Cycles 101

4th Montreal-Toronto Workshop in Number Theory

Eyal Goren

Assumptions / Notations

$X, Y, \ldots$ smooth projective complex algebraic varieties (irreducible).

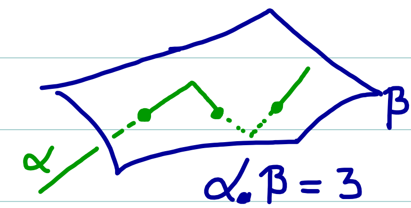
$H^{i,B}$, $H_{i,B}$, $H^{i,dR}$, $H_{i,dR}$, … Betti / de Rham / étale …. (co-homology)

$d = \dim_{\mathbb{C}}(X)$ (so cohomology groups of coherent sheaves die above d).

We also consider X as a topological and real manifold (So Betti / de Rham cohomology dies above $2d$).

Topology

Intersection pairing: $\{r\text{-cycles}\} \times \{(2d-r)\text{-cycles}\} \longrightarrow \mathbb{Z}$
induces a perfect pairing (Poincaré duality)



$$H_{r,B}(X,\mathbb{Q}) \times H_{2d-r}(X,\mathbb{Q}) \longrightarrow \mathbb{Q}$$

Using $H^{r,B}(X,\mathbb{Q}) = [H_{r,B}(X,\mathbb{Q})]^*$, we have

$$H^{r,B}(X,\mathbb{Q}) = H_{2d-r}(X,\mathbb{Q}).$$

The resulting pairing

$$H^{r,B}(X,\mathbb{Q}) \times H^{2d-r,B}(X,\mathbb{Q}) \longrightarrow \mathbb{Q}$$

is equal to the cup pairing in cohomology.

Differential Geometry

X – being a real C^∞-manifold – carries real differential forms. These are, locally, sums of forms of the form

$$\omega = f(x_1, \dots, x_{2d})\, dx_{i_1} \wedge \cdots \wedge dx_{i_r} \qquad \text{(an } r\text{-form, } f \text{ real } C^\infty)$$

where $x_1, \dots, x_{2d}$ are local coordinates. Forms form an exterior algebra with a differential d:

$$d\omega = \sum_{\nu=1}^{2d} \frac{\partial f(x_1, \dots, x_{2d})}{\partial x_\nu} \cdot dx_\nu \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_r}.$$

image $=:$ exact forms, kernel $=:$ closed forms.

$$H^{r,dR}(X, \mathbb{R}) = \{\text{closed } r\text{-forms}\} / \{\text{exact } r\text{-forms}\}$$

This is a cohomology theory.

de Rham Theory

An r-form can be integrated on an r-cycle. So:

$$\{r\text{-cycles}\} \times \{r\text{-forms}\} \longrightarrow \mathbb{R}$$

exact forms integrate to zero on a cycle by Stoke and closed forms integrate to zero on boundaries. Get:

$$H_{r,B}(X,\mathbb{R}) \times H^{r,dR}(X,\mathbb{R}) \longrightarrow \mathbb{R}.$$

Theorem (de Rham): This is a perfect pairing identifying $H^{r,dR}(X,\mathbb{R})$ with $H^{r,B}(X,\mathbb{R})$. The pairing induced from the intersection pairing is

$$H^{r,dR}(X,\mathbb{R}) \times H^{2d-r,dR}(X,\mathbb{R}) \longrightarrow \mathbb{R}, \quad (\alpha,\beta) \longmapsto \int_X \alpha\wedge\beta.$$

(This gives $H^{r,dR}(X,\mathbb{R})$ a rational structure. A form ω is rational $\Longleftrightarrow$ $\int_\alpha \omega \in \mathbb{Q}$, $\forall\, \alpha \in H_{r,B}(X,\mathbb{Q})$.)

