}} = CH^j(X)_{alg,\mathbb{Q}} \\
\quad \rotatebox{90}{$\subseteq$} \\
N^j CH^j(X)_{\mathbb{Q}} = 0
\end{array}$$

Bloch shows that $AJ : CH^j(X) \to H^1(K, H^{2j-1}(\bar{X}, \mathbb{Q}_\ell)(j))$

is compatible with the filtrations N^i

i.e. maps $N^i CH^j(X) \to H^1(K, N^i H^{2j-1}(\bar{X}, \mathbb{Q}_\ell)(j))$

<u>Refined B-B conj</u> $\operatorname{rank} gr^i CH^j(X)_{\mathbb{Q}} = \operatorname{ord}_{s=j} L(gr^i H^{2j-1}, s)$.

(joint with BERTOLINI & DARMON)

K = imag. quadr. field, $h_K = 1$, disc $K = -D_K$ odd, $\mu_K = \{\pm 1\}$.

(so $D_K = 7, 11, 19, 43, 67, 163$)

ψ = canonical Hecke character of ∞-type $(1,0)$, of cond $\psi = (\sqrt{-D_K})$

ψ^2 = unr. character of type $(2,0)$: $\psi^2((\alpha)) = \alpha^2$.

$$\text{L-f'n of} \to \quad \psi \overset{\psi^2}{\curvearrowright} \psi^3 \overset{\psi^2}{\curvearrowright} \psi^5 \overset{\psi^2}{\curvearrowright} \psi^7, \ \dots$$

$$\text{has sign:} \left\{ \begin{array}{lcccc} D_K = 7 & + & - & + & - \\ D_K \neq 7 & - & + & - & + \end{array} \right.$$

W_r = Kuga - Sato variety $/ X_1(D_K)$ of dim $r+1$

$A_{\mathbb{Q}}$ = e.c. with CM by $\mathcal{O}_K$, min. conductor $= D_K^2$

$$A_{\mathbb{Q}} \leftrightarrow \psi$$

ψ^t, $t = 2r+1$: look at $X_{2r-1,1} := W_{2r-1} \times A$ ($2r+1$ - dim'l)

$$\begin{array}{c} X_{2r-1,1} \\ \cup\!\!\mathsf{I} \\ A^{2r-1} \times A = (A \times A)^{r-1} \times (A \times A) \end{array}$$

$$\mathrm{graph}(\sqrt{-D_K})^{r-1} \times \Delta =: \Delta_{2r-1,1} \in CH^{r+1}(X_{2r-1,1})_0$$

$$f = \theta_{\psi^{2r}} = \sum_{a \subset \mathcal{O}_K} \psi^{2r}(a) e^{2\pi i N(a) z} \in S_{2r+1}(\Gamma_0(D_K), \varepsilon_K)$$

$$M_f \subseteq H^{2r}(W_{2r-1})$$

$$H^{2r+1}(X_r) = H^{2r+1}(W_{2r-1} \times A) \underset{\substack{\uparrow \\ \text{KÜNNETH}}}{\supseteq} \underset{\substack{\shortparallel \\ \psi^{2r}}}{M_f} \otimes M_\psi = M_{\psi^{2r+1}} \oplus M_{\psi^{2r-1}}(-1).$$

<u>Theorem</u> (BDP)

Supp. r is odd if $D_K = 7$; r is even if $D_K \neq 7$.

Then
$$[\Delta_{2r-1,1}] \neq 0 \in CH^{r+1}(X_{2r-1,1})_{0,\mathbb{Q}} / N^1 CH^{r+1}(X_{2r-1,1})_{0,\mathbb{Q}}$$

In particular, it is non-trivial in $Gr^{r+1}(X_{2r-1,1}) \otimes \mathbb{Q}$.

what goes into the proof:

K = # field, X v place of K, $v \mid p$ where X has good red'n

$$\begin{tikzcd}
CH^j(X)_0 \arrow[r] \arrow[ddrr, bend right, "\text{(Niziol, Nekovar)}"'] \arrow[ddddrr, bend right=40, "\alpha"'] \arrow[dddddrr, bend right=50, "\beta"'] & H^1(k, \underbrace{H^{2j-1}(\bar{X}, \mathbb{Q}_p)(j)}_{\overset{\shortparallel}{V}}) \arrow[r] & H^1(\overset{K_v}{\overset{\shortparallel}{K}}, V) \arrow[d, phantom, "\cup"] & \\
 & & & \\
 & & H^1_f(K, V) & \text{(Bloch-Kato subspace)} \\
 & & D_{dR}(V) / \mathrm{Fil}^0 D_{dR}(V) \arrow[u, "\exp_V", "\wr"'] \arrow[d, equal, "\omega_p"] & \\
 & & H^{2j-1}_{dR}(X_{/K}) / \mathrm{Fil}^j \arrow[d, equal, "PD"] & \\
 & & (\mathrm{Fil}^j H^{2j-1}(X_{/K}))^\vee &
\end{tikzcd}$$

Key | <u>Proposition</u>: $\alpha(N^i CH^j(X)_0) \subseteq \mathrm{Fil}^i H^{2j-1}_{dR}(X) / \mathrm{Fil}^j$

So $\alpha(N^1 CH^j(X)_0) \subset \mathrm{Fil}^1(\quad) \xrightarrow{PD}$ Annihilator of $\mathrm{Fil}^{2j-1} H^{2j-1} = H^0(X, \Omega^{2j-1})$

From [BDP2]: $[\beta(\Delta)(\omega_f \wedge \omega_A)]^2 =$ value of p-adic L-f'n outside range of interpolation $= \underbrace{\mathcal{L}_p(*) \cdot \mathcal{L}_p(**)}$

can check non-vanishing by a finite computation!

using congruences to reduce to compute classical L-values in the interpolation