GENERALISED HEEGNER CYCLES AND INTERMEDIATE JACOBIANS OF KUGA-SATO VARIETIES I

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Abstract. These notes are taken from a lecture given by Professor Henri Darmon at the Fourth Montreal-Toronto Workshop in Number Theory at the CRM (April 28-29, 2012). They are expanded slightly and include most of the comments and questions from the audience as footnotes. The scribe takes any responsibility for errors introduced in the copying and typesetting procedure.

Contents

1. The Classical Setting

The goal shall be to investigate certain cycles on Kuga-Sato varieties and similar varieties. Before introducing the cycles we are interested in, we shall first review the classical case in the spirit of Bloch and Schoen. Some of this classical material was already explicated in the series of three afternoon talks on Saturday by Sankaran, Koshravi, and Smithling.

In this classical case, we have some integer $r \geq 0$ and the $2r$-th Kuga-Sato variety W_{2r} that can be described as the $2r$th power of a universal elliptic curve over $X_0(N)$[1]. In this case $\dim(W_{2r}) = 2r + 1$. Let us visualise this as

$$\begin{tikzcd} \mathcal{E}^{2r} \arrow[d, "\pi"] \\ X_0(N) \end{tikzcd}$$

Consider for a closed point $x \in X_0(N)$ some fiber $\pi^{-1}(\{x\}) = E_x \times E_x$ where E_x is some elliptic curve. Generically, the Neron-Severi group $NS(E_x \times E_x)$ of the fiber has rank 3 generated by $0 \times E_x$, $E_x \times 0$, and the diagonal Δ. There are however certain exceptional fibers where the rank of the Neron-Severi group jumps to rank four. Such a case may be $E_x = A$ where A

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[1] Or $X_1(N)$

has complex multiplication by a quadratic imaginary order say $\mathbb{Z}[\sqrt{D}]$ where $D < 0$.

In this case, the extra element is the graph of the multiplication map by $\sqrt{D}$ denoted $\mathrm{graph}(\sqrt{D}) \subseteq E_x \times E_x$ which does not come from specializing a generic element. We can form out of this graph an element

$$\Delta_A = (\mathrm{graph}(\sqrt{D}) - A \times 0 - D(0 \times A))^r \in CH^{r+1}(W_{2r})_0(H_D)$$

of the higher Chow group of nullhomologous cycles, and this element is supported on the fiber above A. We get such a cycle Δ_A for each elliptic curve with complex multiplication arising from such closed fibers. Here in fact the field of definition is H_D, the ray class field attached to the imaginary quadratic order.

We now define $\Delta_D = \mathrm{trace}_{H_D/K}(\Delta_A)$ where $K = \mathbb{Q}(\sqrt{D})$.

There are a few results already known

Theorem 1 (Shoen). *If $r = 1$ and $N = 3$, $\{\Delta_A\}_A$ generates a subgroup of infinite rank in $CH^{r+1}(W_2)_0$.*

This result is proved by looking at the images of the Δ_A under the Abel-Jacobi map. The second result was first formulated by Gross and Zagier and proved in complete generality by Zhang, which relates the heights of these cycles to central critical values of derivatives of L-functions attached to modular forms.

Theorem 2 (Gross-Zagier, Zhang). *Let $f \in S_{2r+2}(\Gamma_0(N))$ and let δ_D^f be an f-isotypic projection of the cycle Δ_D. Then*

$$\hat{h}(\Delta_D^f) = (*)L'(f_{/k}, r+1)$$

where $(*)$ will always stand for some nonzero "fudge" factor.

For $r = 0$ this is the usual Gross-Zagier formula, and for $r \geq 1$ this is the generalisation to forms of even weight greater than four. Of course, here there are a few points that are being glossed over here. For instance, for these cycles we have defined to exist there should be some Heegner hypothesis satisfies which forces the sign in the functional equation of the L-function to be minus one.

2. Generalised Heegner Cycles

Now we come to the main topic. We generalise by replacing W_{2r} with something very similar: the product $W_r \times A^r$, where A is a fixed elliptic curve with CM by $\mathcal{O}_K$, which also is fibered over $X_0(N)$:

$$\begin{array}{c} W_r \times A^r \\ \big\downarrow{\scriptstyle \pi} \\ X_0(N) \end{array}$$

Notice now that W_r can be of even or odd dimension! Here $K = \mathbb{Q}(\sqrt{D})$ is as usual some imaginary quadratic field, and in fact all the examples of A

in **this** lecture will be CM by the maximal order in an imaginary quadratic field with class number 1.

