Stratifications of moduli spaces and p-adic uniformization

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§9. Why are they important?

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§1. How it all began: variation of zeta functions

$X/\mathbb{F}_q$ smooth and proper

$$\zeta_X(t) = \exp\Big(\sum_{n=1}^{\infty} \# X(\mathbb{F}_{q^n}) \cdot \frac{t^n}{n}\Big)$$ (the zeta fc'n)

It has a cohomological interpretation:

$$\zeta_X(t) = \prod_i \det(1 - t\, F_q \mid H^i(X))^{(-1)^{i+1}},$$

where F_q is the q-th power Frobenius and $H^i(X)$ is a Weil cohomology. For example:

$H^i(X) = H^i(X \times_{\mathbb{F}_q} \overline{\mathbb{F}}_q, \mathbb{Q}_\ell)$ for $\ell \neq p$, or $H^i_{crys}(X/W(\mathbb{F}_q))$.

By purity: if $P_i(t) = \det(1 - t F_q \mid H^i(X)) = \prod (1 - \alpha_{ij} t)$ then

$$|\alpha_{ij}|_{complex} = q^{i/2}$$ for any cplx abs. value. (Weil number)

Further, Poincaré duality implies q^n/α_{ij} is also an $\alpha_{i'j'}$, hence an alg. integer.

$\Rightarrow$ all ℓ-adic valuations of α_{ij} are 0.

$\Rightarrow$ Study p-adic valuation of α_{ij}.

$\Rightarrow$ Study the p-adic Newton polygon of P_i, $i = 0, 1, \dots, 2 \cdot \dim(X)$.

If X is an abelian variety then as Galois-modules

$$H^i(X) = \Lambda^i H^1(X).$$

$\Rightarrow$ Enough to study the Newton polygon of the characteristic polynomial of

F_q acting on $H^1(X)$, for example

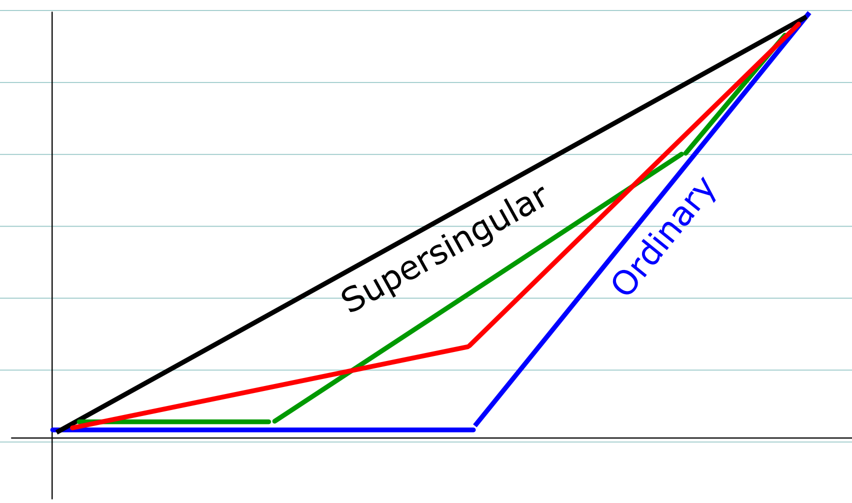
$H^1_{crys}(X/W(\mathbb{F}_q)) =$ Dieudonné module of X.

These Newton polygons are just another way to give the decomposition of the Dieudonné module up to isogeny.

Properties when $X = AV$ of dim'n g:

- Start at $(0,0)$ end at $(2g, g)$.
- Lower convex.
- Integral break points.
- Symmetric $(x \longleftrightarrow 1-x)$

(And vice-versa)



§2. Newton polygon stratification

What we know today for the moduli space $\mathcal{A}_g$ of pr. pol. AV of dim'n g: (Katz, but mostly Oort)

1) Newton polygons "go up" under specialization.

Given a Newton polygon $\mathcal{P}$:

2) There is a locally closed set $W_{\mathcal{P}}$ of $\mathcal{A}_g \otimes \mathbb{F}_p$ whose geometric pts have that Newton polygon.

3) This gives a stratification of $\mathcal{A}_g \otimes \overline{\mathbb{F}}_p$: $\overline{W_{\mathcal{P}}} = \bigcup_{\mathcal{P}' \geq \mathcal{P}} W_{\mathcal{P}'}$.