Hodge Theory

Let $z_1, \dots, z_d$ be complex local coordinates on X. Then, $z_1, \dots, z_d, \bar{z}_1, \dots, \bar{z}_d$ are real coordinates. A complex-valued form of type (p,q) is a sum of

$$\omega = f(z,\bar{z})\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q} \qquad p+q=r$$

This notion allows the Hodge decomposition of $H^{r,dR}(X,\mathbb{C}) = H^{r,dR}(X,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$:

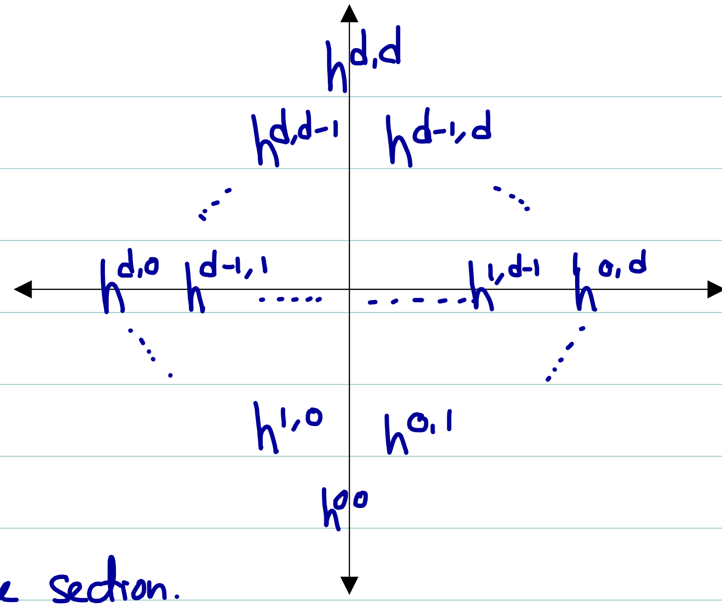
$$H^{r,dR}(X,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X) \qquad (\leftarrow \text{cplx v. spaces})$$

and gives: $\overline{H^{p,q}(X)} = H^{q,p}(X)$.

Let $h^{p,q} = \dim_{\mathbb{C}}(H^{p,q}(X))$ (the Hodge numbers). From construction $h^{p,q} = 0$ if $p>d$ or $q>d$. So: we have the Hodge diamond:

Two axis of symmetry:

① $\overline{H^{p,q}(X)} = H^{q,p}(X)$.



② X is projective:

$X \subseteq \mathbb{P}^N$, L a hyperplane section.

$L \cap X$ = cycle of codimension 2. $L: H_{r,B}(X,\mathbb{Q}) \longrightarrow H_{r-2}(X,\mathbb{Q})$

passing to dual, iterating and identifying $B = dR$,

$$L^{d-r}: H^{r,dR}(X,\mathbb{R}) \longrightarrow H^{2d-r,dR}(X,\mathbb{R}).$$

Hard Lefschetz theorem: This is an isomorphism.

It is not hard to see that $H^{p,q} \longrightarrow H^{p+d-r,q+d-r}$, so: $h^{p,q} = h^{d-q,d-p}$.

Examples:

Curve of genus g

$$\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$$

abelian surface

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array}$$

$\mathbb{P}^2$

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

K3 surface

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array} \quad \xrightarrow{\Sigma} \quad \begin{array}{c} b^4 \\ b^3 \\ b^2 \\ b^1 \\ b^0 \end{array}$$

$$e = b^4 - b^3 + \dots + b^0$$
$$\chi = h^{2,0} - h^{1,0} + h^{0,0}$$
$$c_2 = e$$
$$c_1^2 = 12\chi - e$$

(e = Euler char., χ = holomorphic Euler char.)

<u>The Hodge index theorem:</u> • Assume $d = \dim(X)$ is even. The intersection form on $H^{d,dR}(X,\mathbb{R})$ is symmetric, real and non-degenerate.
There is an orthogonal decomposition, $H^{d,dR}(X) = \bigoplus_{\substack{p+q=d \\ 0 \le s \le d/2 \\ p \le q}} S^{pqs}$, where

$$S^{pqs} = L^s \cdot \left(H^{p-s,q-s,prim}(X) + H^{q-s,p-s,prim}(X)\right)(\mathbb{R}).$$

The intersection form on S^{pqs} is <u>definite</u> of sign $(-1)^{p-s}$.

(If d is <u>odd</u>, the intersection form is alternating non-degenerate so $\sim \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$.)

<u>Example</u>: If X is a surface, consider $H^{1,1}(X)$. The 1-dim'l space $\mathbb{R}\cdot L$ is positive definite. On the complement, the intersection form is negative definite.

