Schoen's work III

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Review of notation

$D =$ the discriminant of an order in an imaginary quadratic field
$\mathscr{O}_D = \mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right]$, the order of discriminant D
$k_D = \operatorname{Frac} \mathscr{O}_D = \mathbb{Q}(\sqrt{D})$
$H_D = k_D\big(j(\mathscr{O}_D)\big)$, the ring class field attached to the order $\mathscr{O}_D$.
$\mathfrak{H} \subset \mathbb{C}$ the upper half-plane
$\mu_n =$ group of nth roots of unity
$\widetilde{W} =$ the smooth projective 3-fold over $\operatorname{Spec} \mathbb{Q}$ from the previous 2 talks, fibered over $X = \mathbb{P}^1_{\mathbb{Q}}$
For $F/\mathbb{Q}$ a quadratic extension, $\delta(F) = 1$ if F is real and unramified at 3 or imaginary and ramified at 3, and $\delta(F) = -1$ otherwise.

Let F be a number field. This and the previous two talks are interested in the instance of the Beilinson–Bloch conjecture

$$\operatorname{rank} CH^2(\widetilde{W}_F)_{\text{hom}} \stackrel{?}{=} \operatorname{ord}_{s=2} L_F\big(H^3_{\text{ét}}(\widetilde{W}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l), s\big),$$

say especially when $[F : \mathbb{Q}] \leq 2$.

An important consequence of Amir-Khosravi's talk is that for $F = \mathbb{Q}$ or F quadratic with $\delta(F) = 1$,

$$\operatorname{rank} CH^2(\widetilde{W}_F)_{\text{CM}} = 0,$$

where we recall

$$CH^2(\widetilde{W}_F)_{\text{CM}} = \ker[CH^2(\widetilde{W}_F)_{\text{hom}} \longrightarrow CH^2(\widetilde{W}_F \times_X \eta_X)].$$

A big remaining task is to study the L-function side of the story, and in particular to see that $\delta(F)$ gives the sign in the functional equation for L_F.

For F a number field, write

$$G_F := \mathrm{Gal}(\overline{\mathbb{Q}}/F)$$

and

$$\rho_{F,l} \colon G_F \longrightarrow GL\big(H^3_{\text{ét}}(\widetilde{W}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)\big).$$

Since $\widetilde{W}_F$ has good reduction at primes $\mathfrak{p}$ of F not dividing 3, the representation $\rho_{F,l}$ is unramified at primes $\mathfrak{p} \nmid 3l$. By Deligne's proof of the Weil conjectures, for such $\mathfrak{p}$ the polynomial

$$P_{\mathfrak{p}}(t) := \det\big(\mathrm{id} - t\,\mathrm{Frob}_{\mathfrak{p}}^{-1} \bigm| H^3_{\text{ét}}(\widetilde{W}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)\big)$$

has coefficients in $\mathbb{Z}$ and is independent of l.

We form the partial L-series

$$L_F\big(H^3(\widetilde{W}), s\big) := \prod_{\substack{\mathfrak{p} \text{ prime of } F \\ \mathfrak{p}\nmid 3}} \frac{1}{P_{\mathfrak{p}}(N\mathfrak{p}^{-s})},$$

which converges absolutely for $\Re(s) > \frac{5}{2}$. The missing factors, for primes $\mathfrak{p} \mid 3$, converge absolutely and are nonvanishing for $\Re(s) > \frac{3}{2}$, and therefore do not affect $\operatorname{ord}_{s=2} L_F\big(H^3(\widetilde{W}), s\big)$.

We can regard $L_F\big(H^3(\widetilde{W}), s\big)$ as the (again, partial) Artin L-series $L(\rho_F^*, s)$ attached to the contragredient representation ρ_F^* of $\rho_{F,l}$. By the general formalism of L-series,

$$L(\rho_F^*, s) = L(\operatorname{Ind}_{G_F}^{G_{\mathbb{Q}}} \rho_F^*, s) = L(\rho_{\mathbb{Q}}^* \otimes \operatorname{Ind}_{G_F}^{G_{\mathbb{Q}}} \mathbf{1}, s).$$

