Travaux de Bruinier

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Notes

The basic problem

The basic problem is to find a generalization of the work of Borcherds described in earlier lectures, for example, to arbitrary totally real fields.

The basic setup is the following:

$$F = \text{a totally real field with } |F:\mathbb{Q}| = d.$$

$$V = \text{a quadratic space over } F$$

$$\mathrm{sig}(V) = ((n,2),(n+2,0),\dots,(n+2,0))$$

$$L = \text{(even) integral lattice in } V,\ L' = \text{dual lattice.}$$

$$S_L \subset S(V(\hat{F}))$$

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S_L &\subset S(V(\hat{F}))
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This subspace is stable under the Weil representation action $\omega = \omega_\psi$ of $\mathrm{SL}_2(\hat{\mathbb{Z}})$.

We define a representation ω_L of $\mathrm{SL}_2(\mathbb{Z})$ on S_L via restriction. Also, let $\bar{\rho}_L = \omega_L$, i.e.,

$$\rho_L(\gamma)\varphi = \overline{\omega_L(\gamma)\bar{\varphi}}.$$

Recall

$$\mathbb{D} = \text{oriented negative 2-planes in } V_{\infty_1}.$$

$$\varphi(\tau, z) = \text{Gaussian}, \ \tau \in \mathfrak{H}^d, \ z \in \mathbb{D}.$$

$$\varphi = \varphi_\mu \in S_L$$

$$\theta(\tau, z, h, \varphi) = \sum_{x \in V(F)} \varphi(\tau, z; x)\, \varphi(h^{-1}x), \qquad h \in G(\mathbb{A}_f),$$

$$= \text{Siegel theta function}$$

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The Siegel theta function $\theta(\tau, z, h, \varphi)$ has weight

$$(\frac{n}{2} - 1, \frac{n}{2} + 1, \ldots, \frac{n}{2} + 1).$$

We view

$$\theta(\tau, z, h) : S_L \longrightarrow \mathbb{C}$$

as a distribution on S_L or $S(V(\mathbb{A}_f))$.

Let

$$f = \begin{matrix} \text{Hilbert modular form} \\ \text{of weight } (1 - \frac{n}{2}, \frac{n}{2} + 1, \ldots, \frac{n}{2} + 1). \end{matrix}$$

but anti-holomorphic in $(\tau_2, \ldots, \tau_d)$, S_L-valued and type ω_L.

Let

$$\theta(z, h; f) = \int_{\Gamma \backslash \mathfrak{H}^d} f(\tau) \cdot \theta(\tau, z, h) \, (v_2 \ldots v_d)^{\frac{n}{2}+1} \, d\mu(\tau)$$

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Allowing f to be 'meromorphic' at the cusps, we would like to see $\log ||\Psi(f)||^2$ as an output for a meromorphic form $\Psi(f)$ on X_K with an explicitly given divisor.

- **Problem I.** Why this signature?
  For example, you could consider the case

$$\mathrm{sig}(V) = (\underbrace{(n,2),\ldots,(n,2)}_{r},\underbrace{(n+2,0),\ldots,(n+2,0)}_{d-r}).$$

  Our present vision of the relation between automorphic forms and Chow groups is too naive.
- **Problem II.** Koecher's principle: Once a function is holomorphic on $\Gamma\backslash\mathfrak{H}^d$, it is holomorphic at the cusps, i.e., there are no nonzero f's of the type we need!

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Bruinier has found a way to get around this problem. At the moment, the method involves a more analytic approach:

- J. H. Bruinier, *Regularized theta lifts for orthogonal groups over totally real fields*, arXiv:0908.3076v2

There is also a more recent paper of Bruinier and Yang in which they give arithmetic applications:

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Review of the Borcherds case

First we review the case $F = \mathbb{Q}$, $d = 1$.

Here the input functions can be either:

- *weakly holomorphic forms*, Borcherds' original input, or
- *harmonic weak Maass forms*, introduced by Bruinier and Funke.

