STEVE KUDLA

Stratification of Unitary Shimura Varieties.

↳ application of p-adic groups for studying structure of moduli spaces

$k = \mathbb{Q}(\sqrt{\Delta})$, $\Delta < 0$, O_k : ring of integers (σ: Galois autom $\neq$ id)

INTRODUCTION

Consider: (A, ι, λ) ; · A abelian variety of dim n
· $\iota : O_k \longrightarrow \mathrm{End}(A)$
· $\lambda : A \xrightarrow{\sim} A^\vee$ principal polarization

+ actions

$O_k \curvearrowright A$, $O_k \curvearrowright A^\vee$: $a \in O_k$ $\iota(a^\sigma)^\vee = \iota^\vee(a)$ s.t.

· λ is O_k-linear (equiv: Rosati inv. is just complex conj)
· 'Signature condition':

$$\mathrm{char}(\,\cdot \mid \mathrm{Lie}(A))(T) = (T-a)^{n-1}(T-a^\sigma)$$

Note: A: abelian ~~variety~~ var / $\mathbb{C}$ $\longrightarrow$ complex torus

↳ the moduli space is represented by

$$\mathcal{M}(\mathbb{C}) \simeq \coprod \Gamma/D$$

→ choice for any ramified prime.

D: complex unit ball in $\mathbb{C}^{n-1}$
$\Gamma < U(V)$ arithm grp
V : herm v.s / k of signature $(n-1, n)$

Finer arithmetic structure ⟶ moduli problem over scheme.

↳ consider (A, ι, λ) : $A \longrightarrow S$ $(S \to \mathrm{Spec}(O_k))$ with properties above and signature condition:

$$\mathrm{char}(\;\mid \mathrm{Lie}(A))(T) = i_*((T-a)^{n-1}(T-a^\sigma)) \quad i : O_k \to O_S$$

↓

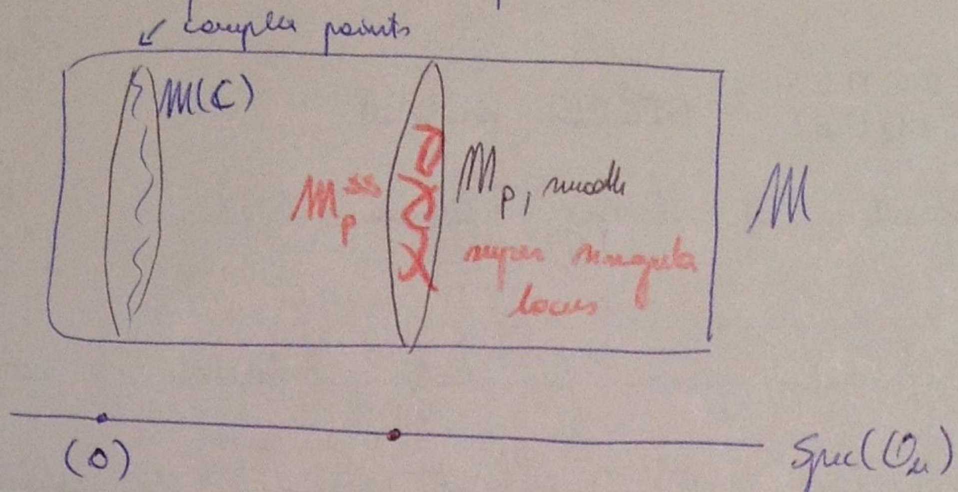
problems at certain primes (ramified)

and hence in general we need additional conditions

→ we will specialise on p inert and $p \neq 2$:

PICTURE

What happens at a prime p inert, $\neq 2$



↳ $\mathcal{M}(\mathbb{C})$ have a uniformisation, i.e.:

(union of shim. varieties)

$$\coprod G(\mathbb{Q}) \backslash D \times G(\mathbb{A}_f) / K \qquad G = U(V)$$

↳ p-adic uniformisation:

NO GROUP THEORETIC DESCRIPTION of FIBER at p.