• Suppose $r = p+q$ is even, $r \le d$, $\alpha \in H^{p,q}(X) + H^{q,p}(X)$ real, primitive, then

$$(-1)^p \, I(\alpha, L^{d-r}\alpha) > 0.$$

$\left(I(\alpha,\beta) = \int_X \alpha \wedge \beta \right)$.

Lefschetz Theorem

Hodge's theorem implies:

- (Hard Lefschetz) Let X be a smooth projective variety of dim'n d. L a hyperplane section. Then
$$L^{d-r} : H^{r,dR}(X) \longrightarrow H^{2d-r,dR}(X) \quad \text{an } \underline{\text{isomorphism}}.$$

- (Weak Lefschetz) Let Y be a smooth hyperplane section of X. Then
$$H^i(X) \longrightarrow H^i(Y)$$
is an isomorphism for $i < d-1$, injective for $i = d-1$.

This generalizes to <u>complete intersections</u>. For the general case, we have:

<u>Barth's Theorem:</u> Let Y be a non-singular subvariety of $\mathbb{P}^N$. Then
$$H^i(\mathbb{P}^N) \longrightarrow H^i(Y)$$
is an isomorphism for $i \leq 2d - N$.

Example: N odd, $d=(N+1)/2 \Rightarrow H^1(Y)=\{0\}$. Thus, one cannot embed an abelian threefold in $\mathbb{P}^5$ (even as a non complete intersection. No abelian variety of $\dim^n > 1$ is a complete intersection, by weak Lefschetz and using $b_1 = 2\cdot\dim$.)

Example: Hard Lefschetz $\Rightarrow h^{p,q} \leq h^{p+1,q+1}$ if $p+q < \frac{1}{2}d$.

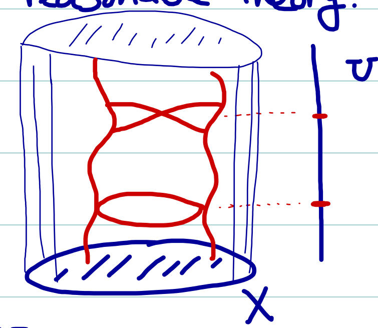
Cycles, the cycle map and equivalence relations.

X = projective algebraic variety of dim'n d.

An irreducible effective cycle Z of codim'n r, is a closed irreducible subvariety of codim'n r. The abelian group generated by those is the group $Z^r(X)$ of codim'n r cycles.

- $\alpha, \beta \in Z^r(X)$ are **rationally equivalent** if $\exists\, U \subseteq \mathbb{P}^1$, $\gamma \in Z^r(X \times U)$ s.t. $\forall u \in U$, $\gamma_u \in Z^r(X)$ and $\exists\, u_0 \neq u_1 \in U$ s.t. $\gamma_{u_0} = \alpha$, $\gamma_{u_1} = \beta$.

This is an equivalence relation — the weakest giving a reasonable theory. The quotient is the r-th Chow group:



$$CH^r(X) := Z^r(X) / \underset{rat}{\sim}.$$

- **Algebraic equivalence** is defined the same, using $U \subseteq C$, any curve.

The cycle map: Let Z be irreducible of codim'n r. The map

$$H_{2d-2r,B}(Z,\mathbb{Q}) \longrightarrow H_{2d-2r,B}(X,\mathbb{Q})$$

provides us with a chain of dim'n $2d-2r$ and, by Poincaré duality, with a class

$$[Z] \in H^{2r,B}(X,\mathbb{Q}).$$

This gives the cycle map

$$cl : CH^r(X) \longrightarrow H^{2r}(X,\mathbb{Q}).$$

Suppose that X is defined over a number field K. We have an isomorphism

$$H^{2r,B}(X^{an},\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell \cong H^{2r,et}(X \otimes_K \bar{K}, \mathbb{Q}_\ell)$$

But then a Z, which is K-rational, gives an invariant class $[Z] \in H^{2r,et}(\bar{X},\mathbb{Q}_\ell)(r)$.

At any rate, one can show (not too hard) that the cycle map cl factors

through algebraic equivalence: $cl : Z^r(X)/_{\sim alg.} \longrightarrow H^{2r,B}(X,\mathbb{Q})$.

- $\alpha, \beta \in Z^r(X)$ are **homologically equivalent** if $cl(\alpha) = cl(\beta)$.

