Witt Vectors and Dieudonné Rings

Luiz Kazuo Takei
(*caveman*)

McGill University

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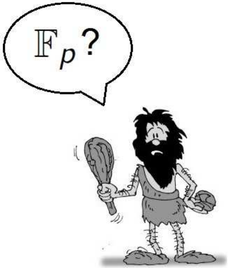
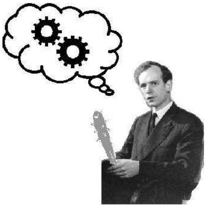
Motivation



- You know, $\mathbb{Z}_p$...

- What are you saying, stranger? Do you mean $\mathbb{F}_p$?

Motivation II



Elements of $\mathbb{Z}_p$: $\displaystyle\sum_{i=0}^{\infty} a_i p^i = a_0 + a_1 p + a_2 p^2 + \cdots + a_n p^n + \cdots$
where $a_i \in \{0, \ldots, p-1\}$ " $=$ " $\mathbb{F}_p$.

So

$$\mathbb{Z}_p = \prod_{i=0}^{\infty} \mathbb{F}_p.$$

This is true as sets, but not as rings!

How can we understand the ring structure of $\mathbb{Z}_p$ viewed in $\displaystyle\prod_{i=0}^{\infty} \mathbb{F}_p$?

The Definition (as a set)

Throughout this talk, p is fixed.

Let A be a ring and define

$$W(A) := W_p(A) := \prod_{i=0}^{\infty} A$$

If $x \in W(A)$, we will denote its coordinates by x_n, i.e.,

$$x = (x_n)_{n=0}^{\infty} \in W(A).$$

The Witt polynomials

Definition

For each $n \in \mathbb{Z}_{\geq 0}$, define the n-**th Witt polynomial** to be

$$w_n := \sum_{i=0}^{n} p^i X_i^{p^{n-i}} \in \mathbb{Z}[X_0, \dots, X_n].$$

Example

$$\begin{aligned} w_0 &= X_0 \\ w_1 &= X_0^p + pX_1 \\ w_2 &= X_0^{p^2} + pX_1^p + p^2 X_2 \\ w_3 &= X_0^{p^3} + pX_1^{p^2} + p^2 X_2^p + p^3 X_3 \end{aligned}$$

Ghost components

The w_n's define maps

$$w_n : W(A) \longrightarrow A, \text{ and}$$

$$w_* := (w_n)_{n=0}^{\infty} : W(A) \to A^{\mathbb{Z}_{\geq 0}}.$$

Terminology

Let $x = (x_n)_{n=0}^{\infty} \in W(A)$.

- **Witt components**: x_n
- **ghost components**: $w_n(x)$

Remark

If p is not a zero-divisor in A, then w_* is injective.

The definition (as a ring)

Theorem (Main Theorem)

There is a unique covariant functor $W = W_p : \mathrm{Alg}_{\mathbb{Z}} \longrightarrow \mathrm{Alg}_{\mathbb{Z}}$ *such that for any ring* A,

(i) $W_p(A) = \prod_{i=0}^{\infty} A$ *and, for any ring homomorphism* $f : A \to B$,

$$W(f)((a_n)_{n=0}^{\infty}) = (f(a_n))_{n=0}^{\infty}.$$

(ii) *The maps* $w_n : W(A) \to A$ *are homomorphism of rings for all* $n \in \mathbb{Z}_{\geq 0}$.

Moreover, the zero element of $W(A)$ *is* $(0, 0, \dots)$ *and the unit element is* $(1, 0, 0, \dots)$.

Definition

The ring $W(A)$ is the **ring of Witt Vectors** with coefficients in A.

Theorem

Let K be a perfect ring of characteristic p, and let R be the strict p-ring with residue ring K and Teichmüller representatives $\tau : K \to R$. Then we have a ring isomorphism

$$\begin{array}{rccc} f: & W(K) & \xrightarrow{\sim} & R \\ & (x_n) & \longmapsto & \sum_{n=0}^{\infty} \tau(x_n^{1/p^n}) p^n \end{array}$$

Corollary (Cavemen reconstruct $\mathbb{Z}_p$ from $\mathbb{F}_p$!!!)

Let $\tau : \mathbb{F}_p \to \mathbb{Z}_p$ be the Teichmüller character. Then we have a ring isomorphism

$$\begin{array}{rccc} f: & W(\mathbb{F}_P) & \xrightarrow{\sim} & \mathbb{Z}_p \\ & (x_n) & \longmapsto & \sum_{n=0}^{\infty} \tau(x_n) p^n \end{array}$$

Universal polynomials for $+$ and $\cdot$

Let $R = \mathbb{Z}[X_0, X_1, \ldots, Y_0, Y_1, \ldots]$.
Take $X = (X_n)$ and $Y = (Y_n) \in W(R)$.
Let $(S_n) = S = X + Y$ be the sum in $W(R)$. Then

$$w_n(S) = w_n(X + Y) = w_n(X) + w_n(Y).$$

We can EXPLICITLY solve those equations for S_r and obtain, for each $r \in \mathbb{Z}_{\geq 0}$, $S_r \in \mathbb{Z}[X_0, \ldots, X_r, Y_0, \ldots, Y_r]$.

Now, if A is any ring and $(x_n), (y_n) \in W(A)$, we can compute the sum $x + y \in W(A)$ just by specializing the polynomials S_r, i.e,

$$(x_n) + (y_n) = (S_n(x, y)) .$$

So the polynomials S_n are universal polynomials giving us addition in $W(A)$ for any ring A! Same thing for multiplication!

Reduction to the Universal Case

Idea: We want to prove something about $W(A)$ for any ring A.

