# Kudla: The supersingular locus of unitary Shimura varieties.

$k = \mathbb{Q}(\sqrt{\Delta})$, $\Delta < 0$, $\mathcal{O}_K$. $\sigma$ = Gal. auto

$(A, \iota, \lambda)$ $A$ = ab. var. dim$^n$ $n$, $\iota: \mathcal{O}_K \hookrightarrow \mathrm{End}(A)$, $\lambda: A \to A^\vee$ pr. pol$^n$ s.t.: $\underset{\mathcal{O}_K}{\circlearrowleft}\ \underset{\iota(a)^\vee}{\circlearrowleft}$

for $\iota^\vee(a) := \iota(\sigma(a))^\vee$, $\lambda$ is $\mathcal{O}_K$-linear.

Signature: $\mathrm{char}(\iota(a) \mid \mathrm{Lie}(A))(T) = (T-a)^{n-1}(T-a^\sigma) \rightsquigarrow$ signature $(n-1, 1)$.

($A^\vee$ has same signature)

even pr. pol$^n$ doesn't determines $V$ uniquely leading to many components.

/$\mathbb{C}$ Represented by $\mathcal{M}(\mathbb{C}) \cong \underbrace{\coprod \Gamma \backslash \mathbb{D}}_{\text{a union of Shimura varieties}}$, where $\mathbb{D}$ = cplx unit ball in $\mathbb{C}^{n-1}$

and $\Gamma \subset U(V)$ arith. group, where $V$ is a hermitian v.sp. /$k$ of sg. $(n-1, 1)$.

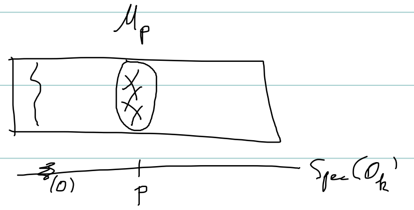
Generalizes to schemes:

$S$/ Sch/Spec$(\mathcal{O}_k)$, $A/S$, ...., signature condn:

$\mathrm{Char}(\iota(a) \mid \mathrm{Lie}(A))(T) = \iota_*((T-a)^{n-1}(T-a^\sigma))$, $\iota: \mathcal{O}_k \to \mathcal{O}_S$.

and some add'l cond$^{ns}$ at primes ramified in $\mathcal{O}_k$.

$\exists\ \mathcal{M}$ = moduli stack/Spec$(\mathcal{O}_k)$, smooth over Spec $\mathcal{O}_k[\Delta^{-1}]$.




cplx pts have a uniformization

$$\mathcal{M}(\mathbb{C}) = \coprod G(\mathbb{Q}) \backslash \mathbb{D} \times G(\mathbb{A}_f)/K$$

$\{G = U(V)\}$

Hope: $\mathcal{M}_p$ has p-adic uniformization. Not quite....

$\mathcal{M} \supset \mathcal{M}_p^{ss}$ supersingular points.

$\hat{\mathcal{M}}_p^{ss}$ = formal completion of $\mathcal{M}$ along $\mathcal{M}_p^{ss}$. has relative dim$^{\underline{n}}$ $n-1 = \dim \mathcal{M}(\mathbb{C})$.

Let $\mathbb{F} = \overline{\mathbb{F}}_p$, $W = W(\mathbb{F})$. $\hat{\mathcal{M}}_p^{ss}$ is a formal scheme over Spf $W$.

There is a formal scheme $\mathcal{N}$ (RZ-space) such that

$$\hat{\mathcal{M}}_p^{ss} \cong \coprod \underbrace{G'(\mathbb{Q}) \backslash \mathcal{N} \times G(\mathbb{A}_f^p)}_{\Gamma' \backslash \mathcal{N}} / K^p.$$

$\longleftarrow$ analogue of $\Gamma \backslash D$.

__The RZ space $\mathcal{N}$__

We assume $p$ inert in $k$, $p \neq 2$. Reset notation:

$k/\mathbb{Q}_p$ unram. quad. ext, $\delta \in k^\times$, $\delta^\sigma = -\delta$, $\delta$ = unit.

$\varphi_0 : k \to W$, $\varphi_1 = \varphi_0 \circ \sigma$.

Let $S \in \mathrm{Nilp}$ = sch. / Spf$(W)$, $p$. loc. nilpotent.

$(X, \iota_X, \lambda_X, \rho_X)$ : $X$ $p$-div. gp over $S$ of dim$^{\underline{n}}$ $n$ and ht $2n$.