- $\alpha, \beta \in Z^r(X)$ are **numerically equivalent** if $\forall \gamma \in Z^{d-r}(X)$

$$\alpha . \gamma = \beta . \gamma .$$

The following relations hold:

$$\begin{array}{ccccccc} \text{Rationally equiv.} & \overset{\Longrightarrow}{\not\Longleftarrow} & \text{Algebraically equiv.} & \overset{\Longrightarrow}{\not\Longleftarrow} & \text{homologically equiv.} & \overset{\Longrightarrow}{\overset{?}{\Longleftarrow}} & \text{numerically equiv.} \end{array}$$

The implications $\Longrightarrow$ are clear.

<u>Example</u> • Let X = proj. irred. curve of genus $g > 0$. Any two points are algebraically equivalent.

$$Z^1(X)/_{\sim \text{alg.}} \cong \mathbb{Z}, \qquad CH^1(X) = Z^1(X)/_{\sim \text{rat.}} \hookleftarrow \mathrm{Jac}(X).$$

- Homological equiv. $\not\Longrightarrow$ Algebraic equiv. (Example of Griffiths, Z^2(3-fold)).

The main conjectures

For $z \in Z^r(X)$, $cl(z) \in H^{r,r}(X,\mathbb{R}) \cap H^{2r,B}(X,\mathbb{Q})$.

Hodge Conjecture: $H^{r,r}(X,\mathbb{R}) \cap H^{2r,B}(X,\mathbb{Q}) = cl(Z^r(X))$.

The Hodge conjecture is wide-open. Some known cases:

* For $r = d = \dim(X)$.
* For $r = 0$.
* For $r = 1$ (divisors).
* For $r = d-1$ (by duality)

So, for curves & surfaces and 3-folds

* Generic abelian varieties.
* Other sporadic cases.
* Certain Kuga varieties.

* The main challenge at present seems to be dim 4.

Grothendieck's standard conjectures:

Let X be a projective smooth variety over a field k. L, a hyperplane section.

1. Homological equivalence $\equiv$ numerical equivalence. (In pos. characteristic use ℓ-adic cohomology).

2. Let $A^i(X) \subseteq H^{2i}(X)$ be $cl(Z^i(X))$. Then cup with L^{d-2i} induces an isomorphism
$$A^i(X) \xrightarrow[\cong]{\cup L^{d-2i}} A^{d-i}(X).$$

(Here the cohomology is B or dR in char. 0, ét or crys in pos. char.)

3. For all $r \leq d/2$, the $\mathbb{Q}$-valued pairing on $A^{r,\mathrm{prim}}(X)$,
$$\alpha \longmapsto (-1)^r \langle L^{d-2r}\alpha . \alpha \rangle$$
is positive definite.

What's known?

* ① $\Longrightarrow$ ② .   ② + ③ $\Longrightarrow$ ①

* In char. 0, ③ holds and ①,② hold if the Hodge conjecture holds.
  (Here we work with deRham cohomology.)
  Thus, for divisors ① holds: numerical equiv. $\equiv$ homo. equiv.

(In fact, numerical equiv. $=$ algebraic equiv. for <u>divisors</u>)

* Assume the conjectures for $X/\mathbb{F}_q$ and all its powers. Let $H^i(X)$ be the ith cohomology in some Weil cohomology theory. Then Fr acts semisimply on X and has char. polynomial with integer coef. that are independent of the cohomology theory. Furthermore, the eigenvalues all have absolute value $q^{i/2}$.

Examples

<u>Notation</u>: $\mathrm{Pic}(X) = CH^1(X) =$ divisors mod. rat'l equivalence. (Picard group)

$NS(X) =$ divisors mod. algebraic (homol./numer.) equivalence.

$\rho(X) =$ rank of the Néron–Severi group (the Picard number).

$NS(X)$ may have torsion in general.

$\mathrm{Pic}^0(X) = \left(\text{divisors} \underset{\text{alg.}}{\sim} 0\right) \Big/ \left(\text{divisors} \underset{\text{rat.}}{\sim} 0\right)$

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow NS(X) \longrightarrow 0$$

<u>X = curve of genus g.</u>

g dim'l AV

$$0 \longrightarrow \mathrm{Pic}^0(X) = \mathrm{Jac}(X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0$$

$$H^{1,B}(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$$

$$H^{1,dR}(X,\mathbb{C}) = H^0(X,\Omega_{X/\mathbb{C}}) \oplus H^1(X,\mathcal{O}_X) \qquad \text{☆}$$

Here we make use of a general theorem:

$$H^{p,q}(X) = H^q(X,\Omega^p_{X/\mathbb{C}}), \qquad \Omega^p_{X/\mathbb{C}} = \Lambda^p \Omega_{X/\mathbb{C}}.$$

So ☆ is the Hodge decomposition.