Here are the definitions:
For a function

$$\mathbf{f} : \mathfrak{H} \longrightarrow S_L,$$

define the slash operator of weight k

$$(\mathbf{f}|_{k,\bar{\rho}_L}[\gamma])(\tau) = j(\gamma,\tau)^{-k}\,\bar{\rho}_L(\gamma)^{-1}\,\mathbf{f}(\tau).$$

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Harmonic weak Maass forms:

$$\mathcal{H}_{k,\bar\rho_L}$$

(i) For all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\mathbf{f}|_{k,\bar\rho_L}[\gamma] = \mathbf{f}.$$

(ii) $\Delta_k \mathbf{f} = 0$ where

$$\Delta_k = -v^2\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) + ik\,v\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

(iii) There is a finite sum

$$P_{\mathbf{f}}(\tau) = \sum_{m\le 0} c^+(m)\,q^m, \qquad c^+(m) \in S_L$$

so really just an element of $\mathbb{C}[q^{-1}] \otimes_{\mathbb{C}} S_L$, such that

$$\mathbf{f}(\tau) - P_{\mathbf{f}}(\tau) = O(e^{-\epsilon v}), \qquad \text{as } v \longrightarrow \infty,$$

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Note that

$$-\Delta_k = R_{k-2} \circ L_k = L_{k+2} \circ R_k + k.$$

where the raising and lowering operators are

$$R_k = 2i \frac{\partial}{\partial \tau} + kv^{-1}, \qquad L_k = -2i\, v^2 \frac{\partial}{\partial \bar{\tau}}.$$

Weakly holomorphic forms:

$$\mathcal{M}^!_{k,\bar{\rho}_L} \quad \subset \quad \mathcal{H}_{k,\bar{\rho}_L}.$$

Here **f** is holomorphic, i.e., killed by L_k, and satisfies (i) and (iii). Condition (iii) then just says that **f** is meromorphic at the cusp.

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Consider the conjugate lowering operator

$$\delta_k : \mathbf{f} \mapsto v^{k-2}\overline{L_k(\mathbf{f})} = 2i\, v^k \overline{\frac{\partial \mathbf{f}}{\partial \bar{\tau}}}.$$

Observing the identity

$$L_{2-k}(v^{k-2}\bar{\phi}) = v^k\, \overline{R_{k-2}\phi}.$$

we have

$$L_{2-k}(\delta_k(\mathbf{f})) = v^k\, \overline{R_{k-2} \circ L_k(\mathbf{f})} = v^k\, \overline{\Delta_k(\mathbf{f})} = 0,$$

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Indeed:

Proposition (Bruinier-Funke)

There is an exact sequence

$$0 \longrightarrow \mathcal{M}^!_{k,\bar{\rho}_L} \longrightarrow \mathcal{H}_{k,\bar{\rho}_L} \xrightarrow{\delta_k} \mathcal{S}_{2-k,\rho_L} \longrightarrow 0.$$

where $\mathcal{S}_{2-k,\rho_L}$ is the space of holomorphic cusp forms of type ρ_L and weight $2-k$.

Remark: In particular, the space of weakly holomorphic forms has finite codimension in the space of harmonic weak Maass forms.

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Relation to Poincaré series

To define Poincaré series, we begin with simpler spaces of functions, which are only required to have translation invariance.

The first of these, for $s \in \mathbb{C}$, is

$$A_{k,\bar{\rho}_L}(s)$$

the space of smooth functions

$$f : \mathfrak{H} \longrightarrow S_L$$

(i) For all $\gamma \in \Gamma_\infty^u \quad \subset \Gamma_\infty, \quad$ (Note that $|\Gamma_\infty : \Gamma_\infty^u| = 2$.)

$$f|_{k,\bar{\rho}_L}[\gamma] = f$$

(ii)

$$\Delta_k f = \frac{1}{4}(k - 1 + s)(k - 1 - s)\, f,$$

Note that for the values $s = \pm s_0$, $s_0 = 1 - k$, such functions are annihilated by Δ_k.