[A question was asked (see footnote[2]).]

For $r = 1$ (which is in some sense the minimal case) here, we get the fiber over a closed point $\pi^{-1}(x) = E_x \times A$. In this case, generically the rank of the Neron-Severi group is 2. As in Section 1, we wish to know the points at which the rank jumps. In fact, the points for which the NS-rank becomes bigger are exactly those x for which there is an isogeny $\varphi : A \to E_x =: A'$ for instance, then we get two extra elements of $NS(E_x \times A)$: the graphs $\mathrm{graph}(\varphi)$ and $\mathrm{graph}(\varphi \circ \sqrt{D})$. In this case, we define

$$\Delta_{\varphi,r} = (\mathrm{graph}(\varphi) - 0 \times A - \deg(\varphi)A' \times 0)^r$$

We can think of $\Delta_{\varphi,1}$ living in $A \times A'$, where $A \times A'$ is viewed as a fiber of the variety $W_1 \times A$. Then $\Delta_{\varphi,r} \in CH^{2r+1}(X_r)_0(H_D)$ where $X_r = W_r A^r$. Here we have used the subscript 0 for the Chow group again: by a similar argument as in the classical case in Section 1 we can indeed show that $\Delta_{\varphi,r}$ is a nullhomologous cycle.

This now gives us a method of constructing a large number of different cycles on these Kuga-Sato-like varieties

We want to understand the behaviour of these cycles and applications.

3. Gross-Zagier-like Formulae

Assume that A has CM by $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{D})$ is imaginary quadratic and such that the class number of $\mathcal{O}_K$ is trivial [i.e. has trivial reduced K_0 group], that the discriminant $D(K)$ is odd, and that $\mathcal{O}_K^\times = \{\pm 1\}$.

[A question was asked (see footnote[3])]

Consider a cusp form $f \in S_{r+2}(\Gamma_0(N))$, which corresponds to some middle cohomology class. We can define the classical Hecke character ψ on ideals by

$$\psi((\alpha)) = \left(\frac{\alpha}{\sqrt{D}}\right)\alpha \tag{1}$$

where $\left(\frac{\alpha}{\sqrt{D}}\right)$ is the Legendre symbol. This does not depend on the choice of generator α of (α) and this character is of weight $(1, 0)$.

We will consider the special case where the isogeny φ is actually the identity, and for notational convenience we define $\Delta_r = \Delta_{1,r}$. Again we can look at the f-isotypic projection (i.e. onto the f-component). Furthermore, this f-component further decomposes by looking at the cohomology of A^r according to certain Hecke characters that can be formed from the basic one defined in Equation (1). For instance, there is a piece where the underlying ℓ-adic representation corresponds to $\psi^j \overline{\psi}^{r-j}$.

[2]Q:Is it possible that the derivative of the L-function could vanish? A: In this setting, yes. It certainly happens for weight 2. There are examples of higher order vanishing.

[3]Q: How dependent is this on class number 1? A: This is just for notational simplicity. In fact in our paper we formulate our results for arbitrary K, but in the general case the notation just becomes more messy.

Then we conjecture that for $0 \leq j \leq r$,

$$\hat{h}\left(\Delta_r^{f,\psi^j\overline{\psi}^{r-j}}\right) = (*)L'(f \times \theta_{\psi^{r-2j}}, r+1-j).$$

Here the $\theta_{\_}$ denotes the corresponding θ series attached to a character and the product denotes Rankin convolution. Note that $\mathrm{wt}(\theta_{\psi^{r-2j}}) = 1+r-2j < \mathrm{wt}(f) = r+2$. The novelty of this conjecture is that we capture such values of f convoluted not just with finite order characters in the imaginary quadratic field (i.e. θ series of weight 1 attached to finite group characters), but with θ series of more general weights attached to characters of more general weights. We remark that this formula is not in the literature as it is not proved, but one could imagine that the techniques used by Zhang to prove the Gross-Zagier formula could also be used to prove this formula.

