An Introduction to Kuga Fiber Varieties

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Notation

- G a $\mathbb{Q}$-simple algebraic group of *Hermitian* type.
- K a maximal compact subgroup of $G(\mathbb{R})$.
- Γ a torsion free arithmetic subgroup.
- $Y := G(\mathbb{R})/K$ the corresponding Hermitian symmetric space and X its quotient by Γ.

Let ρ be a representation of G defined over $\mathbb{Q}$ into a $2n$-dimensional vector space V. Let L be a maximal lattice inside of V, and assume $\rho(\Gamma) \subseteq \mathrm{Aut}(L) = GL(2n, \mathbb{Z})$.

A Commutative Diagram

Set $T = V_{\mathbb{R}}/L$, a real analytic torus. Γ acts on V by ρ and preserves L, so our assumptions above result in the following diagram:

$$\begin{tikzcd} Y \times T \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{\pi}"] & \Gamma\backslash(Y \times T) =: W \arrow[d, dotted, "\pi"] \\ Y \arrow[r, "p"] & \Gamma\backslash Y =: X \end{tikzcd}$$

W is a smooth torus fiber bundle over X.

Families of Abelian varieties

Recall that an n-dimensional polarized complex abelian variety corresponds to the real torus T plus:

- A complex structure J on V
- A symplectic form A on V such that $A(v, Jw)$ is positive definite and $A(L, L) \subseteq \mathbb{Z}$.

Suppose there is a symplectic form A on L defined over $\mathbb{Z}$ such that $\rho(G) \subset Sp(V_{\mathbb{Q}}, A)$. Then there is a maximal compact $K' \subset Sp(V(\mathbb{R}), A)$ containing $\rho(K)$. The space $Sp(V, A)/K' = \mathfrak{h}_n$ is the Siegel upper half plane of genus n parameterizing J as above.
Let $E_\Gamma = \rho(\Gamma)\backslash(\mathfrak{h}_n \times V/L)$, then the map ρ from G to $Sp(V, A)$ induces the following smooth maps of fiber bundles.

$$
\begin{tikzcd}
W \arrow[r] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & E_\Gamma \arrow[d, "\pi'"'] \\
X \arrow[r, "\tau"] \arrow[d, phantom, "||"] & \rho(\Gamma)\backslash\mathfrak{h}_n \arrow[d, phantom, "||"] \\
\Gamma\backslash G(\mathbb{R})/K \arrow[r, "\rho"] & \rho(\Gamma)\backslash Sp(\mathbb{R})/K'
\end{tikzcd}
$$

The fiber bundle on the right is a bundle of abelian varieties, with complex structure on fiber $\{J\} \times V/L \subset E_\Gamma$ equal to J and polarization A. Furthermore, π' is a map of quasi-projective complex varieties. Since X is also a complex, quasi-projective variety, two questions immediately arise.

2 Questions

- When is τ (the *Eichler map*) holomorphic?
- When does W have a "good" complex structure?

Call a complex structure $\mathfrak{J}$ on W "good" if

- the map $\pi : W \to X$ is holomorphic;
- $\mathfrak{J}$ restricted to each fiber agrees with the complex structure induced from the map into E_Γ;
- the complex structure induced by $\mathfrak{J}$ on the universal covering manifold $Y \times V$, is a holomorphic complex vector bundle over Y.

Theorem (Kuga)

W has good complex structures if and only if τ is holomorphic and in this case such a structure is unique. Moreover, if X is compact then W is a Hodge variety (hence projective).

Kuga Fiber Varieties

More generally, when τ is holomorphic the W are smooth quasi-projective algebraic varieties, called *Kuga fiber varieties*. They provide families of polarized abelian varieties parameterized by X. "Kuga fiber variety" may also refer to a compactification of such a space.

Examples:

- Universal abelian varieties over appropriate moduli spaces.
- Abelian families of Hodge type.
- Abelian families of PEL type.
- Products.

Kuga Fiber Varieties (Cntd.)

Why do we care?

- Provide generalizations of well studied modular curves
- The high degree of structure imposed on Kuga varieties makes them ideal for test cases of certain conjectures.
- Can get relations between abelian varieties in the same Kuga fiber variety / between families over the same base.