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The first of these, for $s \in \mathbb{C}$, is

$$A_{k,\bar{\rho}_L}(s)$$

the space of smooth functions

$$f : \mathfrak{H} \longrightarrow S_L$$

(i) For all $\gamma \in \Gamma_\infty^u \quad \subset \Gamma_\infty$, (Note that $|\Gamma_\infty : \Gamma_\infty^u| = 2$.)

$$f|_{k,\bar{\rho}_L}[\gamma] = f$$

(ii)

$$\Delta_k f = \frac{1}{4}(k-1+s)(k-1-s)\,f,$$

Note that for the values $s = \pm s_0$, $s_0 = 1-k$, such functions are annihilated by Δ_k.

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Such functions have a Fourier expansion of the following form

$$f(\tau) = a(0,s)\, v^{(1-k-s)/2} + b(0,s)\, v^{(1-k+s)/2}$$
$$+ \sum_{m\neq 0} a(m,s)\, \mathcal{W}_s(4\pi m v)\, e(mu)$$
$$+ b(m,s)\, \mathcal{M}_s(4\pi m v)\, e(mu),$$

where $\mathcal{W}_s(t)$ and $\mathcal{M}_s(t)$ are Whittaker functions, and $a(m,s)$ and $b(m,s)$ are in S_L.

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Relation to Poincaré series

The key facts about them are

(a) Formulas:

$$\mathcal{M}_s(v) = |v|^{-k/2}\, M_{\nu,\mu}(|v|),$$

$$\mathcal{W}_s(v) = |v|^{-k/2}\, W_{\nu,\mu}(|v|),$$

$$\nu = \mathrm{sgn}(v)\, k/2, \qquad \mu = s/2.$$

Note that the only dependence on the sign of v is in the index ν. Here $M_{\nu,\mu}(|v|)$ and $W_{\nu,\mu}(|v|)$ are classical Whittaker functions.

(b) Asymptotics:

$$M_{\nu,\mu}(t) = t^{\mu+\frac{1}{2}}(1+O(t)), \qquad W_{\nu,\mu}(t) = O(t^{-\mu+\frac{1}{2}}), \qquad t \longrightarrow 0,$$

$W_{\nu,\mu}(t) = O(e^{-\epsilon t})$, and

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(c) Special values at $s_0 = 1 - k$, (Remark: Usually, $k < 1$.)

$$\mathcal{W}_{s_0}(v) = \begin{cases} e^{-v/2} & \text{if } v > 0, \\ e^{-v/2}\,\Gamma(1-k, |v|) & \text{if } v < 0, \end{cases}$$

for $\Gamma(s, x)$ the incomplete Γ-function, and

$$\mathcal{M}_{s_0}(v) = (-\mathrm{sgn}(v))^{k-1}\, e^{-v/2} \left(\Gamma(2-k) - (1-k)\,\Gamma(1-k; -v) \right),$$

Note that

$$\mathcal{W}_{s_0}(v) \asymp e^{-|v|/2}, \qquad |v| \to \infty,$$

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- Next, the space of **Whittaker forms** is the subspace of $A_{k,\bar{\rho}_L}(s)$ spanned by functions of the form

$$b(m,s)\,\mathcal{M}_s(4\pi mv)\,e(mu),$$

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- Taking $s = s_0 = 1 - k$, the space of **harmonic Whittaker forms**

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- Very explicitly, for any $s \in \mathbb{C}$, the space of Whittaker forms is the span of the functions

$$f_{m,\mu}(\tau, s) = \Gamma(s+1)^{-1} \mathcal{M}_s(4\pi m v)\, e(mu)\, \varphi_\mu.$$

for $m \in \mathbb{Q}$, $m < 0$, $\mu \in L'/L$, with $m + Q(\mu) \in \mathbb{Z}$.

- Let

$$\delta_k(f) = v^{k-2} \overline{L_k(f)}.$$

Then, for $f \in H_{k,\bar{\rho}_L}$, $\delta_k(f) \in S_{2-k,\rho_L}$, the space of holomorphic functions $g : \mathfrak{H} \to S_L$ satisfying (i) and of exponential decay as $v \to \infty$.