$\iota_X : O_k \hookrightarrow \mathrm{End}(X)$, $\lambda : X \to X^\vee$ $p$-princ. pol$^{\underline{n}}$, $O_k$-linear.

Signature: $\mathrm{char}(\iota(a) \mid \mathrm{Lie}(X)) = \iota_*\left((T-a)^{n-1}(T-a^\sigma)\right)$.

framing: Choose a "framing object" $(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}})/\mathbb{F}$

__Concretely__: $E$ = s.s. ell. curve$/\overline{\mathbb{F}}_p$, $\mathbb{X}_0 = E(p)$

End$(\mathbb{X}_0) = O_{\mathbb{B}}$ , $\mathbb{B}/\mathbb{Q}_p$ division.

Choose $O_k \hookrightarrow O_{\mathbb{B}}$ s.t. action on Lie$(\mathbb{X}_0)$ is by $\varphi_0$.

$\exists$ can. pr. pol'r.

$$\mathbb{X} = \mathbb{X}_0^{n-1} \times \overline{\mathbb{X}}_0 , \quad \overline{\mathbb{X}}_0 = \mathbb{X}_0 \text{ by } i_{\overline{\mathbb{X}}_0} = i_{\mathbb{X}_0} \circ r.$$

Let $\bar{S} = S \times_W \mathbb{F}$, $\bar{X} = X \times_S \bar{S}$ , $\mathbb{X}_{\bar{S}} = \mathbb{X} \times_{\mathbb{F}} \bar{S}$

$\rho_X : \bar{X} \longrightarrow \mathbb{X}_{\bar{S}}$ is a $O_k$-linear q. isogeny of ht. $= 0$.

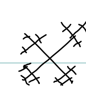
s.t. $\rho_X^*(\lambda_{\mathbb{X}}) \sim \lambda_{\bar{S}}$.

$\mathcal{N}(S)$ = set of iso. classes of such collections.

<u>Thm</u> (RZ): pro.-rep. by a formal scheme $\mathcal{N}/W$,

$\mathcal{N}$ sep. loc. form. of finite type of rel. dim$^{\underline{n}}$ $n-1$.

<u>Recall</u>: Drinfeld space $\hat{\Omega}$, $(\hat{\Omega})_{red} = \times$ $\mathbb{P}^1$'s building for $PGL_2(\mathbb{Q}_p)$.



<u>Description of $\mathcal{N}_{red}$</u> (Vollaard, Vollaard-Wedhorn)

Set theoretically: $\mathcal{N}_{red}(\mathbb{F}) = \mathcal{N}(\mathbb{F})$.

$(X, i_X, \lambda_X, \rho_X)/\mathbb{F}$ , $\mathbb{F} = \overline{\mathbb{F}}_p$. $\rightsquigarrow$ Dieudonné module $M(X)$ (covariant)

$M(X)$ is a free $W$-module of rank $2n$.

$M(\mathbb{X}_{\mathbb{Q}}) = M(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p =: N$ – the isocrystal assoc. to $\mathbb{X}$.

$\rho_X$ gives $M(X) \longrightarrow N$. (image is a lattice in $N$)

$N$ comes with $F, V$, $\langle , \rangle$ alter. $W$-bilinear form, $\langle Fx, y\rangle = \langle x, Vy\rangle^{\sigma}$.

$O_k$-action $\iff$ $N = \underset{\substack{\varphi_0 \\ \text{e.sp.}}}{N_0} \oplus \underset{\substack{\varphi_1 \\ \text{e.sp.}}}{N_1}$, $F, V$ have degree 1.

$N_0, N_1$ are isotropic w.r.t. pol$^{\underline{n}}$.

We want to study $M \subset N$, $W$ module (rk $2n$ ✓) stable under $F, V$ etc, ...

$M = M_0 \oplus M_1$

$\mathrm{Lie}(X) = M/VM = \underset{1}{M_0/VM_1} \oplus \underset{n-1}{M_1/VM_0}$ (should be $n-1, 1$, but we change to conform w. the references)

and $\overset{\perp}{M} = M$ ($\perp$ ← dual w.r.t. pol$^{\underline{n}}$), that is $\langle M^{\perp}, M\rangle = W$ ($M^{\perp\perp} = \tau(M)$, $\tau = FV^{-1} = pV^{-2}$).

On $N_0$ define $\{x, y\} = p^{-1}\delta^{-1}\langle x, Fy\rangle$, non-deg. form on $N_0$.

$\{\tau(x), \tau(y)\} = \{x, y\}^{\sigma^2}$, $\{y, x\} = \{x, \tau^{-1}y\}^{\sigma}$.