The exponential sequence of sheaves

$$0 \longrightarrow \underline{\underline{\mathbb{Z}}} \longrightarrow \mathcal{O}_X \xrightarrow{e^{2\pi i(\cdot)}} \mathcal{O}_X^* \longrightarrow 1$$

gives

$$\begin{array}{c} 0 \to H^{1,B}(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow[\text{"}cl(\cdot)\text{"}]{} H^{2,B}(X,\mathbb{Z}) \\ \qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Pic}(X) \end{array}$$

So: $\mathrm{Pic}^0(X) = H^1(X,\mathcal{O}_X) / H^1(X,\mathbb{Z})$, a complex torus, whose tangent space is $H^1(X,\mathcal{O}_X)$.

$$0 \to \mathrm{Tgt}_{\mathrm{Jac}(X),0} \to H^{1,dR}(X/\mathbb{C}) \to \mathrm{Tgt}_{\mathrm{Jac}(X)^\vee,0} \to 0.$$

X = surface

We know the Hodge conjecture for divisors. Thus,

$$H^{1,1}(X,\mathbb{R}) \cap H^{2,0}(X,\mathbb{Q}) = Z^1(X)/\text{alg. equivalence}.$$

$H^{2,0} = H^0(X, \Lambda^2\Omega_{X/\mathbb{C}}) = H^0(X,K)$, where K is a canonical divisor.
$h^{2,0}$ is the geometric genus. $h^{1,0}$ is called the irregularity and $h^{2,0}-h^{1,0}$ is called the arithmetic genus.

$$\begin{array}{ccccc} & & 1 & & \\ & q & & q & \\ p_g & & h^{1,1} & & p_g \\ & q & & q & \\ & & 1 & & \end{array}$$

Not everything is possible.

$c_1^2 = 12(p_g - q + 1) - e \qquad e = \sum_i (-1)^i b_i$

$c_2 = e$

<u>Then:</u> (Noether) $p_g \leq \frac{1}{2} c_1^2 + 2$.

(BMY) $c_1^2 \leq 3c_2$, for general type.

But, in general, q and p_g are not restricted much further.

For $X = \mathbb{P}^2$ we even know the Chow groups.

$$CH^i(\mathbb{P}^2) = \mathbb{Z}, \quad i = 0, 1, 2.$$

But for a general surface they are complicated. $CH^0(X)$ is reflecting rational lines on the surface. Its structure is very complicated. For example:

<u>Theorem</u> (Mumford): If $p_g > 0$ the following is <u>false</u>: " $\exists\, n$ s.t. $\forall$ 0-cycles α with $\deg(\alpha) \geq n$, $\alpha \sim_{\text{rat.}} \beta$, β effective". In particular, $CH^0(X)$ is not representable.

<u>Conjecture</u> (Bloch): If $p_g = 0$, $CH^0(X)$ <u>is</u> representable.

Rigid Calabi-Yau 3-folds:

A smooth projective 3-fold is Calabi-Yau if $\mathcal{K}_X$ is trivial and $h^{0,1} = h^{1,2} = 0$. It is rigid if $H^1(X, T_X) = 0$, which implies by Serre's duality that $h^{1,2} = 0$. So:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{2,2} & & 0 & \\
1 & & 0 & & 0 & & 1 \\
 & 0 & & h^{1,1} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

$h^{1,1} = h^{2,2}$, $h^{11} + h^{2,2} = e$ (that could be very large)

Hodge: $h^{2,2}$ is accounted for by divisors/$_{\text{alg.}}$. $h^{2,2} = b_4 = b_2$. Often can be analyzed "easily".

OTOH, $H^{3,et}(\bar{X}, \mathbb{Q}_\ell)$ gives a 2-dim'l Galois representation, not coming from alg. cycles.