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Assume that $k < 0$. Then we can form Poincaré series:

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where the first two vertical arrows are given by

$$f \mapsto \mathrm{PS}_{k,\bar\rho_L}(f) = \sum_{\gamma\in\Gamma^u_\infty\backslash\Gamma} f|_{k,\bar\rho_L}[\gamma],$$

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The space $M^!_{k,\bar{\rho}_L}$ is the kernel of the composite map

$$\xi_k = \mathrm{PS}_{2-k,\rho_L} \circ \delta_k.$$

Proposition

$$\mathrm{PS}_{k,\bar{\rho}_L} : H_{k,\bar{\rho}_L} \xrightarrow{\sim} \mathcal{H}_{k,\bar{\rho}_L},$$

with inverse given by

$$\mathbf{f} \mapsto \sum_{m<0} \sum_{\mu} c^+(m)\, f_{m,\mu}(\tau, s_0).$$

where

$$P_{\mathbf{f}}(\tau) = \sum_{m \le 0} c^+(m)\, q^m, \qquad c^+(m) \in S_L.$$

In particular, the weakly holomorphic forms have a simple description:

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The Borcherds lift of a Poincaré series

- The upshot of the previous discussion is that every $\mathbf{f} \in \mathcal{H}_{k,\bar{\rho}_L}$ can be written uniquely as a Poincaré series for $f \in H_{k,\bar{\rho}_L}$.
- If $\mathbf{f} = \mathrm{PS}_{k,\bar{\rho}_L}(f)$, then

$$\Phi(z, h; \mathbf{f}) = \int_{\Gamma\backslash\mathfrak{H}}^{\mathbf{reg}} \mathbf{f}(\tau) \cdot \theta(\tau, z, h)\, d\mu(\tau)$$

$$= \int_{\Gamma_\infty^u\backslash\mathfrak{H}}^{\mathbf{reg}} f(\tau) \cdot \theta(\tau, z, h)\, d\mu(\tau). \tag{0.1}$$

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Here is Bruinier's key idea. In the case $F = \mathbb{Q}$,

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M^!_{k,\bar\rho_L} & \longrightarrow & H_{k,\bar\rho_L} & \xrightarrow{\delta_k} & S_{2-k,\rho_L} \\
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\theta^{\mathrm{reg}}\downarrow & & \theta^{\mathrm{reg}}\downarrow & & \theta^{\mathrm{KM}}\downarrow \\
\text{Borcherds forms} & \longrightarrow & \text{harmonic Green functions} & \xrightarrow{dd^c} & A^{(1,1)}(X)
\end{array}$$

where the lower right square is the Theorem of Bruinier-Funke for θ^{KM} the classical geometric theta lift. To be precise

$$dd^c(\theta^{\mathrm{reg}}(\mathbf{f})) = \theta^{\mathrm{KM}}(\delta_k(\mathbf{f})) + a^+(0)\,\Omega.$$

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Here is Bruinier's key idea. In the case $F = \mathbb{Q}$,

$$\begin{array}{ccccc}
M^!_{k,\bar\rho_L} & \longrightarrow & H_{k,\bar\rho_L} & \xrightarrow{\delta_k} & S_{2-k,\rho_L} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{M}^!_{k,\bar\rho_L} & \longrightarrow & \mathcal{H}_{k,\bar\rho_L} & \xrightarrow{\delta_k} & \mathcal{S}_{2-k,\rho_L} \\
\theta^{\mathrm{reg}} \downarrow & & \theta^{\mathrm{reg}} \downarrow & & \theta^{\mathrm{KM}} \downarrow \\
\begin{array}{c}\text{Borcherds}\\ \text{forms}\end{array} & \longrightarrow & \begin{array}{c}\text{harmonic}\\ \text{Green functions}\end{array} & \xrightarrow{dd^c} & A^{(1,1)}(X)
\end{array}$$

where the lower right square is the Theorem of Bruinier-Funke for θ^{KM} the classical geometric theta lift. To be precise

$$dd^c(\theta^{\mathrm{reg}}(\mathbf{f})) = \theta^{\mathrm{KM}}(\delta_k(\mathbf{f})) + a^+(0)\,\Omega.$$