We list a few examples of results regarding these spaces.

Cohomology

- When X is compact, the groups $H^r(W) \cong \sum_{p+q=r} H^p(X, H^q(T)) = \sum_{p+q=r} H^p(X, (\wedge^q \rho)^*)$.
- (Abdulali) If $W \to X$ is a Kuga fiber variety satisfying the H_2 condition, then for any $x \in X$, the space $H^{2p}(W_x, \mathbb{Q})^\Gamma$ consists of Hodge cycles.
- (Abdulali) Suppose $W_i \to X$ for $i = 1, 2$ are two Kuga fiber varieties over the same base satisfying the H_2-condition and defined over a number field k_0. For a sufficiently large extension k of k_0, if $x \in X(k)$ the following holds:
  If $F_1 \subset H^{b_1}(W_{1,x}, \mathbb{Q})$, $F_2 \subset H^{b_2}(W_{2,x}, \mathbb{Q})$ are irreducible and isomorphic as $\pi_1(X^{an}, x)$-modules then

  $$F_1 \otimes \mathbb{Q}_\ell \cong F_2 \otimes \mathbb{Q}_\ell((b_2 - b_1)/2)$$

  as $\pi_1^{\mathrm{alg}}(X_k, x)$-submodules of $H_{et}^{b_i}(W_{i,x}, \mathbb{Q}_\ell)$.

Abdulali's results imply certain relations among the zeta functions of W_1 and W_2 [2]. Zeta functions of Kuga varieties have been studied by Kuga, Shimura, Deligne, Langlands, Ohta, etc.

- Sheaves correspond to modular forms: For $N \geq 3$ let $E(N)$ denote the universal elliptic curve of level N over $\Gamma(N)\backslash\mathfrak{h}_1$. Let $\overline{E(N)}^k$ be a sufficiently nice smooth compactification of k^{th} fiber product of $E(N)$. Then

$$S_{k+2}(\Gamma(N)) \cong H^0(\overline{W(N)}^k, \Omega^{k+1}) \otimes \mathbb{C}.$$

When $k = 2$ and $\overline{E(N)}^2$ is a rigid Calabi-Yau threefold, we have

$$L(\overline{W(N)}^2, s) = L(f, s)$$

where f is the normalized cusp form of weight 4 on $\Gamma(N)$.

Holomorphicity

If one wants to classify these fiber varieties, the question arises:

Find all representations $\rho : G \to Sp(2n, \mathbb{Q})$ such that τ is holomorphic.

Algebraic setting: Recall $\mathrm{Lie}(G(\mathbb{R})) = \mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ with the natural map $G(\mathbb{R}) \to Y$ sending $\mathfrak{p}$ isomorphically to $T_o(Y)$. Then there exists unique $H_0 \in Z(\mathfrak{k})$ such that $\mathrm{ad} H_0|_{\mathfrak{p}}$ is the complex structure on $T_o(Y)$. For $Sp(V, A)$ and base point $\tau(o) = J \in \mathfrak{h}_n$, the corresponding H_0' is $J/2 \in \mathrm{Lie}(Sp(V, A))$. Holomorphicity of τ reduces to

$$(H_1): \quad [d\rho(H_0) - H_0', d\rho(g)] = 0 \quad \text{for all } g \in \mathfrak{g}_{\mathbb{R}}.$$

An even stronger condition is

$$(H_2): \quad d\rho(H_0) = H_0'.$$

Kuga varieties satisfying the H_2 condition have especially nice properties as we have seen.

We now discuss Addington's system of classification of Kuga varieties arising from non-split quaternion algebras in terms of its "chemistry". We begin by recalling some theory.

Fix a field $F \hookrightarrow \overline{F}$ of characteristic 0 and a finite extension F'.

Definition

The *restriction of scalars* functor $R_{F'/F}$ from the category of F'-varieties to the category of F-varieties has the following properties:

1. $(R_{F'/F}X)(F) = X(F')$
2. Let Φ be the set of distinct embeddings of F' over F. Then

$$R_{F'/F}X \cong \prod_{a \in \Phi} X^a \qquad (\text{over } \overline{F}).$$

Unitary groups

Fix a totally real number field k. Let D over k be a division quaternion algebra (with involution $d \to \overline{d}$). Fix a right vector space V over D of dimension n.