When $F/\mathbb{Q}$ is Galois, we have $\operatorname{Ind}_{G_F}^{G_{\mathbb{Q}}} \mathbf{1} \simeq \bigoplus_\xi \xi^{\dim \xi}$, where the sum runs through the irreducible representations ξ of $\mathrm{Gal}(F/\mathbb{Q})$. Hence

$$L(\rho_F^*, s) = \prod_\xi L(\rho_{\mathbb{Q}}^* \otimes \xi, s)^{\dim \xi}.$$

Recall that the Beilinson–Bloch conjecture asserts that

$$\dim CH^2(\widetilde{W}_F)_{\mathrm{hom}} \otimes \overline{\mathbb{Q}}_l = \mathrm{ord}_{s=2} L_F\left(H^3_{\text{ét}}(\widetilde{W}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l), s\right).$$

Schoen uses the identity $L(\rho_F^*, s) = \prod_\xi L(\rho_{\mathbb{Q}}^* \otimes \xi, s)^{\dim \xi}$ to formulate the following refinement to B–B in the case at hand.

For each $\overline{\mathbb{Q}}_l$-representation V and irreducible representation ξ of $\mathrm{Gal}(F/\mathbb{Q})$, let V^ξ denote the ξ-isotypic component of V, and let ν_ξ denote the multiplicity of ξ in V. Then

Conjecture

$$\dim\left(CH^2(\widetilde{W}_F)_{\mathrm{hom}} \otimes \overline{\mathbb{Q}}_l\right)^\xi = \mathrm{ord}_{s=2} L\left(\rho_{\mathbb{Q}}^* \otimes (\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \mathbf{1})^\xi, s\right),$$

or equivalently,

$$\nu_\xi\left(CH^2(\widetilde{W}_F)_{\mathrm{hom}} \otimes \overline{\mathbb{Q}}_l\right) = \mathrm{ord}_{s=2} L\left(\rho_{\mathbb{Q}}^* \otimes \xi, s\right).$$

We first consider $F = \mathbb{Q}$.

Recall from Sankaran's talk the Dedekind η-function

$$\eta(\tau) = \exp(2\pi i\tau/24) \prod_{n\geq 1} \big(1 - \exp(2\pi i n\tau)\big)$$

and the global 3-form on $\widetilde{W}^{\text{an}}$

$$\omega = \frac{2\pi i}{3}\eta^8(\tau)\, d\tau\, dz_1\, dz_2,$$

which is a generator for $H^{3,0}(\widetilde{W}^{\text{an}}, \mathbb{C})$.

Let $q = \exp(2\pi i\tau/3)$, and consider the q-expansion $\eta^8(\tau) = \sum_{n\geq 1} a_n q^n$ and associated Dirichlet series

$$L\big(\eta^8(3\tau), s\big) = \sum_{n\geq 1} a_n n^{-s}.$$

$\eta^8(3\tau)$ is the unique normalized weight 4 newform on $\Gamma_0(9)$.

By Deligne, for primes $p \neq 3$, we have

$$\mathrm{Tr}\big(\rho_{\mathbb{Q}}(\mathrm{Frob}_p^{-1}) \mid H^3_{\text{ét}}(\widetilde{W}_{\overline{\mathbb{Q}}}, \mathbb{Q}_l)\big) = \mathrm{Tr}\big(T_p \mid H^{3,0}(\widetilde{W}^{\mathrm{an}}, \mathbb{C})\big).$$

Since $H^{3,0}(\widetilde{W}^{\mathrm{an}}, \mathbb{C})$ is 1-dimensional with generator ω, the RHS equals a_p. It follows from this that the Euler product expansion for $L\big(\eta^8(3\tau), s\big)$ agrees (at primes $p \neq 3$) with that for $L(\rho^*_{\mathbb{Q}}, s)$.

We shall exploit this by expressing $L\big(\eta^8(3\tau), s\big)$ as a Hecke L-series. Let ζ_3 be a primitive 3rd root of unity and $K := \mathbb{Q}(\zeta_3)$. Let $I_{\sqrt{-3}}$ denote the group of fractional ideals for $\mathbb{Z}[\zeta_3]$ relatively prime to $\sqrt{-3}$, and define the Größencharakter

$$\begin{aligned} \Psi\colon\ I_{\sqrt{-3}} &\longrightarrow K^\times \\ \mathfrak{a} &\longrightarrow a^3 \end{aligned}$$

where $\mathfrak{a} = (a)$ with $a \equiv 1 \bmod \sqrt{-3}$.

