Toroidal compactifications of Hilbert modular varieties

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Montréal, April 9, 2011
1. Torus embeddings

2. Hilbert modular varieties and their boundary components

3. Toroidal compactification — analytic theory

4. Algebraic theory
affine Torus embeddings

\[ k \quad \text{a field} \]
\[ M \quad \text{a lattice } (\cong \mathbb{Z}^n) \]
\[ T = \text{spec}(k[M^*]) \quad \text{a split torus over } k \]
\[ M = X_*(T) \quad \text{the cocharacter group of } T \]

We want to look at a certain type of (partial) compactifications of \( T \), called \textbf{torus embeddings}.

\[ R \quad \text{a discrete valuation ring } (k\text{-algebra}) \]
\[ K \quad \text{quotient field} \]
\[ x \quad \text{a point in } T(K) \text{ which does not extend to } R \]

**Goal:** Look for open embeddings \( T \hookrightarrow \overline{T} \) such that \( x \) extends to a section \( x \in \overline{T}(R) \).

The valuation on \( K \) induces a linear morphism \( \nu_x : M^* \to \mathbb{Z} \). Therefore the subring defining the open embedding above must not contain \( m^* \) with \( \nu_x(m^*) < 0 \).
affine Torus embeddings

- \( \overline{T} := \text{spec}(k[\nu^{-1}_x\mathbb{Z}_{\geq 0}]) \).
- more generally: Replace \( \nu^{-1}_x\mathbb{Z}_{\geq 0} \) by any saturated (in order to get an open embedding) submonoid \( \subset M \) such that \( \nu_x \) is non-negative.

All these \( \overline{T} \) have the property that the action of \( T \) (by multiplication) extends to them.

**Proposition**

The following are equivalent

1. affine open dense embeddings \( T \hookrightarrow \overline{T} \) such that the action of \( T \) extends
2. finitely generated submonoids of \( M^* \)
3. polyhedral cones \( \sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_n \subset M_{\mathbb{R}} \), which do not contain a line (here \( v_i \in M \))

\[ \sigma \mapsto \sigma^\vee = \{ y \in M^* \mid \langle x, y \rangle \geq 0 \ \forall x \in \sigma \} \mapsto T_\sigma := \text{spec}(k[\sigma^\vee]) \]

E.g.: \( \mathbb{R}_{\geq 0}v_x \leftrightarrow \text{spec}(k[\nu^{-1}_x\mathbb{Z}_{\geq 0}]) \)
Drop the assumption of affine — can we get something proper?

\[ \sigma, \tau \subset M_{\mathbb{R}} \]

\[ \sigma \subset \tau \]

\[ \leadsto \text{ natural map } T_{\tau} \rightarrow T_{\sigma} \]

Given a collection \( \Delta := \{ \sigma_i \} \) with \( \sigma_i \subset M_{\mathbb{R}} \), we can glue along all these natural maps (i.e. take the direct limit). Call this \( T_{\Delta} \).

\( T_{\Delta} \text{ separated} \), if \( \sigma \cap \tau \) for two r.p.c. in \( \Delta \) is either empty or a common face of \( \sigma \) and \( \tau \) which is included in \( T_{\tau} \). We call \( \Delta \) a partial polyhedral cone decomposition in this case (and assume also that with \( \sigma \in \Delta \), it contains all of the faces).

\( T_{\Delta} \text{ proper} \), if for all possibe \( \nu_x \) as above there is a cone \( \sigma \) with \( \nu_x \in \sigma \), i.e. if \( \bigcup_{\sigma \in \Delta} \sigma = M_{\mathbb{R}} \).
Theorem

The following define equivalent categories

1. open dense embeddings $T \hookrightarrow \overline{T}$ such that the action of $T$ extends
2. partial polyhedral cone decompositions $\Delta$ of $M^*$.

Morphisms in the second case are *refinements*.

$T_\Delta$ **smooth**, if all r.p.c. in $\Delta$ are generated by part of a basis of $M^*$.