The Borcherds lift of a Poincaré series

- For F with $d > 1$, the first two spaces on the bottom row are zero, by Koecher's principal.
- First, Bruinier defines the space $H_{\mathbf{k},\bar{\rho}_L}$.
  Of course, the Poincaré series for functions in this space diverge wildly.
- The main idea is to bypass the 'missing spaces' and to define directly a map

$$\theta^{\text{Bruinier}} : H_{k,\bar{\rho}_L} \longrightarrow \begin{array}{c}\text{harmonic}\\ \text{Green functions}\end{array}$$

by regularizing the unfolded version (0.1):

$$f \quad \mapsto \quad \int_{\Gamma^u_\infty \backslash \mathfrak{H}}^{\mathbf{reg}} f(\tau) \cdot \theta(\tau, z, h)\, d\mu(\tau).$$

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The Borcherds lift of a Poincaré series

Here is the picture:

$$
\begin{tikzcd}
M^{!}_{\mathbf{k},\bar{\rho}_L} \arrow[r] \arrow[d, dotted] & H_{\mathbf{k},\bar{\rho}_L} \arrow[r, "\delta_{\mathbf{k}}"] \arrow[d, dotted, "\theta^{\text{Bruinier}}"'] & S_{2-\mathbf{k},\rho_L} \arrow[d, "\mathrm{PS}_{2-k,\rho_L}"'] \\
 & & \mathcal{S}_{2-\mathbf{k},\rho_L} \arrow[d, "\theta^{\mathrm{KM}}"'] \\
\begin{array}{c}\text{Borcherds}\\ \text{forms}\\ ??\end{array} \arrow[r] & \begin{array}{c}\text{harmonic}\\ \text{Green functions}\\ ??\end{array} \arrow[r, "dd^c"] & A^{(1,1)}(X).
\end{tikzcd}
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The basic spaces of Whittaker functions

To carry out this construction for a totally real field F, we first need the analogues of the spaces $A_{k,\bar{\rho}_L}(s)$ and $H_{k,\bar{\rho}_L}$ in the Hilbert modular case.

Recall:

$$\mathrm{sig}(V) = ((n,2),(n+2,0),\dots,(n+2,0))$$

$$\text{Siegel weight} = (\frac{n}{2}-1,\frac{n}{2}+1,\dots,\frac{n}{2}+1)$$

$$\mathbf{k} = (1-\frac{n}{2},\frac{n}{2}+1,\dots,\frac{n}{2}+1)$$

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Now, for $s \in \mathbb{C}$,

$$A_{\mathbf{k},\bar{\rho}_L}(s)$$

is the space of smooth functions

$$f : \mathfrak{H}^d \longrightarrow S_L.$$

(1) For all $\gamma \in \Gamma_\infty^u$,

$$f|_{\mathbf{k},\bar{\rho}_L}[\gamma] = f$$

Note that Γ_∞^u now has infinite index in Γ_∞.

(2) In the first variable, for $k = 1 - \frac{n}{2}$,

$$\Delta_k f = \frac{1}{4}(k-1+s)(k-1-s)\, f,$$

(3) f is anti-holomorphic in $(\tau_2, \dots, \tau_d)$.

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Such functions have a Fourier expansion of the following form

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\begin{aligned}
f(\tau) = a(0,s)\, v_1^{(1-k-s)/2} + b(0,s)\, v_1^{(1-k+s)/2} \\
+ \sum_{m\neq 0} a(m,s)\, \mathcal{W}_s(4\pi m v)\, e(\mathrm{tr}(mu)) \\
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$$\boldsymbol{\mathcal{W}}_s(t) = \mathcal{W}_s(t_1)\ \exp(\frac{1}{2}(t_2 + \cdots + t_d))$$

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are Whittaker functions, $m \in F$, and $a(m, s)$ and $b(m, s)$ are in S_L.

Here, for example, m constrains the support of $a(m, s)$:

$$a(m, s) = \sum_{\substack{\mu \in L'/L \\ m+Q(\mu) \in \partial_F^{-1}}} a(m, \mu, s)\, \varphi_\mu \ \in S_L.$$

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The space of **Whittaker forms** is the subspace of $A_{\mathbf{k},\bar{\rho}_L}(s)$ spanned by functions of the form

$$b(m,s)\,\mathcal{M}_s(4\pi mv)\,e(\mathrm{tr}(mu)),$$

for $m \ll 0$.