By a lemma of Shimura (going back to Hecke), $L(\Psi, s)$ is the Mellin transform of a weight 4, normalized newform on $\Gamma_0(9)$, namely of $\eta^8(3\tau)$. Hence

$$L\big(\eta^8(3\tau), s\big) = L(\Psi, s) = \prod_{\substack{\mathfrak{p} \text{ prime of } K \\ \mathfrak{p} \neq (\sqrt{-3})}} \frac{1}{1 - \Psi(\mathfrak{p}) N\mathfrak{p}^{-s}}.$$

For $F/\mathbb{Q}$ quadratic, let $\gamma\colon G_{\mathbb{Q}} \to \mu_2$ be the character with kernel G_F. Then

$$L(\rho_F^*, s) = L(\rho_{\mathbb{Q}}^* \otimes \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \mathbf{1}, s) = L(\rho_{\mathbb{Q}}^*, s) L(\rho_{\mathbb{Q}}^* \otimes \gamma, s)$$

and (up to finitely many bad Euler factors)

$$L(\rho_{\mathbb{Q}}^* \otimes \gamma, s) = L\big(\Psi \cdot (\gamma \circ N_{K/\mathbb{Q}}), s\big),$$

where the γ on the RHS is viewed as a character on ideals via class field theory.

We shall also consider the case that F is the ring class field $H_{9D} = k_D\big(j(\mathscr{O}_{9D})\big)$, where D is the discriminant of an order in an imaginary quadratic field. Recall from Amir-Khosravi's talk that the irreducible representations of $\mathrm{Gal}(H_{9D}/\mathbb{Q})$ consist of characters $\gamma\colon \mathrm{Gal}(H_{9D}/\mathbb{Q}) \to \mu_2$ and 2-dimensional representations of the form

$$\mathrm{Ind}_{\mathrm{Gal}(H_{9D}/k_D)}^{\mathrm{Gal}(H_{9D}/\mathbb{Q})} \kappa$$

for $\kappa \neq \kappa^{-1}$ a character of $\mathrm{Gal}(H_{9D}/k_D)$.

Hence

$$L(\rho^*_{H_{9D}}, s) = \prod_{\kappa} L(\rho^*_{\mathbb{Q}} \otimes \mathrm{Ind}_{G_{k_D}}^{G_{\mathbb{Q}}} \kappa, s) \prod_{\gamma} L(\rho^*_{\mathbb{Q}} \otimes \gamma, s).$$

The factor $L(\rho^*_{\mathbb{Q}} \otimes \gamma, s)$ is expressible as a Hecke L-series as on the previous slide. Furthermore one has (again up to finitely many bad Euler factors)

$$L(\rho^*_{\mathbb{Q}} \otimes \mathrm{Ind}_{G_{k_D}}^{G_{\mathbb{Q}}} \kappa, s) = L\big((\Psi \circ N_{Kk_D/K}) \cdot (\kappa \circ N_{Kk_D/k_D}), s\big) \quad \text{if} \quad K \neq k_D,$$

and

$$L(\rho^*_{\mathbb{Q}} \otimes \mathrm{Ind}_{G_{k_D}}^{G_{\mathbb{Q}}} \kappa, s) = L(\Psi \cdot \kappa, s) L(\Psi \cdot \kappa^{-1}, s) \quad \text{if} \quad K = k_D.$$

Analytic continuation and the functional equation for all of the L-series on previous slides now follow from that for Hecke L-series.

We'd now like to see that for F quadratic, $\delta(F)$ gives the sign in the functional equation for $L(\rho_F^*, s)$. In fact we shall give the sign for all of the Hecke L-series on previous slides.