$\mathcal{M}_p \supset \mathcal{M}_p^{ss}$ (n points - i.e. products of n elliptic curves)

↳ may take the formal completion of $\mathcal{M}$ along $\mathcal{M}_p^{ss}$:

$$\widehat{\mathcal{M}}_p^{ss}$$

↳ as a topological space ✓ but as rings they are big:

$\widehat{\mathcal{M}}_p^{ss}$ has rel. dimension $n-1$

Take now $W = W(\overline{\mathbb{F}_p})$

↳ I want to look at $\widehat{\mathcal{M}}_1^{ss}$ as a formal scheme $/W$.

p-adic uniformisation (Rapoport - Zink)

There is a formal scheme $\mathcal{N}$ (R-Z space) s.t.

$$\widehat{\mathcal{M}}_p^{ss} \simeq \coprod G'(\mathbb{Q}) \backslash \mathcal{N} \times G(\mathbb{A}_f^p) / K^p$$

(analogue to the Hall quotients

The RZ space $\mathcal{M}$

$k/\mathbb{Q}_p$: unramified quadratic ext, $\delta \in k^\times$, $\delta^\sigma = -\delta$, δ : unit

fix embeddings

$$\begin{cases} \varphi_0 : k \longrightarrow W \\ \varphi_1 = \varphi_0 \circ \sigma \end{cases}$$ (lifting of residue field's embeddings into $\overline{\mathbb{F}_p}$, they are two)

We are going to make a moduli problem for p-divisible groups:

S / $\mathrm{Nilp} = {}^{\mathrm{Sch}}/_{\mathrm{Spf}(W)}$, p: loc. nilpotent

$(X, \iota_X, \lambda_X, \rho_X)$:

- X : p-div gp / S of dim n and ht $2n$
- $\iota : \mathcal{O}_k \longrightarrow \mathrm{End}(X)$
- $\lambda : X \longrightarrow X^\vee$ p-princ polarisation $\mathcal{O}_k$-linear

signature condition

$$\mathrm{char}(\iota(a) \mid \mathrm{Lie}(X)) = i_*((T-a)^{n-1}(T-\bar{a}))$$

- ρ_X : 'framing' : must choose a framing object

$(\mathbb{X}, \iota_{\mathbb{X}}, \lambda_{\mathbb{X}}) / \mathbb{F}$

Concretely:

E: an elliptic curve $/ \overline{\mathbb{F}_p}$, $\mathbb{X}_0 = E(p^\infty)$

$\mathrm{End}(\mathbb{X}_0) = \mathcal{O}_{\mathbb{B}}$, $\mathbb{B}/_{\mathbb{Q}_p}$ division

I can choose an embedding

$$\mathcal{O}_k \hookrightarrow \mathcal{O}_{\mathbb{B}}$$

s.t. the action of $\mathrm{Lie}(\mathbb{X}_0)$ is by φ_0

$\exists$ princ. polarisation ... $\iota_{\mathbb{X}_0} = \iota_{\mathbb{X}_0} \circ \sigma$

$\rightarrow$ so as a framing object I choose

$$\mathbb{X} = \underbrace{\mathbb{X}_0 \times \ldots \times \mathbb{X}_0}_{n-1} \times \mathbb{X}$$

S_s: $\bar{S} = S \times_W \overline{\mathbb{F}_p}$ lift

$$\begin{array}{ccc} \rho_X : \bar{X} & \longrightarrow & \mathbb{X}_S \\ \downarrow & & \downarrow \\ \bar{S} & = & \bar{S} \end{array}$$

quasi-isogeny + $\mathcal{O}_k$ linear

+

additional condition:

of ht 0.,

+ $\rho_X^*(\lambda_X) \sim \lambda_{\bar{S}}$.

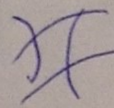
↳ this controls the way the isogeny class of X varies over the special fibre.

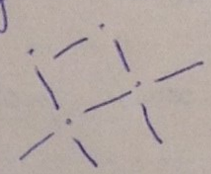
<u>Coll</u> : $\mathcal{M}(S)$: the set of isom classes of such collections $(X, \iota_X, \lambda_X, \rho_X)$.

<u>Thm</u> (R-Z): It is pro-representable by a formal scheme $\mathcal{M}/W$

Structure of reduced locus of this space.