$T_\Delta$ **projective**, if there is a piecewise linear function $\mu : \bigcup \sigma \rightarrow \mathbb{R}$ with integral values on $M$, such that the $\sigma \in \Delta$ are the maximal sets on which $\mu$ is linear (together with the faces of those) which satisfied a certain convexity property.
For each cone $\sigma \in \Delta$ there is an associated stratum $T[\sigma]$, isomorphic to the quotient of $T$ having cocharacter lattice $M/(\langle \sigma \rangle \cap M)$. It embeds by the obvious map

$$\text{spec}(k[\sigma^\perp \cap M^*]) \rightarrow \text{spec}(k[\sigma^\vee \cap M^*])$$

Properties:

- $T_\Delta = \bigcup_{\sigma \in \Delta} T[\sigma]$.
- $\sigma \subseteq \tau \iff \overline{T[\tau]} \subseteq \overline{T[\sigma]}$.
- $\kappa = \sigma \cap \tau \iff \overline{T[\sigma]} \cap \overline{T[\tau]} = \overline{T[\kappa]}$. 
The functor

Obviously:

\[ T(S) = \{ \pi : M_S \to (\mathcal{O}_S, \times) \text{ morphism of monoids} \} \]

We have:

\[ T_\Delta(S) = \begin{cases} 
M' \subset M_S \\
\pi : M' \to (\mathcal{O}_S, \times) \\
\forall s \in S: M'_s = \sigma^\vee \cap M \text{ for some } \sigma \in \Delta 
\end{cases} \]

\( \phi \) a strict morphism of monoids if \( \phi(e) = e \) and \( \phi(x) \) invertible \( \iff \) \( x \) invertible.
Given a holomorphic map (with "bounded image")

\[ \mu : B_1^*(0) \to T(\mathbb{C}) = M_\mathbb{C}/M \]

defines a monodromy element \( x \in M \) (image of 1 under the monodromy representation \( \mathbb{Z} = \pi_1(B_1^*(0)) \to M \)).

**Lemma**

\( \mu \) extends to \( B_1(0) \to T(\mathbb{C})_\Delta \) if and only if \( x \in \text{supp}(\Delta) \).

Roughly: Via the embedding \( B_1^*(0) \hookrightarrow \mathbb{G}_m(\mathbb{C}) \), \( \mu \) "looks like" the cocharacter \( x \).
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The symmetric space

Fix the following data:

- $F$: totally real field of degree $n$
- $\mathcal{L}$: fixed ideal of $\mathcal{O}_F$
- $V$: projective $\mathcal{O}_F$-module of rank 2 with a $\mathcal{L}^{-1}$-valued symplectic form $\langle \cdot, \cdot \rangle_{\mathcal{O}_F}$
  
  i.e. an isomorphism $\Lambda^2_{\mathcal{O}_F} V \cong \mathcal{L}$

- $\langle v, w \rangle_{\mathcal{O}_F} = \text{tr}_{F\mid \mathbb{Q}} \langle v, w \rangle_{\mathbb{Q}}$

- $G$: 
  
  \{ $g \in \text{Res}_{F\mid \mathbb{Q}} \text{GL}_F(V_{\mathbb{Q}}) \mid \det(g) \in \mathbb{G}_{m, \mathbb{Q}}$ \}

  \begin{align*}
  &= \text{Res}_{F\mid \mathbb{Q}} \text{GL}_F(V_{\mathbb{Q}}) \cap \text{GSp}(V_{\mathbb{Q}}) \\
  &= G(\mathbb{R})^+ / K \cdot \mathbb{Z}
  \end{align*}

- $D$: 
  
  \{ polarized $\mathcal{O}_F$-Hodge structures on $V_{\mathbb{C}}$ \}

  $V_{\mathbb{R}} = \bigoplus_{\rho \in \text{Hom}(F, \mathbb{R})} V^\rho$ with

  $V^\rho$ isomorphic to $\mathbb{R}^2$ with the $F$-representation induced by $\rho$.  