Taking $s = s_0 = 1-k$, the space of **harmonic Whittaker forms**

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$$\delta_k(f) = v_1^{k-2}\overline{L_k(f)},$$

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$$\kappa = (\frac{n}{2}+1, \dots, \frac{n}{2}+1),$$

and, noting that $2-k=\frac{n}{2}+1$, define the lowering Poincaré operator

$$\xi_{\mathbf{k}} = \mathrm{PS}_{\kappa,\rho_L} \circ \delta_k : H_{\mathbf{k},\bar{\rho}_L} \longrightarrow \mathcal{S}_{\kappa,\rho_L}.$$

Then there is an exact sequence:

$$0 \longrightarrow M^!_{\mathbf{k},\bar{\rho}_L} \longrightarrow H_{\mathbf{k},\bar{\rho}_L} \xrightarrow{\xi_{\mathbf{k}}} \mathcal{S}_{\kappa,\rho_L} \longrightarrow 0,$$

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The main results

Now suppose that f is a Whittaker form in $A_{\mathbf{k},\bar{\rho}_L}(s)$ define the *regularization*:

$$\int_{\Gamma_\infty^u\backslash\mathfrak{H}}^{\mathbf{reg}} f(\tau,s)\cdot\theta(\tau,z,h)\,d\mu(\tau)$$

$$= \int_{(\mathbb{R}_+^\times)^d}\left(\int_{O_F\backslash\mathbb{R}^d} f(\tau,s)\cdot\theta(\tau,z,h)\,du\right)(v_2\dots v_d)^{\frac{n}{2}-1}\,N(v)^{-2}\,dv. \quad (0.2$$

For example, for the 'standard' vector for $m \gg 0$

$$f_{-m,\mu}(\tau,s) = C\,\mathcal{M}_s(-4\pi mv)\,e(-\mathrm{tr}(mu))\,\varphi_\mu,$$

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Here are Bruinier's basic analytic results:

Theorem

(i) For $\mathrm{Re}(s) > s_0 + 2$, the regularized integral (0.2) converges for (z, h) outside of a set of measure 0 and defines an integrable function on X_K.

More precisely,

$$\Phi_{m,\mu}(z, h, s) = \sum_{\substack{x \in h(\mu+L) \\ Q(x)=m}} \phi(x, z, s), \tag{0.3}$$

where

$$\phi(x, z, s) = \frac{\Gamma(\frac{1}{2}(s+s_0))}{\Gamma(s+1)} R(x, z)^{\frac{1}{2}(s+s_0)}$$
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$$R(x,z) = \frac{Q(x_1)}{Q(\mathrm{pr}_{z^\perp}(x_1))},$$

and

$$F(a,b,c;z) = \text{Gauss's hypergeometric function.}$$

(ii) (Offending terms) On the complement of the special cycle $Z(m,\mu)$, the series on the right hand side of (0.3) converges for $\mathrm{Re}(s) > s_0$. In a neighborhood of any point (z_0, h_0),

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$$R(x,z) = \frac{Q(x_1)}{Q(\mathrm{pr}_{z^\perp}(x_1))},$$

and

$$F(a,b,c;z) = \text{Gauss's hypergeometric function.}$$

(ii) (Offending terms) On the complement of the special cycle $Z(m,\mu)$, the series on the right hand side of (0.3) converges for $\mathrm{Re}(s) > s_0$. In a neighborhood of any point (z_0, h_0),

$$\Phi_{m,\mu}(z,h,s) - \sum_{\substack{x \in h_0(\mu+L) \\ Q(x)=m \\ (x_1,z_0)=0}} \phi(x,z,s)$$

is a C^∞-function.

The main results

(iii) On $X_K - Z(m,\mu)$, the function $\Phi_{m,\mu}(z,h,s)$ a meromorphic analytic continuation in s with a simple pole at $s = s_0$ with residue

$$A(m,\mu) = 2\,\frac{\deg(Z(m,\mu))}{\mathrm{vol}(X_K)}.$$

Moreover, for a fixed s away from the poles, $\Phi_{m,\mu}(z,h,s)$ is real analytic on $X_K - Z(m,\mu)$.