Recall: Drinfeld's space $\hat{\Omega}$

$(\hat{\Omega})_{red}$  copies of $\mathbb{P}^1$'s (building for $PGL_2(\mathbb{Q}_p)$)

$p = 3$: 

↳ <u>Analogue in the R-Z case</u> : Description of $\mathcal{M}_{red}$ (Vollaard, Vollaard-Wedhorn)

SET-THEORETIC DESCRIPTION

$$\mathcal{M}_{red}(\overline{\mathbb{F}}_p) = \mathcal{M}(\overline{\mathbb{F}}_p)$$

$(X, \iota_X, \lambda_X, \rho_X)/\mathbb{F}$ ⟿ Covariant Dieud. module

$\mathbb{M}(X)$: free W-mod

similarly, take the dieud. mod of the framing object:

$\mathbb{M}(\mathbb{X})$ in fact since it's related through a quasi-isogeny, I consider the ISO-CRYSTAL:

$$\mathbb{M}(\mathbb{X})_{\mathbb{Q}} = \mathbb{M}(\mathbb{X}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = N$$

the map $\rho_X : X \longrightarrow \mathbb{X}$ quasi-isogeny, gives us

$\mathbb{M}(X) \longrightarrow N$ (no map to the Dieud. module)

↳ we get a collection of Dieud. lattices

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DE RECHERCHES
MATHÉMATIQUES

→ Analyse the iso-crystal: $M = \mathbb{M}(X)_{\mathbb{Q}}$: F, V,
the polarisation gives us $\langle\, ,\, \rangle$ alt. bilinear form $/W$ s.t.
$\langle Fx, y\rangle = \langle x, Vy\rangle^{\sigma}$

the $\mathcal{O}_K$-action is equiv. to $M = M_0 \oplus M_1$ φ_0, φ_1
F, V have deg 1 and
M_0, M_1 is isotropic with respect to $\langle\, ,\, \rangle$.

<u>In order to study $\mathcal{M}(\overline{\mathbb{F}}_p)$ it's enough to translate the data into 'Dieudonné data'.</u>

$M \subset N$ is a Dieud. lattice (i.e. W-mod, stable under F, V)

$M = M_0 \oplus M_1$ ($\mathcal{O}_K$-action) $M^{\perp} = M$

<u>Signature condition:</u> $Lie(X) = M/VM = \underbrace{M_0/VM_1}_{\text{dim } n-1} \oplus \underbrace{M_1/VM_0}_{\text{dim } 1}$ DECOMPOSITION

'inverted' signature condition $\}$ twisted by σ

→ Conversely starting from such a set of data, I obtain a p-div group '+ structure' as above.

<u>Caution</u> with duals : $\sigma^2 \neq id$ on N.

↳ def. of dual: $\langle M^{\perp}, M\rangle = W$, $(M^{\perp})^{\perp} = \tau(M)$,
where: $\tau = FV^{-1} = pV^{-2}$ (σ^2-linear)

We are left with a linear algebra problem: SECOND REDUCTION:
↳ we try to translate everything in terms of N_0.

On N_0 define $\{x, y\} = p^{-1}\delta^{-1}\langle x, Fy\rangle$
↓
N_0 isotropic with resp to $\langle\, ,\, \rangle$

this is a non-deg form.

Property: $\{\tau(x), \tau(y)\} = \{x,y\}^{\sigma^2}$; $\{y,x\} = \{x, \tau^{-1}y\}^{\sigma}$

NOT QUITE a Hermitian form (on N_0, σ has infinite order)

For $A \subset N_0$ W-lattice, $A^\vee = \{x \in N_0 \mid \{x, A\} \subset W\}$.

→ if I have $M = M_0 \oplus M_1$ (D-lattice as above),

$$A = M_0$$

$$A \underset{1}{\supset} A^\vee \underset{n-1}{\supset} pA$$ get an equiv. with signature condition for M and ~~self~~ self-duality.

↳ for going backwards, take $M_1 = F^{-1}A^\vee$.

↳ so we arrive to a collection of lattices $A = M_0 \subset N_0$ + properties.

<u>Pb</u>: $\{ , \}$ is not Hermitian.