The symmetric space

**Definition of $\mathcal{O}_F$-Hodge structure**

$F^0 := V^{-1,0}$ defines a $\mathcal{O}_F$-Hodge structure if one of the equivalent conditions hold

1. The representation $h : S \to GL(V_\mathbb{R})$ giving the Hodge structure factorizes via $G_\mathbb{R}$
2. The action of $\mathcal{O}_F$ on $V$ induces endomorphims of Hodge structures
3. For each $\rho$, there is a $F^0,\rho \subset V_\mathbb{C}^\rho$ (1-dimensional) such that $F^0 = \bigoplus \rho F^0,\rho$.
4. The complex torus $A := V_\mathbb{Z}\backslash V_\mathbb{C}/F^0$ is an $\mathcal{O}_F$-complex torus.
The symmetric space

Definition of polarization for $O_F$-Hodge structures

Each $F^0,\rho \subset V^\rho_C$ induces a sign $\text{sgn} \frac{\langle v, \overline{v} \rangle}{2\pi i}$, where $v \in F^0,\rho$ is any non-zero element. We call the Hodge structure polarized if:

1. All signs above are positive
The Borel embedding

Relaxing the condition on $F^0 := V^{-1,0}$ of polarized Hodge structure — but not the condition of $\mathcal{O}_F$-compatibility — $F^0$ is just defined by a collection of 1-dimensional subspaces $F^{0,\rho} \subset V^\rho_C$. Therefore we get the open Borel embedding:

$$D \hookrightarrow D^\vee = (\text{Res}_{F:Q} \mathbb{P}_F(V_Q))(\mathbb{C}) \cong \prod_{\rho} \mathbb{P}(V^\rho_C)$$

The closure $\overline{D}$ of the image decomposes into boundary components, which are products of boundary components of the $H \subset \mathbb{P}(V^\rho_C)$ (either a real point or $H$ itself). To each such boundary component one associates a real parabolic in $G_R$. If $G_R$ is simple only $G_R$ and maximal parabolics of occur, otherwise products of those. For the compactification of the quotients $D/G(Z)^+$ only those boundary components whose parabolic is defined over $\mathbb{Q}$ matter. These are just the points

$$I \in (\text{Res}_{F:Q} \mathbb{P}_F(V_Q))(\mathbb{Q}) = \mathbb{P}_F(V_Q).$$
Siegel domain realization

The study of boundary components is intimately related to the realizations of $D$ as a Siegel domain (of the first kind).

Consider the filtration given by $I$:

$$0 \subset I \subset V$$

of saturated $\mathcal{O}_F$-lattices. Since $\Lambda^2(V) \cong \mathcal{L}$, the lattice

$$U^I = I \otimes \Lambda^2 \otimes \mathcal{O}_F \mathcal{L}$$

acts as square zero elements shifting the filtration by 1. We let the algebraic group $\mathbb{G}_a(U^I_\mathbb{Q})$ act unipotently via exponentials of these.

Define

$$P^I = \{ g \in G \mid gI \subseteq I \}, \text{ the parabolic associated with } I$$

$$= \mathbb{G}_m \cdot \text{Res}_{F:Q} \mathbb{G}_m \cdot \mathbb{G}_a(U^I_\mathbb{Q})$$

$$G^I = \mathbb{G}_a(U^I_\mathbb{Q}) \ltimes \mathbb{G}_m \subseteq P^I$$

$$D^I = \{ \mathcal{O}_F\text{-mixed Hodge structures w.r.t. } I \} \cong U^I_\mathbb{C}$$
Ad hoc definition:
$F^0 \mathcal{O}_F$-mixed Hodge structures w.r.t. $I \Leftrightarrow F^0,\rho \neq I^\rho \forall \rho$.
This condition may also be formulated as $h : S_C \to \text{GL}_C$ (appropriately defined) factorizing through $G^I(\mathbb{C})$.

Boundary map
We have an inclusion
$$D \to D^I$$
such that
$$D = \{ x \in D^I \cong U^I_C \mid \Im(x) \in C^I \}$$
where $C^I \subset U^I_\mathbb{R}$ is the cone of totally positive elements. This is a Siegel domain of the first kind.
Note: $\Im(x)$ well-defined.
The analytic boundary and the Baily-Borel compactification

On \( D \subset D^l \cong U_C^l \) one may define the “distance” to the boundary point \( I \) as follows

\[
d^l(x) = \frac{1}{|N_{F:Q} \mathcal{O}(x)|},
\]

where we chose any \( F \)-linear identification of \( U_Q \) with \( F \).