Definition

Let $\epsilon = \pm 1$. A map $h : V \times V \to D$ is a D-valued ϵ-Hermitian form on V wrt. the given involution if

1. h is k-bilinear
2. $h(v, w\alpha) = h(v, w).\alpha$ for $\alpha \in D$
3. $h(v, w) = \epsilon \overline{h(w, v)}$.

Assume that V is equipped with such an h and it is nondegenerate.

Let $SU(V, h)$ be the special unitary group of h, ie. $g \in \mathrm{Aut}_D V$ with reduced norm 1 and preserving h.

Algebraic Groups of Type II and III

With notation as above, let S be the set of infinite places φ of k.
Definition

1. An algebraic group of type III.1 is a group $R_{k/\mathbb{Q}}Sp(V', A)$ where V' is a 2n dimensional k-vector space.
2. A group of type III.2 is a group $R_{k/\mathbb{Q}}SU(V, h)$ such that $\epsilon = 1$ and when φ does not split D, h^{φ} on $V \otimes_{\varphi} \mathbb{R}$ is definite.

To define groups of type II we need the following: Suppose D splits over k', then $SU(V_{k'}, h)$ is isomorphic to a special matrix group preserving some form B on $(k')^{2n}$. As a matrix, $B^t = -\epsilon B$.

3. An algebraic group of type II is a group $R_{k/\mathbb{Q}}SU(V, h)$ with $n \geq 3$ such that $\epsilon = -1$ and when φ splits D, the real symmetric form B_{φ} corresponding to h^{φ} is definite.

Over $\mathbb{R}$, these groups G look like

$$\begin{array}{ll} III.1 & \prod_{\varphi\in S} Sp(2n,\mathbb{R}) \\ III.2 & \prod_{D_\varphi\cong M_2(\mathbb{R})} Sp(2n,\mathbb{R}) \times \prod_{D_\varphi\cong\mathbb{H}} SU_{\mathbb{H}}(n) \\ II & \prod_{D_\varphi\cong M_2(\mathbb{R})} SO(2n,\mathbb{R}) \times \prod_{D_\varphi\cong\mathbb{H}} SO^*(n) \end{array}$$

$$(SU_{\mathbb{H}}(n) = SU(2n)\cap Sp(2n,\mathbb{C}), SO^*(n) = SU(n,n)\cap SO(2n,\mathbb{C})).$$

The groups of type II and III are $\mathbb{Q}$-simple and give rise to products of Hermitian symmetric spaces of type II and III respectively. For a group of type II or III, let S_0 correspond to the non-compact factors and S_1 the compact. The complex dimension of the corresponding spaces $G(\mathbb{R})/K$ is

- $|S_0|n(n-1)/2$ if G is type II
- $|S_0|n(n+1)/2$ if G is type III.

Chemistry

Let G, S, S_0, S_1 be as above. An *atom* is an element of S, a *molecule* a set of atoms and a *polymer* a formal sum of molecules. Let k' be the Galois closure of k and $\mathfrak{G} = \mathrm{Gal}(k', \mathbb{Q})$.
$\mathfrak{G}$ acts on atoms transitively. This induces an action on molecules and polymers.

A polymer is called *stable* if

(i) it is invariant under $\mathfrak{G}$

(ii) each molecule contains at most one atom from S_0.

A stable polymer is *rigid* if each molecule contains exactly one atom from S_0.

Recall that over $\mathbb{C}$, $G = R_{k/\mathbb{Q}} G_0 \cong \prod_{a \in S} G_0^a$, with $G_0^a(\mathbb{C}) \cong Sp(2n, \mathbb{C})$ or $SO(2n, \mathbb{C})$. Denote this representation by id. We define complex representations of G corresponding to atoms, molecules and polymers as follows:

Chemistry (Cntd.)

- For $a \in S$, let ρ_a be the projection representation $G(\mathbb{C}) \to G_0^a(\mathbb{C})$, so $\rho_a = \mathrm{id} \circ \mathrm{proj}_a$.
- For a molecule $M = \{a_1, \ldots, a_m\}$, define $\rho_M = \rho_{a_1} \otimes \ldots \otimes \rho_{a_m}$.
- For a polymer $P = M_1 + \ldots + M_d$, define $\rho_P = \rho_{M_1} \oplus \ldots \oplus \rho_{M_d}$.