Example:
X = resolution by blow-up of $\overbrace{\text{a}}^{\text{certain}}$ compactification of the affine 3-fold

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0$$

Here $h^{1,1} = h^{2,2} = 70$

The Galois representation is associated to the modular form

$$[\eta(2\tau)\eta(4\tau)]^4 \in S_2(\Gamma_0(8)). \qquad \text{(H. Verrill).}$$

The argument uses point counting in characteristic p to deduce the L-fcn of $H^{3,et}(X)$ from the zeta function. The analysis of the contribution of H^2_{et}, H^4_{et} is based on X being fibered by K_3 surfaces over $\mathbb{P}^1$ and

$$\rho(X) = \rho(\text{gen'l fibre}) + 1 + \sum_s (n(X_s) - 1)$$

$\uparrow$ # of components of the fibre X_s

Moreover, this involves understanding the structure of $H^{2,0}(X)$.

This example yields periods $F(x) = \sum_{n=0}^{\infty} V_n x^{2n+1}$,
where
$$V_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \quad \text{(\#s appearing in Apéry's work on } \zeta(3)\text{).}$$

Mazur's Theorem on Hodge / Newton polygons

p prime. $\overline{\mathbb{F}}_p \supseteq \mathbb{F}_q \supseteq \mathbb{F}_p$, $q = p^a$.

X_s = smooth, projective variety over $\mathbb{F}_q$. Then

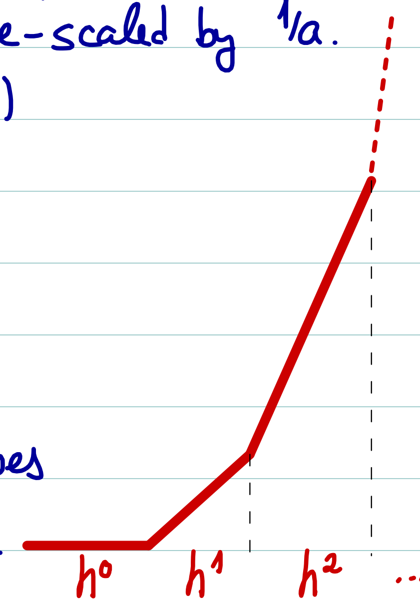
$$H^{i,\mathrm{crys}}(X/W(\mathbb{F}_q)) \qquad \text{f. rank free } W(\mathbb{F}_q)\text{-modules.}$$

Frob: $X \longrightarrow X^{(\varphi)}$ induces a σ-linear map on $H = H^{i,\mathrm{crys}}(X, W(\mathbb{F}_q))$, but Frob^a induces a linear map. We associate to it its Newton polygon, re-scaled by $1/a$. (a slope $\frac{\lambda}{a}$ of length b means b roots of valuation λ.)

On the other hand, $H/\mathrm{Frob}(H)$ has elementary divisors

$$\underbrace{p^0, \dots, p^0}_{h^0}, \underbrace{p^1, \dots, p^1}_{h^1}, \underbrace{p^2, \dots, p^2}_{h^2}, \dots \dots$$

We can construct the Hodge polygon having segments of slopes $0, 1, 2, \dots$ of length over x-axis equal to the multiplicity.

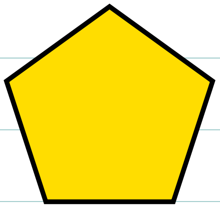


Theorem: Assume that X_s admits a lifting to a smooth proper scheme $\mathfrak{X} \to \mathrm{Spec}(W(\mathbb{F}_q))$ and that $H^q(X, \Omega^p_{X/W(\mathbb{F}_q)})$ are torsion-free $W(\mathbb{F}_q)$-modules. Then:

The multiplicity of the slope j in the Hodge polygon of $H^{i,\mathrm{crys}}(X_s/W(\mathbb{F}_q))$ is $h^{j,i-j}$, where $h^{p,q}$ are the Hodge numbers of $X_0 \otimes_{W(\mathbb{F}_q)[\frac{1}{p}]} \mathbb{C}$.

Furthermore, the Newton polygon lies above (or on) the Hodge polygon.

Remark: Assuming the standard conjectures the eigenvalues of Frob^a are units outside p. The theorem gives information about their p-adic valuation in terms of Hodge numbers of the generic fiber. This, in some instances, gives information about $\# X(\mathbb{F}_{q^r})$, $r = 1, 2, 3, \dots$.



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