Let Φ denote any of the Größencharaktere Ψ, $\Psi \cdot (\gamma \circ N_{K/\mathbb{Q}})$, $(\Psi \circ N_{Kk_D/K}) \cdot (\kappa \circ N_{Kk_D/k_D})$, $\Psi \cdot \kappa$, $\Psi \cdot \kappa^{-1}$ appearing previously, and let

$$\chi(\mathfrak{p}) := \frac{\Phi(\mathfrak{p})}{N\mathfrak{p}^{3/2}}$$

and

$$L(\chi, s) := \prod_{\mathfrak{p}} \frac{1}{1 - \chi(\mathfrak{p}) N\mathfrak{p}^{-s}} = L(\Phi, s + 3/2).$$

Let

$$
\begin{aligned}
2r &:= \text{the degree of the field } M = K \text{ or } Kk_D \text{ for which } \Phi \text{ is a Größencharakter},\\
\mathfrak{D} &:= \text{the different of } M/\mathbb{Q}, \text{ and}\\
\mathfrak{c} &:= \text{the conductor of } \Phi.
\end{aligned}
$$

Let

$$L^\sharp(\chi, s) := (N_{M/\mathbb{Q}}\mathfrak{c}\mathfrak{D})^{rs/2}(2\pi)^{r(s+3/2)}\Gamma(s+3/2)^r L(\chi, s).$$

Then $L^\sharp$ admits an analytic continuation and functional equation

$$w(\chi)L^\sharp(\chi, s) = L^\sharp(\overline{\chi}, 1-s).$$

In all cases under consideration the character χ satisfies $\chi(\overline{\mathfrak{p}}) = \overline{\chi}(\mathfrak{p})$
$\implies L^\sharp(\chi, s) = L^\sharp(\overline{\chi}, s)$
$\implies w(\chi) = \pm 1.$

Set $w(\Phi) := w(\chi)$.

Root number calculations

Theorem

1. $w(\Psi) = 1$ *and* $L(\Psi, 2) \neq 0$.
2. *Let* F *be a quadratic field and* $\gamma\colon G_{\mathbb{Q}} \to \mu_2$ *the Dirichlet character with kernel* G_F. *Then* $w\big(\Psi \cdot (\gamma \circ N_{F/\mathbb{Q}})\big) = \delta(F)$.

Let κ *be a ring class character for* k_D, *and assume in (3)–(5) that* $k_D \neq K$.

3. *If* κ *is unramified at* 3, *then* $w\big((\Psi \circ N_{Kk_D/K}) \cdot (\kappa \circ N_{Kk_D/k_D})\big) = \delta(L)$, *where* $L \subset Kk_D$ *is the maximal subfield unramified at* 3.
4. *If* κ *is tamely ramified at* 3, *then* $w\big((\Psi \circ N_{Kk_D/K}) \cdot (\kappa \circ N_{Kk_D/k_D})\big) = 1$.
5. *If* κ *is wildly ramified at* 3, *then* $w\big((\Psi \circ N_{Kk_D/K}) \cdot (\kappa \circ N_{Kk_D/k_D})\big) = -1$.

Now assume that $k_D = K$.

6. *If* κ *ramifies at* 3, *then* $\kappa \neq \kappa^{-1}$ *and* $w(\Psi \cdot \kappa) = -w(\Psi \cdot \kappa^{-1})$.
7. *If* κ *is unramified at* 3, *then* $w(\Psi \cdot \kappa) = \epsilon(b)\kappa^{-1}\big((\sqrt{-3})\big)$, *where* $b\mathbb{Z}[\zeta_3]$ *is the conductor of* κ, $b \in \mathbb{Z}$, *and* $\epsilon\colon G_{\mathbb{Q}} \to \mu_2$ *is the Dirichlet character associated to* $K/\mathbb{Q}$. *If* κ *has odd order, then* $\kappa\big((\sqrt{-3})\big) = 1$.

About the proof

Nonvanishing of $L(\Psi, 2)$ is verified by numerical computation.

The main ingredient in the rest of the proof is a result of Tate asserting that $w = \prod_\nu w_\nu$, where w_ν can be expressed explicitly in terms of local differents and conductors.