[A question was asked (see footnote[4])]

Let p be a prime that splits in K. Now we turn to a p-adic variant of this formula and the corresponding p-adic L-functions, which will also play a role in the next lecture given by Kartin Prasanna. In this case, we will consider the p-adic version of the Abel-Jacobi map. Instead of looking at the heights of these generalised Heegner cycles, we will look at the images of Δ_r under this p-adic variant. Also, instead of the classical L-series on the right hand side, we will have a p-adic variant constructed by Hida.

Recall that the classical complex Jacobi map (see the lecture of Patrick Wall) was defined as a map

$$\mathrm{AJ}_{\mathbb{C}} : CH^{r+1}(X)_0 \to \frac{\mathrm{Fil}^{r+1} H_{dR}^{2r+1}(X/\mathbb{C})^\vee}{\mathrm{Im}\, H_{2r+1}(X(\mathbb{C}), \mathbb{Z})}.$$

The notation Fil^* indicates the Hodge filtration and $-^\vee$ indicates the dual. We saw yesterday in Wall's lecture how this map was defined on cycles essentially by integrating. Note that in this setting the torsion is dense in the target. If we use the p-adic version of this Abel-Jacobi map

$$\mathrm{AJ}_p : CH^{r+1}(X)_0(\mathbb{Q}_p) \to \mathrm{Fil}^{r+1}(H_{dR}^{2r+1}(X/\mathbb{Q}_p))^\vee$$

then we have the following advantages:

(1) The target is now torsion-free, so that any torsion element in the Chow group must map to 0 in the target. Hence, torsion is easier to detect as in principal it is just a finite computation!
(2) More closely related to etale cohomology and Galois representations: AJ_p factors through the etale AJ map, which was illustrated briefly in Zong's lecture ("Bloch's recurring fantasy"). This is useful for arithmetic applications (e.g. Euler systems).

p-adic L functions arise from some classical L-function by looking at critical values. Dividing these values by the appropriate periods one obtains

[4] Q: Is your height the Bloch-Beilinson height pairing on the central middle Chow group? A: Exactly.

some algebraic numbers, and then we would like to know whether these algebraic parts of special values of L-functions extend p-adically to a larger domain.

For p-adic L-functions, the special values that will be interpolate are the special values in the Gross-Zagier formula.

$$L(f_{/K} \times (\psi^i \times (\psi^*)^j)^{-1}, 0).$$

for some fixed modular form f and ψ varying over Hecke characters with $\psi^* = \psi \circ c$, with c being complex conjugation. Here we have just normalised via twisting by the norm character so that the special value occurs at 0. If we look at this collection of central critical values as i and j varies, we can visualize this in a diagram shown in Figure 1.

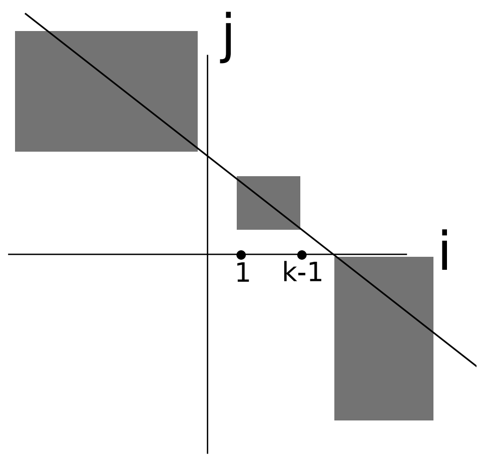

FIGURE 1. The shaded regions show where the central critical values occur. The line is the central critical line reflected under the classical functional equation.