4) Given a Newton polygon $\mathcal{P}$ and a corresponding locally closed set $W_{\mathcal{P}}$:

$$\dim(W_{\mathcal{P}}) = \#\{(x,y) \in \mathbb{Z}^2,\ y < x \leq g,\ (x,y) \text{ lies above, or on, } \mathcal{P}\}$$

For example: $\dim(\text{ordinary locus}) = \dim(\mathcal{A}_g \otimes \overline{\mathbb{F}}_p) = \sum_{x=1}^{g} \sum_{y=0}^{x-1} 1 = \sum_{x=1}^{g} x = g(g+1)/2$

$\dim(\text{sup. sing.}) = \sum_{x=1}^{g} \sum_{y=\lceil x/2 \rceil}^{x-1} 1 = \frac{g(g+1)}{2} - \sum_{x=1}^{g} \lceil x/2 \rceil = \lfloor g^2/4 \rfloor$

5) $W_{\mathcal{P}}$ is irreducible.

Finer structures:

In a given Newton strata one can distinguish two phenomena:

Let $x \in W_{\underline{P}}$.

1) We have all $y \in W_{\underline{P}}$ s.t. $A_x \sim A_y$. That is, we can find some isogeny b/w $A_x[p^\infty]$ and $A_y[p^\infty]$ (and such exists by definition) that comes from an isogeny of the abelian varieties.

Example: Consider the $\overline{\mathbb{F}}_p$-points of $\mathcal{A}_g^{ord}$. The p-divisible group is in fact always isomorphic to

$$(\mu_{p^\infty})^g \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^g.$$

Since every abelian variety over $\overline{\mathbb{F}}_p$ has complex multiplication by an etale CM algebra E

this is an invariant of the isogeny class of a point. For example, if E is a CM field of degree $2g$ in which p splits completely then such E arizes this way.

2) Oort's central leaf (studied by Chai and Oort)

$$C_x = \{ y \in W_{\underline{p}}(\overline{\mathbb{F}}_p) : A_y[p^\infty] \cong A_x[p^\infty] \}$$

Note a certain "transversality":

- The prime-to-p Hecke orbit is contained in C_x.

<u>Theorem</u> (Chai & Oort): This orbit is <u>dense</u> in C_x.

(Open problem for general Shimura varieties).

- The p-Hecke orbit tends to "head away" from C_x.

More precisely, if we only consider p-isogenies with kernel which is a successive extension of α_p's then Oort proved that

"the central and isogeny leaves almost give a product structure on an irreducible component of a Newton polygon stratum"

5.3. **Theorem** ("central and isogeny leaves almost give a product structure on an irreducible component of a Newton polygon stratum"). *Let $d \in \mathbb{Z}_{\geq 1}$, let ξ be a symmetric Newton polygon, and let $W'' \subset \mathcal{A}_{g,d} \otimes k$ be an irreducible component of the open Newton polygon stratum $\mathcal{W}^0_\xi(\mathcal{A}_{g,d} \otimes k)$. There exist integral schemes T and J of finite type over k and a finite surjective k-morphism*

$$\Phi : \quad T \times J \quad \twoheadrightarrow \quad W'' \subset \mathcal{A}_{g,d} \otimes k$$

such that

$$\forall u \in J(k), \quad \Phi(T \times \{u\}) \quad \textit{is a central leaf in} \quad W'',$$

every central leaf in W'' can be obtained in this way, and

$$\forall t \in T(k), \quad \Phi(\{t\} \times J) \quad \textit{is an isogeny leaf in} \quad W'',$$

and every isogeny leaf in W'' can be obtained in this way.

§3 Ekedahl-Oort Stratification.

This is based on the following observation:

Consider self-dual commutative group schemes of rank p^{2g} over an alg. closed field k of char. p. There are precisely 2^g such group schemes and they can be classified combinatorially in a way independent of k.

Let G be such a group scheme. First form its canonical filtration.

It consist of all the subgroups of G one can form using V and $\perp$. Here

$$V : G^{(p)} \longrightarrow G \qquad \text{Verschiebung}$$

For example:

$$V(G) \quad \text{means} \quad V(G^{(p)})$$

$$VV(G) \quad \text{means} \quad V^2(G^{(p^2)})$$

$$V\perp V(G) \quad \text{means} \quad V\Big(\big(\perp V(G^{(p)})\big)^{(p)} \Big)$$

The key lemma is that this gives a filtration on G, the canonical filtration, that can be refined (non-canonically) to a final filtration :

$$G = G_{2g} \supset G_{2g-1} \supset \dots \supset G_1 \supset G_0 = \{0\}, \qquad \mathrm{rank}(G_i) = p^i, \quad \perp G_i = G_{2g-i}$$

and the filtration is <u>stable</u> <u>under</u> <u>V</u>.

final sequence:

$$\Psi : \{0,\dots,2g\} \longrightarrow \{0,\dots,2g\}, \quad V(G_j) = G_{\Psi(j)}.$$

Have: $\Psi(0)=0, \quad \Psi(i) \leq \Psi(i+1) \leq i+1$

$\Psi(i+1)=\Psi(i)+1 \iff \Psi(2g-i) = \Psi(2g-(i+1))$ $\Big\}$ "admissible"

Lemma (EO): Ψ is independent of the choice of final filtration. It determines the isomorphism class of G. In particular:

$$f(G) = \log_p \# G(k) = \max\{ i : 0 \le i \le g, \Psi(i) = i \}$$

$$a(G) = g - \Psi(g).$$

There are precisely 2^g such Ψ, as they are determined by the gaps

$\Psi(1) - \Psi(0), \quad \Psi(2) - \Psi(1), \quad \ldots, \quad \Psi(g) - \Psi(g-1)$ that belong to $\{0,1\}$.