Root numbers and representation multiplicities

The preceding theorem combines with Amir-Khosravi's talk to yield the following. Given an irreducible representation ξ of $\mathrm{Gal}(H_{9D}/\mathbb{Q})$, let $w(\rho^*_{\mathbb{Q}} \otimes \xi)$ denote the root number of $L(\rho^*_{\mathbb{Q}} \otimes \xi, s)$. Recall the definition of $\mathscr{Z}(D)$: $CM_D = \{ x \in \dot{X}(\overline{\mathbb{Q}}) \mid \mathrm{End}(Y_x) = \mathscr{O}_D \}$, $Z(D)$ is the subgroup of $\bigoplus_{x \in CM_D} NS(\widetilde{W}_x)$ generated by complex multiplication cycles, and $\mathscr{Z}(D)$ is a certain quotient of $Z(D)$ defined in terms of the $SL_2(\mathbb{F}_3)$-action on $Z(D)$.

Corollary

Case I: $k_D \neq K$.

1. *Suppose* $9 \nmid D$. *If $w(\rho^*_{\mathbb{Q}} \otimes \xi) = 1$, then $\nu_\xi\left(\mathscr{Z}(D) \otimes \overline{\mathbb{Q}}_l\right) = 0$. If $w(\rho^*_{\mathbb{Q}} \otimes \xi) = -1$, then $\nu_\xi\left(\mathscr{Z}(D) \otimes \overline{\mathbb{Q}}_l\right) = 1$.*
2. *Suppose* $9 \mid D$. *If $w(\rho^*_{\mathbb{Q}} \otimes \xi) = 1$, then $\nu_\xi\left(\mathscr{Z}(D/9^j) \otimes \overline{\mathbb{Q}}_l\right) = 0$ for all $j \geq 0$. If $w(\rho^*_{\mathbb{Q}} \otimes \xi) = -1$, then $\nu_\xi\left(\mathscr{Z}(D/9^j) \otimes \overline{\mathbb{Q}}_l\right) = 1$ for exactly one $j \geq 0$.*

Case II: $k_D = K$.

3. *Suppose* $\dim(\xi) = 1$. *Then the same conclusions as in (1) apply.*
4. *Suppose* $\xi = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \kappa$ *with κ ramified at $(\sqrt{-3})$. Then $w(\rho^*_{\mathbb{Q}} \otimes \xi) = -1$, and $\nu_\xi\left(\mathscr{Z}(D/9^j) \otimes \overline{\mathbb{Q}}_l\right) = 1$ for exactly one $j \geq 0$.*
5. *Suppose* $\xi = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \kappa$ *with κ unramified at $(\sqrt{-3})$. Then $L(\rho^*_{\mathbb{Q}} \otimes \xi, s)$ is the product of two L-functions with the same root number, which is computed in part (7) of the previous theorem. Furthermore $\nu_\xi\left(\mathscr{Z}(D) \otimes \overline{\mathbb{Q}}_l\right) = 0$.*

Refined Beilinson–Bloch revisited

Now recall $CH^2(\widetilde{W}_F)_{\mathrm{CM}} = \ker[CH^2(\widetilde{W}_F)_{\mathrm{hom}} \to CH^2(\widetilde{W}_F \times_X \eta_F)]$. The previous corollary paired with Schoen's refined Beilinson–Bloch conjecture raises the question of for which D and ξ one has

$$\nu_\xi\left(CH^2(\widetilde{W}_{H_{9D}})_{\mathrm{CM}} \otimes \overline{\mathbb{Q}}_l\right) = \mathrm{ord}_{s=2} L(\rho^*_{\mathbb{Q}} \otimes \xi, s).$$

One expects that the following hypotheses will often hold:

- $\mathrm{ord}_{s=2} L(\rho^*_{\mathbb{Q}} \otimes \xi, s) = 0$ or 1 according as $w(\rho^*_{\mathbb{Q}} \otimes \xi) = 1$ or -1.
- The map $(\mathscr{Z}(D) \otimes \overline{\mathbb{Q}}_l)^\xi \to \left(CH^2(\widetilde{W}_{H_{9D}})_{\mathrm{CM}} \otimes \overline{\mathbb{Q}}_l\right)^\xi$ is injective.