Now the idea is that we can recover certain values of the p-adic L function as images of Δ_r under the p-adic Abel-Jacobi map:

$$\mathrm{AJ}_p(\Delta_r)(\omega_f \wedge \omega_{A^j}\eta_A^{r-j}) = (*)L_p(f \times \psi^{j+1}\psi^{*r+1-j}$$

Here

$$\begin{aligned} \omega_f &\in \Omega^{r+1}(W_r) \subseteq \mathrm{Fil}^{r+1} H^{r+1}_{dR}(W-r) \\ \omega_A &\in \Omega^1(A) \subseteq H^1_{dR}(A) \\ \eta_A &\in H^1_{dR}(A/\mathbb{Q}_p), \langle \omega_A, \eta_A \rangle = 1 \end{aligned}$$

Now η_A is defined to be in the eigenspace for the $\mathbb{Q}_p$-linear operator Frobenius acting on the de Rham cohomology of $A/\mathbb{Q}_p$. [A question was asked (see footnote[5])]

[5]Q: Is there a relation between the complex and p-adic Abel-Jacobi map? A: That's a very good question. We will see some ideas of this later in the talk. There should be

4. Applications: An Interesting Hodge Cycle

Let $D(K) \in \{7, 11, 19, 43, 67, 163\}$. If our imaginary quadratic field K has such a discriminant, then K has class number one. Also assume that $r \geq 1$ is odd. We are now going to consider a special case of this Kuga-Sato variety: $W_r = \mathcal{E}^r \to X_0(D)$ and $X_r = W_r \times A^r$, where A has CM by $\mathcal{O}_K$ with the conductor of A equal to D^2 (i.e. the minimal case).

As before (Equation (1), we consider the Hecke character ψ. If we take even powers ψ^{r+1} of ψ then ψ^{r+1} is an unramified character of K and so $\theta_{\psi^{r+1}} \in S_{r+2}(\Gamma_0(D), \varepsilon_K)$ where ε_K is the quadratic odd Dirichlet character associated to the quadratic field K.

Now we have

$$H_{et}^{r+1}(\overline{W_r}, \overline{\mathbb{Q}_\ell})^{\theta_{\psi^{r+1}}} \cong \mathrm{Ind}_K^{\mathbb{Q}} \psi_1^{r+1} \cong \varepsilon H_{et}^{r+1}(\overline{A}^{r+1}, \mathbb{Q}_\ell)$$

where ε is a suitable projector. Here the bar notation means extension to the algebraic closure. The latter two isomorphisms as representations of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ shows that the two varieties $\overline{W_r}$ and $\overline{A^{r+1}}$, which geometrically look very different share in common a piece of their cohomology. This in turn translates to the existence of a Tate cycle which lives in the middle etale cohomology of the product $\overline{W_r \times A^r}$, and it is invariant under the action of $G_{\mathbb{Q}}$. We shall denote this Tate cycle by

$$\pi_{et} \in H_{et}^{2r+2}(\overline{W_r \times A^{r+1}}, \mathbb{Q}_\ell)(r+1)^{G_{\mathbb{Q}}}.$$

Given this Tate cycle, there is a corresponding Hodge cycle

$$\pi_{Hd} \in H_{dR}^{r+1,r+1}(W_r \times A^{r+1}/\mathbb{C}) \cap H_B^{2r+2}(W_r \times A^{r+1}, \mathbb{Q})$$

where H_B is just the Betti cohomology. This Hodge cycle π_{Hd} can be obtained by studying the periods of the theta series. We can then formulate a conjecture.

Conjecture 1 (C)**.** *There is an algebraic cycle $\pi \in CH^{r+1}(W_r \times A^{r+1})$ such that $c\ell_{et}(\pi) = \pi_{et}$ and $c\ell(\pi) = \pi_{Hd}$.*

This is only known in the following cases:

- $r = 0$: then W_r is a modular curve, and A is an elliptic curve. Then the existence of π is just the parametrization from the modular curve to the elliptic curve. Follows from the Tate conjecture for curves.
- $r = 1$, $D = 7$: then W_r is an extremal $K3$ surface with maximal Picard rank. The Shioda-Inose theory gives for an involution τ on the surface W_r a correspondence $W/(\tau) \cong \mathrm{Kummer}(A \times A)$.
- $r = 2$, $D = 3$. Here Shoen proved that π exists!

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a relation, but there is nothing obvious. It is quite deep; in fact if we could show the vanishing of the complex version implies the vanishing of the p-adic version, this would be fantastic progress! In fact, in the next part you will see that we know much more about the p-adic images under AJ_p of some generalised Heegner cycles compared to the complex images under $\mathrm{AJ}_{\mathbb{C}}$. However, we know that the vanishing one one is equivalent to the vanishing of the other in the elliptic curve case.