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Finally, for $f \in H_{\mathbf{k},\bar{\rho}_L}$, write

$$f = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu)\, f_{-m,\mu}(\tau),$$

and let

$$\Phi(z,h,s,f) = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu)\, \Phi_{m,\mu}(z,h,s).$$

Definition: The **regularized theta lift** $\Phi(z,h,f)$ of f is the constant term in the Laurent expansion of $\Phi(z,h,s,f)$ at $s = s_0$.

$$\Phi(z,h,f) = \mathrm{CT}_{s=s_0} \Phi(z,h,s,f).$$

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The following result says that, indeed,

$$\theta^{\text{Bruinier}} : H_{\mathbf{k},\bar{\rho}_L} \longrightarrow \text{harmonic Green functions.}$$

Theorem

Let

$$Z(f) = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu)\, Z(m,\mu),$$

Then the differential form $dd^c\Phi(f)$ on $X_K - Z(f)$ extends to a smooth $(1,1)$-form $\omega(f)$ on X_K which is a harmonic Poincaré dual to $Z(f)$. Moreover, Then

$$dd^c[\Phi(f)] + \delta_{Z(f)} = [\omega(f)],$$

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Bruinier extends the result of Bruinier-Funke relating this regularized lifting of $f \in$ to the θ^{KM} lift of $\xi_{\mathbf{k}}(f)$. I will omit this.

It plays a crucial role in the construction of Borcherds forms.

Recall the exact sequence:

$$0 \longrightarrow M^!_{\mathbf{k},\bar{\rho}_L} \longrightarrow H_{\mathbf{k},\bar{\rho}_L} \xrightarrow{\xi_{\mathbf{k}}} \mathcal{S}_{\kappa,\rho_L} \longrightarrow 0,$$

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Borcherds forms

When applied to the subspace $M^!_{\mathbf{k},\bar{\rho}_L}$, θ^{Bruinier} yields the desired generalization of Borcherds forms.

Theorem For $f \in M^!_{\mathbf{k},\bar{\rho}_L}$, with

$$f = \sum_{\mu \in L'/L} \sum_{m \gg 0} c(m,\mu)\, f_{-m,\mu}(\tau),$$

and $c(m,\mu) \in \mathbb{Z}$, recall that

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Then there is a meromorphic function $\Psi(z, h, f)$ on $\mathbb{D} \times G(\mathbb{A}_f)$ such that

(i) $\Psi(z, h, f)$ is modular of weight $-B(f)$ and a multiplier system of finite order, i.e., it is left $G(\mathbb{Q})$-invariant and transforms under K by a unitary character of finite order.

(ii)

$$\operatorname{div}(\Psi(f)) = Z(f).$$

(iii)

$$-\log ||\Psi(z, h, f)||^2 = \Phi(z, h, f).$$

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Arithmetic applications

Bruinier's results open the way for a number of arithmetic applications. In particular, several nice results that depended on Borcherd's work, and hence were only available over $\mathbb{Q}$, can now be proved in a much more systematic way.

For example:

(1) Modularity of the generating function for the classes of the $Z(m, \mu)$'s in the Chow group $\mathrm{CH}^1(X_K)$.

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Define the generating function

$$\phi^{\mathrm{CH}}(\tau) = \sum_{\mu \in L'/L} \sum_{m \gg 0} [Z^0(m,\mu)]\, q^m\, \varphi_\mu,$$

where $[Z^0(m,\mu)]$ denotes the class of $Z^0(m,\mu)$ in the Chow group $\mathrm{CH}^1(X_K)$.

Theorem

$$\phi^{\mathrm{CH}}(\tau) \quad \in S_{\kappa,\rho_L} \otimes \mathrm{CH}^1(X_K)$$

is an $S_L \otimes \mathrm{CH}^1(X_K)$-valued modular form of weight κ.

This result as also proved by X. Yuan, W. Zhang and S. Zhang by another method, based on the modularity of the image of this series in cohomology $H^2(X_K)$.

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One last remark is the following:

Consider the map

$$H_{\mathbf{k},\bar{\rho}_L,\mathbb{Z}} \longrightarrow Z^1(X_K), \qquad f \mapsto Z(f).$$

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