In the cases covered in (1)–(4) in the previous corollary, these hypotheses imply that

$$\nu_\xi\left(CH^2(\widetilde{W}_{H_{9D}})_{\mathrm{CM}} \otimes \overline{\mathbb{Q}}_l\right) \geq \mathrm{ord}_{s=2} L(\rho^*_{\mathbb{Q}} \otimes \xi, s).$$

(At least) two difficulties arise when trying to prove the reverse inequality:

- When ξ is the character of an imaginary quadratic field unramified at 3, there are infinitely many D' such that $\left(\mathscr{Z}(D') \otimes \overline{\mathbb{Q}}_l\right)^\xi$ contributes to $\left(CH^2(\widetilde{W}_{H_{9D}}) \otimes \overline{\mathbb{Q}}_l\right)^\xi$.
- Any ξ can be inflated to a representation of $Gal(H_{9n^2D}/\mathbb{Q})$. If $3 \nmid n$, then $\nu_\xi\left(\mathscr{Z}(n^2D) \otimes \overline{\mathbb{Q}}_l\right) = \nu_\xi\left(\mathscr{Z}(D) \otimes \overline{\mathbb{Q}}_l\right)$. One would hope that the images of these vector spaces in $\left(CH^2(\widetilde{W}_{H_{9D}})_{\mathrm{CM}} \otimes \overline{\mathbb{Q}}_l\right)^\xi$ coincide, but it is not clear how to relate them.

Imaginary quadratic fields with $\mathrm{ord}_{s=2} = 1$

Let F be an imaginary quadratic field with $\mathrm{ord}_{s=2} L(\rho_F^*, s) = 1$. Then B–B predicts

$$\mathrm{rank}\, CH^2(\widetilde{W}_F)_{\mathrm{hom}} = 1.$$

Recall again from Amir-Khosravi's talk that if F is unramified at 3, then there are infinitely many D with $k_D \neq F$ such that $Z(D)$ contributes to $CH^2(\widetilde{W}_F)_{\mathrm{hom}}$. Thus B–B predicts relations in the Chow group between the cycles so obtained as D varies, but understanding this appears to be very difficult.

Rather than attempt to address this problem directly, Schoen uses the Abel–Jacobi map

$$CH^2(\widetilde{W}_F)_{\mathrm{hom}} \longrightarrow J^3(\widetilde{W}^{\mathrm{an}}).$$

to move everything to the intermediate Jacobian $J^3(\widetilde{W}^{\mathrm{an}})$.

In the particular case that $D = -4$, i.e. $F = \mathbb{Q}(\sqrt{-1})$, he obtains computer evidence that the Abel–Jacobi images of about 50 CM cycles are all integer multiples of the image of a single CM cycle in the intermediate Jacobian, as we shall now see.

Recall from Walls's talk that for any compact Kähler manifold X of $\mathbb{C}$-dimension n, the intermediate Jacobian $J^{2k-1}(X)$ can be expressed as the complex torus

$$J^{2k-1}(X) = F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C})^\vee \big/ \big(H_{2n-2k+1}(X,\mathbb{Z})/\text{torsion}\big).$$

We take $X = \widetilde{W}^{\text{an}}$, so that $n = 3$, and $k = 2$. Since $h^{3,0} = 1$ and $h^{2,1} = 0$, we have that

$$F^2H^3(\widetilde{W}^{\text{an}},\mathbb{C})^\vee = H^{3,0}(\widetilde{W}^{\text{an}})^\vee$$

is 1-dimensional. The map

$$H_3(\widetilde{W}^{\text{an}},\mathbb{Z})/\text{torsion} \longrightarrow H^{3,0}(\widetilde{W}^{\text{an}})^\vee$$

is given by

$$\xi \longmapsto \left[\varphi \mapsto \int_\xi \varphi\right].$$

Using the description in Sankaran's talk of the action of $SL_2(\mathbb{F}_3)$ on the global 3-form ω, one can see that $H_3(\widetilde{W}^{\text{an}},\mathbb{Z})/\text{torsion}$ is a free, rank one $\mathbb{Z}[\zeta_3]$-module. Fix a basis ξ, and take the basis element dual to ω in $H^{3,0}(\widetilde{W}^{\text{an}})^\vee$. Then we identify

$$J^3(\widetilde{W}^{\text{an}}) = \mathbb{C}/\Omega\mathbb{Z}[\zeta_3],$$

where $\Omega := \int_\xi \omega$.