Need an <u>additional step</u>:

Let $C = N_0^{\tau = 1}$ be a $\mathbb{Q}_{p^2}$-vect space of dim n.

(N_0 has dim n / $\mathbb{Q}_p$)

<u>and</u> $\{ , \}$ is a Hermitian form on C

(this follows from the formula $\{y, x\} = \{x, \tau^{-1}y\}^{\sigma}$.

→ as a vector space, we have:

$$N_0 = W_{\mathbb{Q}} \otimes_{\mathbb{Q}_{p^2}} C$$

In C we have $\mathbb{Z}_{p^2}$-lattices s.t.

$$\Lambda \underset{1}{\supset} \Lambda^\vee \underset{n-1}{\supset} p\Lambda$$

(but most of the A's above will <u>not</u> have this property, indeed it would imply $A = W \otimes_{\mathbb{Z}_{p^2}} \Lambda$, $\tau(A) = A$)

Such points are called 'super-special points'. ↳ $d = 0$

More generally, we have

<u>Zink's Lemma</u> : Given any A as in ⊛, $\exists\, d$ s.t. $0 \leq d \leq \left[\frac{n-1}{2}\right]$

$$\Lambda(A) = A + \tau(A) + \tau^2(A) + \dots + \tau^d(A)$$

is τ-invariant.

↳ It comes (from base-changing) a lattice in C.

If $\Lambda = \Lambda(A)$ we have the property $\Lambda \underset{\uparrow}{\supset} \Lambda^\vee \supset p\Lambda$

$t = 2d+1$: 'type' of Λ

$(1 \leq t \leq n)$

Λ is called a 'vertex of type t' (vertices in the building $B(SU(C))$).

$\boxed{\text{Conclusion}}$: each point in $M(\overline{\mathbb{F}_p})$ is associated such a lattice.

<u>Set-theoretic stratification:</u>

Given a vertex Λ of type t, define $\mathcal{V}^\circ(\Lambda)(\overline{\mathbb{F}_p}) = \{ x \in M(\overline{\mathbb{F}_p}) \mid \Lambda(A_x) = \Lambda \}$.

$\mathcal{V}(\Lambda)(\overline{\mathbb{F}_p}) = \{ \Lambda(A_x) \subset \Lambda \}$

We get a decomposition:

$$M(\overline{\mathbb{F}_p}) = \bigsqcup_{\Lambda} \mathcal{V}^\circ(\Lambda)(\overline{\mathbb{F}_p})$$

<u>Scheme-theoretic version of this result</u>: (VW)

define for each Λ (of type t) $M_\Lambda \subset M_{red}$ closed subscheme, s.t.:

$$\underset{\underset{\mathcal{V}_\Lambda(\overline{\mathbb{F}_p})}{\parallel}}{M_\Lambda(\overline{\mathbb{F}_p})} = M(\overline{\mathbb{F}_p})$$

$\dim(M_\Lambda) = d = \frac{1}{2}(t-1)$, they are smooth and irreducible,

incidence relation: $M_\Lambda \cap M_{\Lambda'} = \begin{cases} \phi \text{ if } \Lambda \cap \Lambda' \\ M_{\Lambda \cap \Lambda'} \end{cases}$

$$M_{red} = \bigcup_{\Lambda : t(\Lambda) \text{ is max}} M_\Lambda \qquad t_{max} = \begin{cases} n \\ n-1 \end{cases} \qquad \dim M_\Lambda = \left[\frac{n-1}{2}\right] \overset{!}{=} \dim M_p .$$

<u>Moreover</u>: we know the group-theoretic structure of these schemes:

the M_Λ's they are 'known varieties'

$$M_\Lambda = \text{Deligne-Lusztig variety for } U(\Lambda/\Lambda^\vee)$$

$$\Lambda \underset{t}{\supset} \Lambda^\vee \supset p\Lambda$$

$\mathbb{F}_{p^2}$ v.s. $\Lambda/\Lambda^\vee$ with Hermitian form

<u>Example</u>: $n = 3$ we get $x^{p+1} + y^{p+1} + z^{p+1} = 0$

equiv. to say $x x^\sigma + y y^\sigma + z z^\sigma = 0$.

('isotropic line')