This distance defines a topology on \( \tilde{D} = D \cap \bigcap_{I \in \mathcal{P}_F(V_Q)} I \) such that \( \tilde{D}/G(\mathbb{Z})^+ \) becomes the structure of a normal projective (but singular) complex variety, the Baily-Borel compactification. Its boundary consists of finitely many cusps (class number of \( \mathcal{O}_F \)).
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For each \( I \in \mathbb{P}_F(V_{\mathbb{Q}}) \), we have

\[
G \hookrightarrow G^I \\
D \hookrightarrow D^I \\
D \subseteq D^I = (D^I)^\vee \subseteq D^\vee
\]

\( D/G(\mathbb{Z})^+ \) Hilbert modular variety — want to compactify it.  
\( D^I/G^I(\mathbb{Z})^+ \) is a torus — know how to compactify it!

Toroidal compactification: Glue the closure of \( D/G^I(\mathbb{Z})^+ \) in \( (D^I/G^I(\mathbb{Z})^+)_{\Delta_I} \) for some r.p.c.d \( \Delta_I \) to \( D/G(\mathbb{Z})^+ \) via the quotient map.
Given a holomorphic map

$$\mu : B_1^*(0) \to D / G(\mathbb{Z})^+$$

which does not extend to $B_1(0)$.

**Lemma**

(up to replacing $\mu$ by a finite cover)

A monodromy element $x \in G(\mathbb{Z})^+$ (unique up to conjugation) is unipotent.

By the very structure of $G(\mathbb{Z})$, $x$ fixes an $O_F$-line $I \subset V$, hence $x \in U^I(\mathbb{Z})$ and $\mu$ lifts along the map

$$D / G^I(\mathbb{Z})^+ \to D / G(\mathbb{Z})^+.$$
Lemma

\( x \) lies automatically in \( C' \)

**Sketch:** If we consider \( x \) as a cocharacter of the torus \( D'/G'(\mathbb{Z})^+ \) then \( \mu \) “looks like” \( x \) via the inclusion \( B_1^*(0) \subseteq \mathbb{G}_m(\mathbb{C}) \). \( \Im(\tilde{x}(z))) \in U'_R \) is well defined (independent of the lift to \( U'_C \)) and is just

\[
\Im(\tilde{x}(z))) = (-\frac{1}{2\pi} \log |z|) \cdot x \in C'.
\]

We have seen that \( \mu \) extends to a map \( B_1(0) \to (D'/U')_{\Delta'} \) if and only if \( x \in \text{supp}(\Delta') \). Hence to compactify \( D/G(\mathbb{Z})^+ \), the support of \( \Delta' \) should cover precisely \( C' \). Such a \( \Delta' \) is called a **rational polyhedral cone decomposition** of \( C' \). In general it will be infinite.
Toroidal compactification over $\mathbb{C}$

For each $I$, choose a r.p.c.d. $\Delta^I$ of $C^I \subset U^I_{\mathbb{R}}$. Define $(D/G^I(\mathbb{Z})^+)_{\Delta^I}$ as the closure of $D/G^I(\mathbb{Z})^+$ in $(D^I/G^I(\mathbb{Z})^+)_{\Delta^I}$.

**Idea:** Construct the quotient by an appropriate equivalence relation on

$$\bigsqcup_{I}(D/G^I(\mathbb{Z})^+)_{\Delta^I}$$

For a $g \in G(\mathbb{Z})^+$ with $gl = J$, we get a map

$$\tilde{g} : D^I/G^I(\mathbb{Z})^+ \to D^J/G^J(\mathbb{Z})^+$$

inducing $g$ on

$$D/G^I(\mathbb{Z})^+ \to D/G^J(\mathbb{Z})^+$$

and hence projects to the **identity** on $D/G(\mathbb{Z})^+$. 
Require that the maps \( \tilde{g} \) extend to maps

\[
(D^I / G^I(\mathbb{Z})^+)_{\Delta I} \rightarrow (D^J / G^J(\mathbb{Z})^+)_{\Delta J}
\]

which is equivalent to the conditions:

- \( \Delta^I \) is invariant under \( P^I(\mathbb{Z}) = \{ g \in G(\mathbb{Z}) \mid gI = I \} \) (which boils down to invariance under \( \text{Res}_{F|Q}(\mathbb{Z}) = \mathcal{O}_F^* \))

- \( \{\Delta^I\} \) is determined by the choice of \( \Delta_{I_k} \) for representatives \( \{I_k\} \) of the ideal classes of \( F \).
We define the following equivalence relation on $\coprod (D/G^I(\mathbb{Z})^+)_{\Delta^I}$: $x_I \sim y_J$ if

- $x_I$ and $y_J$ are in the image of the same element $z \in D$ or
- $x_I = \tilde{g} y_J$ for an element $g \in G(\mathbb{Z})^+$ with $gl = J$.