Relative to our choices, the Abel–Jacobi map

$$\alpha\colon Z^2(\widetilde{W}^{an})_{\mathrm{hom}} \longrightarrow J^3(\widetilde{W}^{\mathrm{an}})$$

now admits the concrete description

$$\alpha(z) = \int_{C_z} \omega \quad \mathrm{mod}\ \Omega\mathbb{Z}[\zeta_3],$$

where C_z is a differentiable 3-chain bounding z.

We shall just give a sketch of Schoen's approach to numerical computation of the Abel–Jacobi map on CM cycles. There are three main ingredients.

The first is the uniformization

$$\Gamma(3)\backslash\mathfrak{H} \xrightarrow{\sim} \dot{X}^{\mathrm{an}} = \mathbb{P}^1_{\mathbb{C}} \smallsetminus \{\infty, 3\mu_3\}$$

discussed in Sankaran's talk. Given $\tau \in \mathfrak{H}$ mapping to a CM point and z_τ the resulting CM cycle, Schoen derives from a previous paper an explicit formula for $\alpha(z_\tau)$.

Computer evaluation of Schoen's formula requires little more than knowledge of the Fourier coefficients of $\eta^8(3\tau) = \sum_{n\geq 1} a_n q^n$, $q = \exp(2\pi i\tau)$. These are easily written down by virtue of the fact that $\eta^8(3\tau)$ is the cusp form associated to the Hecke character Ψ on $\mathbb{Q}(\zeta_3)$ we encountered earlier:

- If $3 \mid n$, then $a_n = 0$.
- If p is prime and $\equiv -1 \bmod 3$, then $a_p = 0$.
- If p is prime and $\equiv 1 \bmod 3$, then $a_p = \pi^3 + \overline{\pi}^3$, where $p = \pi\overline{\pi}$ in $\mathbb{Z}[\zeta_3]$ and $\pi \equiv 1 \bmod \sqrt{-3}$.

The second ingredient in the numerical computation of the Abel–Jacobi map is the choice of a good $\mathbb{Z}[\zeta_3]$-generator ξ for $H_3(\widetilde{W}^{\mathrm{an}}, \mathbb{Z})/\text{torsion}$. Very roughly, one would like to use the homology class of $\widetilde{W}(\mathbb{R})$, but $\widetilde{W}(\mathbb{R})$ is not orientable.

Schoen works around this by constructing an auxiliary scheme $\widehat{W}$ over $\mathbb{Q}$, obtained via a sequence of blow-ups along certain components of the singular fibers of $\widetilde{W} \to X$, with the property that $\widehat{W}(\mathbb{R})$ is orientable and $H^3(\widehat{W}^{\mathrm{an}}, \mathbb{Z}) \simeq H^3(\widetilde{W}^{\mathrm{an}}, \mathbb{Z})$.

Letting w denote the homology class of $\widehat{W}(\mathbb{R})$, it turns that $w/3$ may be taken as the desired generator.

The third and final ingredient is the numerical evaluation of the period

$$\Omega = \frac{1}{3}\int_{w}\omega.$$

Schoen effects this calculation via the complex uniformization from Sankaran's talk and various transformation properties of $\eta^8(\tau)$, which ultimately yield an infinite series converging rapidly to Ω.

Our goal is now to construct lots of CM cycles defined over $\mathbb{Q}(\sqrt{-1})$ and investigate the relations between their images in $J^3(\widetilde{W}^{an})$.

Let $d \not\equiv 0 \bmod 3$ be the discriminant of a real quadratic field. Then $D = -4d$ is the discriminant of the order $\mathscr{O}_D := \mathbb{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ in k_D, and $\mathbb{Q}(\sqrt{-1}) \subset H_D$. Let x denote the image of $\frac{D+\sqrt{D}}{2}$ in $\Gamma(3)\backslash\mathfrak{H}$.

If $D \equiv 1 \bmod 3$, then recall from Amir-Khosravi's talk that

- there are two Heegner points x_0 in the $SL_2(\mathbb{F}_3)$-orbit of x,
- the corresponding CM cycles are defined over H_D, and
- these cycles have the same image in $\mathscr{X}(D)$.