Theorem (Addington, Satake)

Let G be a group of type II with $\dim_D V \geq 5$, or of type III.

(i) *If ρ is a symplectic representation of G over $\mathbb{Q}$ that admits a holomorphic Eichler map, then ρ is equivalent (over $\mathbb{C}$) to a stable polymer representation.*

(ii) *If P is a stable polymer, then there is a positive integer N (determined by the size of the molecules in P) such that ρ_{NP} is a symplectic representation defined over $\mathbb{Q}$ and admits a holomorphic Eichler map.*

Abdulali showed that Addington's chemistry and main theorem could be generalized to representations of products of groups of type III.2 with $\dim_D V = 1$. He also determined that if P is rigid, the corresponding Eichler map satisfies the H_2 condition.

Sketch of (i)

Since G is $\mathbb{Q}$-simple $\Rightarrow$ can assume ρ is $\mathbb{Q}$-primary, ie. if ρ_0 is an irreducible $\overline{\mathbb{Q}}$-subrepresentation of ρ then for some numberfield K and positive integer p, $p.\rho_0$ is defined over K and

$$\rho \cong_{\overline{\mathbb{Q}}} R_{K/\mathbb{Q}} p.\rho_0.$$

Since $G(\mathbb{C}) = \prod_{a \in S} G^a(\mathbb{C})$ are simple groups,

$$\rho_0 \cong \bigotimes_{a \in S} \rho_{0,a} \circ \mathrm{proj}_a.$$

If $\{\tau_1, \ldots, \tau_e\}$ are extensions of the distinct embeddings of K into $\overline{\mathbb{Q}}$, then

$$\rho \cong \sum_{i=1}^{e} p \left(\bigotimes_{a \in S} \rho_{0,a} \circ \mathrm{proj}_a \right)^{\tau_i} \cong \sum_{i=1}^{e} p. \left(\bigotimes_{a \in S} \rho_{0,a}^{\tau_i} \circ \mathrm{proj}_{\tau_i(a)} \right)$$

We must show each $\rho_{0,a}^{\tau_i} = \text{id}$ or trivial.

- When $a \in S_0$, Satake's classification over the reals of symplectic representations for the noncompact simple Lie group $G^a(\mathbb{R})$ [7] says $\rho_{0,a}^{\tau_i}$ is either a multiple of the identity or trivial. But $\rho_{0,a}^{\tau_i}$ is irreducible.
- For $a \in S_1$, use the fact that the Galois group of k' over $\mathbb{Q}$ acts transitively on the atoms to reduce to previous case.

Let $M = \{a \in S : \rho_{0,a} = \text{id}\}$. Then $P = p\sum_{i=1}^{e} \tau_i(M)$ and since $\rho \cong_{\mathbb{C}} \rho_P$ is defined over $\mathbb{Q}$, it follows easily that $P = \sigma(P)$. That each M contains at most one noncompact embedding can be found in [8, Ch. IV. 5] (Follows from the H_1 condition). ■

Remarks Regarding (ii)

- For groups of type III.1, $P = p.\sum_{i=1}^{d} a_i$ and Satake [7] showed ρ_P is already defined over $\mathbb{Q}$ and admits a holomorphic Eichler map. Addington's chemistry is needed for groups of type II and III.1
- ρ_P defined over $\mathbb{Q}$: Reduce to when P is prime (Galois orbit of molecule M). Embed the k' groups $G_0^a = SU(V^a, h^a)$ into

$$E_M = M(n, D \otimes_{a_1} k') \otimes_{k'} \ldots \otimes_{k'} M(n, D \otimes_{a_m} k')$$

E_M is defined over stabilizer k_0 of M and $G \hookrightarrow R_{k_0/\mathbb{Q}} E_M$ over $\mathbb{Q}$.

$$E_M \cong M((2n)^m, k_0) \text{ or } M(n(2n)^{m-1}, D')$$

for D' a division quaternion algebra. The corresponding irreducible representations of G are equivalent over $\overline{\mathbb{Q}}$ to ρ_P or $2\rho_P$ respectively.