**Theorem (Hirzebruch, Mumford)**

If each $\Delta^I$ is smooth, the quotient $(D/G(\mathbb{Z})^+)_{\Delta}$ of this equivalence relation is a smooth compact analytic orbifold.

By introducing levels and requiring the $\Delta^I$ to be projective, one gets smooth projective complex varieties.

**Remark:** The map $(D/G^I(\mathbb{Z})^+)_{\Delta^I} \to (D/G(\mathbb{Z})^+)_{\Delta}$ factors through $(D/G^I(\mathbb{Z})^+)_{\Delta^I}/(P^I(\mathbb{Z})/G^I(\mathbb{Z})^+) \simeq (D/G^I(\mathbb{Z})^+)_{\Delta^I}/O^*_F$. 
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Hilbert modular varieties

$S$ a scheme over $\text{spec}(\mathbb{Q})$

$$X(S) = \left\{ \begin{array}{ll}
A & \mathcal{O}_F\text{-abelian scheme over } S \\
\rho : & \text{Hom}^{\text{sym}}_{\mathcal{O}_F}(A, t^A) \to \mathcal{L}_S \text{ } \mathcal{O}_F\text{-iso. of etale sheaves} \\
& \text{mapping polarizations to totally positive elements} 
\end{array} \right\}$$

defines a Deligne-Mumford stack over $\mathbb{Q}$ with an isomorphism

$$X(\mathbb{C}) \to D / G(\mathbb{Z})^+$$

Recipe: Pullback the natural $\mathcal{O}_F$-Hodge structure on $H^1_{dR}(A)$ along

$$V_{\mathbb{C}} \xrightarrow{\beta_{\mathbb{C}}} H_1(A, \mathbb{Z}) \otimes \mathbb{C} \xrightarrow{\text{period}} H^1_{dR}(A),$$

where $\beta : V \to H_1(A, \mathbb{Z})$ is an isomorphism compatible with $\mathcal{O}_F$-action and polarization.

Note: $\beta$ is precisely determined up to $G(\mathbb{Z})^+$. 
A simple kind of one-motives

**Definition**

An one-motive $M$ of dimension $(n, 0, n)$ is a morphism $\alpha : X \to T$ from a locally constant etale sheaf of lattices $X$ of dimension $n$ to a torus $T$ of dimension $n$.

Idea: Understand degenerating abelian varieties by representing them as quotients $T/\alpha(Y)$ where $\alpha$ is (infintesimally) close to the boundary of $T$ (which we understand).

Define $^t M := (\alpha' : X^*(T) \to Y'^* \otimes \mathbb{G}_m)$.

Morphisms are commutative diagrams:

```
\begin{array}{c c c}
Y & \longrightarrow & Y' \\
\alpha \downarrow & & \alpha' \downarrow \\
T & \longrightarrow & T'
\end{array}
```
A simple kind one-motives over \( \mathbb{C} \)

Define (\( S = \text{spec}(\mathbb{C}) \)):

\[
0 \longrightarrow H_1(T, \mathbb{Z}) \longrightarrow H_1(M, \mathbb{Z}) \xrightarrow{\pi} Y \longrightarrow 0
\]

\[
0 \longrightarrow H_1(T, \mathbb{Z}) \longrightarrow \text{Lie}(T) \xrightarrow{\exp} T(\mathbb{C}) \longrightarrow 0
\]

Define (\( S \) arbitrary):

\[
H_1^{dR}(M) := \text{Lie}(T) \oplus Y \otimes \mathbb{Z} \mathcal{O}_S
\]

Have an isomorphism (\( S = \text{spec}(\mathbb{C}) \)):

\[
\text{period} : H_1(M, \mathbb{Z}) \otimes \mathbb{C} \rightarrow H_1^{dR}(M)
\]

\[
\gamma \mapsto (\omega \mapsto \int_{\gamma} \omega, \pi(\gamma)))
\]
“mixed” Hilbert modular varieties