We define

$$z_D := \sum_{\gamma \in \mathrm{Gal}(H_D/Q(\sqrt{-1}))} z_{x_0}^{\gamma}.$$

If $D \equiv -1 \bmod 3$, then let x_0 be a special point in the $SL_2(\mathbb{F}_3)$-orbit of x. The cycle z_{x_0} is defined over H_{9D}, and its image in $\mathscr{X}(D)$ is again independent of the special point x_0. In his earlier study of the Galois action on CM cycles, Schoen shows that $\mathrm{Gal}(H_{9D}/H_D) \simeq \mathbb{Z}/4\mathbb{Z}$ acts trivially on z_{x_0}.

We define

$$z_D := \frac{1}{4} \sum_{\gamma \in \mathrm{Gal}(H_{9D}/\mathbb{Q}(\sqrt{-1}))} z_{x_0}^{\gamma}.$$

Computer evaluation of the Abel–Jacobi map on the cycles z_D so defined requires some further manipulations, the details of which we omit.

Schoen then computes the Abel–Jacobi image $\alpha(z_D)$ for all $d < 250$, and finds that $\alpha(z_D) = c_d \alpha(z_{\sqrt{-1}})$ for all d, where $c_d \in \mathbb{Z}$ is given in the following table.

Relations $\alpha(z_D) = c_d\alpha(z_{\sqrt{-1}})$ in the intermediate Jacobian

$D = -4d \equiv 1 \bmod 3$

d	D	c_d
5	-20	-3
8	-32	6
17	-68	-3
29	-116	9
41	-164	15
44	-176	-24
53	-212	3
56	-224	12
65	-260	-24
77	-308	-18
89	-356	15
92	-368	24
101	-404	-15
104	-416	-24

$D = -4d \equiv -1 \bmod 3$

d	D	c_d
13	-52	3
28	-112	12
37	-148	-12
40	-160	-6
61	-244	6
73	-292	-15
76	-304	-24
85	-340	27
88	-352	-12
97	-388	21
109	-436	-27
124	-496	12
133	-532	-6
136	-544	30

Relations $\alpha(z_D) = c_d\alpha(z_{\sqrt{-1}})$ in the intermediate Jacobian, cont'd

$D = -4d \equiv 1 \bmod 3$

d	D	c_d
113	−452	−27
137	−548	63
140	−560	24
149	−596	69
152	−608	−48
161	−644	−42
173	−692	9
185	−740	−6
188	−752	24
197	−788	−33
209	−836	−6
221	−884	−42
233	−932	57
236	−944	48
248	−992	36

$D = -4d \equiv -1 \bmod 3$

d	D	c_d
145	−580	3
157	−628	−6
172	−688	0
181	−724	9
184	−736	60
193	−772	24
205	−820	25
217	−868	24
220	−880	−48
229	−916	−27
232	−928	−54
241	−964	21

A modular form of weight 5/2

Given an odd integer $k \geq 5$ and $g \in S_{k/2}\big(\Gamma_0(4N)\big)$ an eigenform for all Hecke operators T_{p^2}, $(p, 4N) = 1$, with corresponding eigenvalues λ_p, the Shimura correspondence associates to g a cusp form $f \in S_{k-1}\big(\Gamma_0(2N)\big)$ satisfying $T_p f = \lambda_p f$.

In our situation we have $f(\tau) = \eta^8(3\tau) \in S_4\big(\Gamma_0(9)\big)$, and a corresponding $g \in S_{5/2}\big(\Gamma_0(36)\big)$ is described by Koblitz.

Let $g_1(\tau)$ denote the unique weight 2 normalized newform on $\Gamma_0(36)$. Write

$$\theta(\tau) := \sum_{n \in \mathbb{Z}} q^{n^2}, \quad q = \exp(2\pi i \tau).$$

Then

$$g := (g_1\theta)|_{1+(1/4)T_4}.$$

Writing $g(\tau) = \sum_{d \geq 1} b_d q^d$, Schoen's computer calculations indicate that for all $d < 250$ the discriminant of a real quadratic field,

$$b_d = -\epsilon(d) 2 c_d,$$

where as before ϵ is the Dirichlet character of the quadratic field $\mathbb{Q}(\zeta_3)$!!! The idea that the integers c_d might arise from Fourier coefficients of a modular form of weight $\frac{5}{2}$ is due to Gross.