← that's why we put $\delta$ in.

For $A \subset N_0$, $W$-lattice, let $A^{\vee} = \{x \in N_0 \mid \{x, A\} \subseteq W\}$.

The point: if $M = M_0 \oplus M_1$ let $A = M_0$ then signature cond^n amounts to

$A \underset{1}{\supset} A^\vee \underset{n-1}{\supset} pA.$

To go backwards, let $M_1 = F^{-1} A^\vee$. So we pass to $A \subset N_0$ $A \underset{1}{\supset} A^\vee \underset{n-1}{\supset} pA.$

Now let $C = N_0^{\tau=1}$. This is a $\mathbb{Q}_{p^2}$-v.sp. of dim^n $n$.

$\{x, y\}$ is a hermitian form on $C$.

$N_0 = W_{\mathbb{Q}} \underset{\mathbb{Q}_{p^2}}{\otimes} C$. In $C$, $\exists \Lambda$ $\Lambda \underset{1}{\supset} \Lambda^\vee \underset{n-1}{\supset} p\Lambda$ $\iff$ $A = W \underset{\mathbb{Z}_{p^2}}{\otimes} \Lambda$ s.t. $\tau(A) = A$.

We call such points superspecial points. (Those do come from sup.sp'l AV)

<u>Zink's lemma:</u> Given any $A$, $\exists \overset{\text{minimal}}{d}$ s.t. $\underbrace{A + \tau(A) + \dots + \tau^d(A)}_{\Lambda(A)}$ is $\tau$ invariant.

And $0 \le d \le [\frac{n-1}{2}]$. $\Lambda(A)$ comes from $C$. But

$\Lambda \underset{t=2d+1}{\supset} \Lambda^\vee \supset p\Lambda$

$1 \le t \le n$ is called the <u>type</u> of $\Lambda$. $\Lambda$ is called a <u>vertex of type $t$</u>.

such $\Lambda$ <u>are</u> vertices in the bldg of $SU(C)$.

If $\Lambda_1 \subset \Lambda_2$ it has smaller type

$\Lambda_2 \supset \Lambda_1 \supset \Lambda_1^\vee \supset \Lambda_2^\vee \supset p\Lambda_2 \supset p\Lambda_1$

Given a vertex of type $t$ define

$$\mathcal{V}^\circ(\Lambda)(\mathbb{F}) = \{ x \in \mathcal{N}(\mathbb{F}) \mid \Lambda(A_x) = \Lambda \}$$
$$\mathcal{V}(\Lambda)(\mathbb{F}) = \{ \dots \Lambda(A_x) \subseteq \Lambda \}$$

$$\mathcal{N}(\mathbb{F}) = \coprod_{\Lambda} \mathcal{V}^\circ(\Lambda)(\mathbb{F}) \quad \text{(infinite union)}$$

Vollaard-Wedhorn: for each $\Lambda$ (of type $t$), $\mathcal{N}_\Lambda \subset \mathcal{N}_{red}$ a closed subscheme such that $\mathcal{N}_\Lambda(\mathbb{F}) \subset \mathcal{N}(\mathbb{F})$, $\dim \mathcal{N}_\Lambda = d = \frac{1}{2}(t-1)$

$$\mathcal{N}_\Lambda(\mathbb{F}) = \mathcal{V}_\Lambda(\mathbb{F})$$

$\mathcal{N}_\Lambda$ are smooth and irreducible <u>and</u>

$$\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda'} = \begin{cases} \emptyset & \text{if } \Lambda \cap \Lambda' \text{ is not a vertex} \\ \mathcal{N}_{\Lambda \cap \Lambda'} & \text{else} \end{cases}$$

$$\mathcal{N}_{red} = \bigcup_{\substack{\Lambda \\ t(\Lambda) = \max}} \mathcal{N}_\Lambda, \quad \text{in particular } \dim \mathcal{N}_\Lambda = \left[\frac{n-1}{2}\right] = \dim \mathcal{M}_{pp}^{ss}.$$

$\mathcal{N}_\Lambda$ are "known varieties" = Deligne-Lusztig varieties for $\Lambda \underset{t}{\supset} \Lambda^\vee \supset p\Lambda$.
$\Lambda/\Lambda^\vee$ is an $\mathbb{F}_{p^2}$ v.sp. with hermitian form $\to U(\Lambda/\Lambda^\vee) \rightsquigarrow$ D. Lusz. varieties.

e.g. $x^{p+1} + y^{p+1} + z^{p+1} = 0$ (b/c $xx^\sigma + yy^\sigma + zz^\sigma = 0$).