Choose \( I \subset V \).

\[
X^I(S) = \left\{ \begin{array}{l}
M : \mathcal{O}_F\text{-one-motive over } S \\
\rho : \text{Hom}^{\text{sym}}_{\mathcal{O}_F}(M, tM) \to \mathcal{L} \quad \mathcal{O}_F\text{-iso. of etale sheaves} \\
\iota : I_S \to Y \quad \mathcal{O}_F\text{-iso. of etale sheaves}
\end{array} \right\} / \text{iso.}
\]

defines a split torus over \( \mathbb{Q} \) (with cocharacter group \( U^I \)) with isomorphism:

\[
X^I(\mathbb{C}) \to D^I / G^I(\mathbb{Z})^+
\]

Recipe: Pullback the natural mixed \( \mathcal{O}_F \)-Hodge structure along:

\[
V_{\mathbb{C}} \xrightarrow{\beta_{\mathbb{C}}} H^1(M, \mathbb{Z}) \otimes \mathbb{C} \xrightarrow{\text{period}} H^1_{dR}(M)
\]

where \( \beta : V \to H_1(M, \mathbb{Z}) \) is an iso. compatible with \( \mathcal{O}_F \)-action and respecting the subspaces \( I \) pointwise.

Note: \( \beta \) is precisely determined up to \( G^I(\mathbb{Z})^+ \).
Comparison over $\mathbb{C}$

Over $\mathbb{C}$, given $(M = (\alpha : Y \to T), \rho)$, we can define

$$A := T(\mathbb{C})/\alpha(Y)$$

Then $(A, \rho')$ is a polarized $\mathcal{O}_F$-abelian variety precisely if

$$(*) \ (M, \rho) \in D^l/G^l(\mathbb{Z})^+ \text{ lies actually in } D/G^l(\mathbb{Z})^+$$

Under this condition, the map “forget the weight filtration”:

$$D^l/G^l(\mathbb{Z})^+ \supset D/G^l(\mathbb{Z})^+ \to D/G(\mathbb{Z})^+$$

maps $(M, \rho)$ to $(A', \rho')$.

We saw: $(*)$ is satisfied, if $(A, \rho)$ is close enough to the boundary in $X^l(\mathbb{C})_{\Delta'}$. 
Algebraic comparison

We can apply the “torus embedding” functor to the algebraic torus $X^I$ to get a torus embedding $X^I \to X^I_{\Delta^I}$ even defined over $\mathbb{Q}$.

- $R$ a complete discrete valuation ring ($\mathbb{Q}$-algebra)
- $K$ quotient field
- $x = (M, \rho, \iota)$ a point in $X^I(K)$ which does not extend to $R$

(**) The corr. point extends to the partial compactification $(X^I)_{\Delta^I}$.

This is the obvious algebraic analogue of (*)

**Theorem (Mumford)**

\[
\begin{cases}
  (A, \rho) \in X(K) \text{ extending to} \\
  \text{an } \mathcal{O}_F\text{-semi-abelian scheme over } R \\
  \text{with } A_{R/I} \cong \mathbb{G}_m \otimes I
\end{cases}
\quad \overset{\sim}{\Rightarrow} \quad
\begin{cases}
  (M, \rho', \iota) \in X^I(K) \\
  \text{s. t. (**)} \text{ is satisfied}
\end{cases}
\]

This construction is compatible with the complex map $D \subset D^I$, e.g. if $R = \text{spec}(\mathbb{C}[[X]])$, $I = (X)$ and the map giving $x = (M, \rho, \iota)$ over $R$ converges on $B^*_1(0)$. 
Theorem (Mumford, Rapoport)

The previous construction (for more general complete rings) can be used to glue an algebraic model $X_\Delta$ of $(D/G(\mathbb{Z})^+)\Delta$ such that there are isomorphisms $\hat{X}'_{\Delta'} \rightarrow \hat{X}_\Delta$ of the formal completions along corresponding boundary strata. Over $\mathbb{C}$ and in the interior the formal isomorphisms converge locally and give just the map $D/G'(\mathbb{Z})^+ \rightarrow D/G(\mathbb{Z})^+$. 