Examples

1) (Satake type polymers): $S = \{1, \ldots, d\}$, $P = \{1\} + \{2\} + \ldots + \{d\}$. These are rigid iff $S_0 = S$. If G is a group of type III.1 ($R_{F/\mathbb{Q}} Sp(V', A)$), ρ_P is defined over $\mathbb{Q}$, otherwise ρ_{2P} is. Kuga varieties of PEL type for abelian varieties of type II and III arise in this way (when $S_0 = S$).

Example: D a totally definite quaternion algebra over a totally real number field k, $[k : \mathbb{Q}] = d$. V an n-dimensional right D vector space and h a non-degenerate skew-hermitian form on V. Then $A := \mathrm{Tr}_{D/\mathbb{Q}} h(x, y)$ is a symplectic form on V (as a $\mathbb{Q}$-vector space). Given a lattice L and torsion free Γ preserving L, the representation $G = R_{k/\mathbb{Q}} SU(V, D) \hookrightarrow Sp(V, A)$ is equivalent (over $\mathbb{C}$) to ρ_{2P}. The inclusion induces a holomorphic τ satisfying the H_2 condition and this realizes $\Gamma \backslash G / K$ as a PEL-type Shimura variety parameterizing $(2dn)$-dimensional abelian varieties of type III.

Examples (Cntd.)

2) (Mumford type polymers): Assume S_0 has 1 element.
$P = M = S = \{1, \ldots, d\}$, so P is rigid.

Example (Mumford): Let $[k : \mathbb{Q}] = 3$ be totally real, D a quaternion division algebra over k. Define $E_M = E$ as:

$$E = (D \otimes_{a_1} k') \otimes_{k'} (D \otimes_{a_2} k') \otimes_{k'} (D \otimes_{a_3} k').$$

$\mathfrak{G}$ acts on E by permuting the factors. Define $\mathrm{Cor}_{k/\mathbb{Q}}(D)$ to be the subalgebra of E fixed by this action (This will be a CSA over $\mathbb{Q}$ satisfying $\mathrm{Cor}_{k/\mathbb{Q}}(D) \otimes k' \cong E$). We have the norm map

$$Nm : D^* \to \mathrm{Cor}_{k/\mathbb{Q}}(D)^*$$
$$Nm(d) = (d \otimes 1) \otimes (d \otimes 1) \otimes (d \otimes 1)$$

Assume

$$\mathrm{Cor}_{k/\mathbb{Q}}(D) \cong M_8(\mathbb{Q}) \tag{1}$$
$$D \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H} \oplus \mathbb{H} \oplus M_2(\mathbb{R}). \tag{2}$$

Take $V = D$, h the standard involution and $G = R_{k/\mathbb{Q}} SU(V, h)$. The norm map provides a representation of G on $\mathbb{Q}^8$ by (1) (By (2), this representation is of "Mumford type"). Over $\mathbb{R}$, the representation factors through the 2-1 map

$$SU(2) \times SU(2) \times SL(2, \mathbb{R}) \twoheadrightarrow SO(4, \mathbb{R}) \times SL(2, \mathbb{R}) \circlearrowleft \mathbb{R}^4 \otimes \mathbb{R}^2.$$

There is a unique symplectic form (up to scalars) A left fixed by the $\mathbb{Q}$-representation D^*.
Let $\varphi_0 : \mathbb{S} \to SO(4, \mathbb{R}) \times SL(2, \mathbb{R}) \subset Sp(\mathbb{R}^8, A)$ send

$$e^{i\theta} \to I_4 \otimes \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right).$$

The map ρ satisfies the H_1 condition for the complex structure induced by φ_0. Thus for any appropriate choice of L and Γ, we get a holomorphic map

$$\Gamma \backslash \mathfrak{h}_1 \hookrightarrow \rho(\Gamma) \backslash \mathfrak{h}_4.$$

Since G is $\mathbb{Q}$-simple, it follows $\rho(G)$ is the Hodge group of a generic conjugate of φ_0, so the Kuga variety associated to it is of Hodge type.
The representation ρ is absolutely irreducible, so if X_φ is an abelian variety with Hodge group $\rho(G)$, X_φ has no non-trivial endomorphisms. Hence the variety is not of PEL type.

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