It is quite difficult to produce the cycle π! This would be a good place to find a counterexample to the Hodge conjecture. One thing we would like to do is get a bit of evidence for the validity of the Hodge conjectures. There is a quote of Weil on the Hodge conjectures that motivates this a bit:

> In spite of the designation of conjecture, there is, as far as I know, not a shred of justification to believe in the Hodge conjectures. One would do geometers a great service if one could dispose of it by means of a counterexample.

If we really want to get rid of the Hodge conjectures, it would be a good place to start to try and find an example where this algebraic cycle is *not* present. This leads to the notion of the last topic in my talk.

5. Test Cycles

The question is, how are we going to test for the presence of this algebraic cycle? If π exists, then it induces a correspondence

$$X_r = W_r \times A^r \rightsquigarrow A$$

This correspondence, which we shall call

$$\pi? : CH^{r+1}(W_r \times A^r) \to A.$$

takes codimension $r + 1$ cycles and maps them to the elliptic curve. We have appended the character π with the question mark to emphasize that π might not exist. This map would respect fields of definition. Now, we have an infinite collection of cycles $\Delta_{\varphi,r}$. Although they are hard to write down explicitly, if we could only map them down to the elliptic curve A then we could work with the corresponding points.

Even if we cannot prove that this cycle π exists, we can still compute the underlying points, since to do this we only need the Hodge or p-adic realizations of this cycle. Thus if π exists we know what the this correspondence must look like![6]

Another remark is that $\pi?$ has unconditional complex and p-adic variants, which we shall denote $\pi_{\mathbb{C}}$ and π_p respectively. Then at the level of complex points, there is a commutative diagram

$$\begin{tikzcd}
CH^{r+1}(X_r)_0(\mathbb{C}) \arrow[r, "\pi?"] \arrow[d, "\mathrm{AJ}_{\mathbb{C}}"] & A(\mathbb{C}) \arrow[d, "\mathrm{AJ}_{\mathbb{C}}"] \\
\mathrm{Fil}^{r+1} H^{2r+1}(X_r)^\vee \arrow[r] & \frac{\Omega^1(A/\mathbb{C})^\vee}{H^1(A(\mathbb{C}),\mathbb{Z})}
\end{tikzcd}$$

Conjecture 2 ($C_{\mathbb{C}}$)**.** $\pi_{\mathbb{C}}(\Delta_r) \in A(K) \otimes \mathbb{Q}$.

[6] Scribe's note: At this point Zong made a remark about whether π was really well-defined since one had to use a moving lemma, but the speaker and Zong waited until after the talk to discuss this, and I did not hear this discussion unfortunately.

This can be tested numerically. In all cases $\pi_{\mathbb{C}}(\Delta_r) = \sqrt{-D}m_r P_A$ where m_r has an interpretation as an L-value and P_A is a generator of $A(\mathbb{Q})$. The L-value interpretation of m_r is more precisely

$$m_r^2 = (*)L(\psi_A^{2r+1}, r+1).$$

There is a table in the paper showing some computations [this was shown in the lecture].

The same statement should hold for the p-adic Abel-Jacobi map. In fact in the p-adic case we actually have a theorem.

Theorem 3 (C_p)**.** *$\pi_p(\Delta_r) = (*)m_r P_A$ for all $r > 1$ and $D \in \{11, \ldots, 163\}$.*

From C_p and $C_{\mathbb{C}}$, one should be able to conclude C, but we are very far away from doing this. Note that in Theorem 3 we have only stated the theorem for $D \geq 11$ in the list of supposed discriminants and we have shown calculations that only apply to this case. What happens when $D = 7$?

The image of the cycle under Abel-Jacobi is controlled by the central critical value of an L-function

$$L(\theta_{\psi^{r+1}/K} \times \psi^r, *) = L(\psi^{2r+1}, s)L(\overline{\psi}, s-r).$$

where the factorization is just a product of two Hecke L-functions. The situations where we can produce a point on the elliptic curve are those in which the first factor has sign -1 nad the second has sign $+1$ in their respective functional equations. This happens at exactly the values of D in Theorem 3.

If $D = 7$, then the first factor vanishes and the second does not, so we need to consider the image of the cycle in a different piece of the intermediate Jacobian of the varieties, and this will be the subject of the next talk.