Strategy:

- Prove it for $W(R)$, where $R = \mathbb{Z}[X_0, X_1, \ldots, Y_0, Y_1, \ldots]$. This is usually easier because p is not a zero-divisor in R (as opposed to $\mathbb{F}_p$) and, therefore, $w_* : W(R) \to R^{\mathbb{Z}_{\geq 0}}$.
- Now, specialize to A.

This is called **reduction to the universal case**.

The Shift-Map (fancy name: Verschiebung)

Define the **Verschiebung** map by

$$\begin{array}{rccc} V: & W(A) & \longrightarrow & W(A) \\ & (x_0, x_1, x_2, \ldots) & \longmapsto & (0, x_0, x_1, \ldots). \end{array}$$

Proposition

The map V respects addition, i.e, $V(x+y) = V(x) + V(y)$.

Proof (*sketch*).

Reduce it to the universal case, that is: prove it for the ring $R = \mathbb{Z}[X_0, X_1, \ldots, Y_0, Y_1, \ldots]$. To do this, first prove that

$$w_n(V(X+Y)) = w_n(V(X) + V(Y)).$$

Then, use that $w_* = (w_n) : W(R) \to R^{\mathbb{Z}_{\geq 0}}$ is injective (this follows because p is not a zero-divisor in R). $\square$

The Frobenius map

From now on, the ring $A = k$ will be a perfect field of characteristic p.

We have the Frobenius automorphism:

$$\begin{array}{rccc} \sigma : & k & \xrightarrow{\sim} & k \\ & \alpha & \longmapsto & \alpha^p \end{array}$$

which, by the main theorem, induces an automorphism (also called **Frobenius**) for $W(k)$

$$\begin{array}{rccc} F = W(\sigma) : & W(k) & \xrightarrow{\sim} & W(k) \\ & (x_n) & \longmapsto & (x_n^p). \end{array}$$

Properties of F and V

Proposition

The maps $F, V : W(k) \to W(k)$ *satisfy the following properties:*

- $FV(x) = VF(x) = p \cdot x$
- $V(F(x)y) = xV(y)$ *(V is F^{-1}-linear).*

Topology on $W(k)$

Let $n \in \mathbb{Z}_{\geq 0}$. Consider

$$W_{(n)}(k) := \prod_{i=0}^{n} k \leq W(k)$$

and

$$\begin{array}{rccc} \pi_n : & W(k) & \longrightarrow & W_{(n)}(k) \\ & (x_r)_{r=0}^{\infty} & \longmapsto & (x_r)_{r=0}^{n}. \end{array}$$

It is clear that

$$W(k) = \varprojlim W_{(n)}(k).$$

Definition

Let k be equipped with the discrete topology. The **standard topology** on $W(k)$ is the inverse limit topology on $\varprojlim W_{(n)}(k)$, which is the same as the product topology on $W(k) = \prod_{i=0}^{\infty} k$.

More about topology

Proposition

(1) *$W(k)$ is a topological ring.*

(2) *$W(k)$ is complete and Hausdorff.*

(3) *The sets $\ker(\pi_n)$ for $n \geq 0$ form a neighborhood of the identity in $W(k)$.*

Remark

Notice that $\ker(\pi_n) = \{(0, \dots, 0, x_{n+1}, x_{n+2}, \dots)\}$. So the standard topology is the p-adic topology!

Proposition

All the maps defined so far (including $W(f)$, for any $f : k \to k'$) are continuous.

Dieudonné Rings

Idea: have $W(k)$ and the maps F and V all together in a single ring.

The following ring is sometimes called **Dieudonné ring**:

$$W(k)[F, V]$$

This is the non-commutative polynomial ring in the variables F and V satisfying:

$$F \cdot V = V \cdot F = p \quad , \quad F \cdot a = {}^{F}a \cdot F \quad , \quad V \cdot {}^{F}a = a \cdot V$$

Sometimes people call **Dieudonné ring** this other ring:

$$W(k)\,((F)),$$

the non-commutative ring of Laurent series in the variable F satisfying

$$F \cdot a = {}^{F}a \cdot F.$$

Remark

Note that V can be viewed inside this ring, namely:

$$V \cdot F = p \implies V = p\frac{1}{F}.$$

Some useful functions

Let $g = \sum_{i=m}^{\infty} a_i F^i \in W(k)\,((F))$ with $a_m \neq 0$.

Define the functions

$$\begin{array}{rccl} d: & W(k)\,((F)) \setminus \{0\} & \longrightarrow & \mathbb{Z} \\ & g & \longmapsto & m \end{array}$$

and

$$\begin{array}{rccl} s: & W(k)\,((F)) \setminus \{0\} & \longrightarrow & \mathbb{Z} \\ & g & \longmapsto & \nu(a_m) \end{array}$$

where ν is the valuation of $W(k)$.

Quasi-Euclidean property

Proposition

The Dieudonné ring $W(k)\,((F))$ is "Euclidean" with respect to the function s. In particular, all left and right ideals are principal.

Proof.

Let $g, h \in W(k)\,((F))$ with $h \neq 0$. We want: q, r such that

$$g = h \cdot q + r \ , \ \ s(r) < s(h) \ \text{ or } \ r = 0.$$

If $s(g) \geq s(h)$, then there is q_1 such that

$$d(g - h \cdot q_1) \geq m + 1.$$

If $s(g - h \cdot q_1) \geq s(g)$, stop. Otherwise, continue.
Note that $d(q_{i+1}) > d(q_i)$. So if this process doesn't stop, we obtain $g = h \cdot \left(\sum_i q_i \right)$. □

References

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THANK YOU!