Theorem (EO): For an admissible Ψ there is a locally closed set EO_Ψ whose geometric pts. are $x \in \mathcal{A}_g(k)$, $A_x[p]$ has final sequence Ψ.

- The sets EO_Ψ form a stratification of $\mathcal{A}_g \otimes \mathbb{F}_p$.
- Each strata is quasi-affine.
- $\dim(EO_\Psi) = |\Psi| := \sum_{i=1}^{g} \Psi(i)$

Examples:

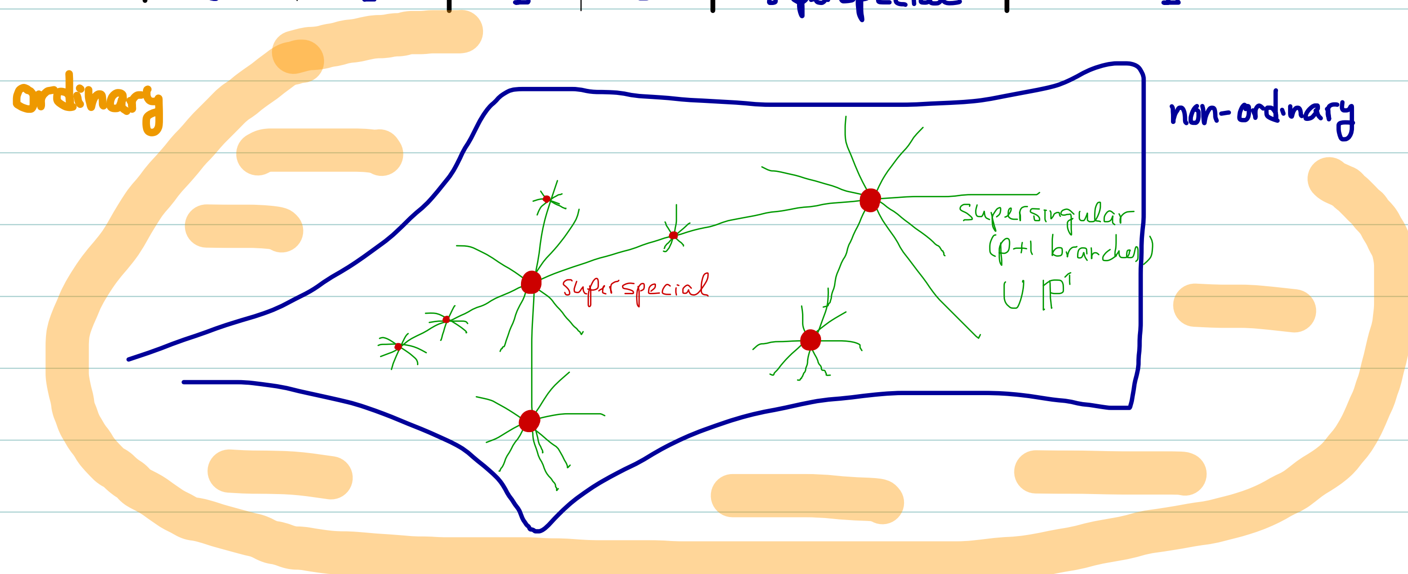
$\boxed{g=1}$

$\mu_p \oplus \mathbb{Z}/p\mathbb{Z}$ $\quad$ $\Psi(1) = 1$ $\quad$ (ordinary)

M, $0 \to \alpha_p \to M \to \alpha_p \to 0$ (non-split) $\quad$ $\Psi(1) = 0$ $\quad$ (supersingular)

$\boxed{g=2}$

group	$\Psi(1)$	$\Psi(2)$	a	f	name	dimension
$\mu_p^2 \oplus \mathbb{Z}/p\mathbb{Z}^2$	1	2	0	2	ordinary	3
$\mu_p \oplus \mathbb{Z}/p\mathbb{Z} \oplus M$	1	1	1	1	non-ordinary	2
$M \oplus M$	0	0	2	0	supersingular	0
H	0	1	1	0	superspecial	1



§4. Tautological classes

Let $\mathcal{A}_g^\dagger$ be a smooth toroidal compactification,

$$\mathcal{X} \underset{e}{\overset{\pi}{\rightleftarrows}} \mathcal{A}_g^\dagger \qquad \text{"universal semi-abelian variety"}$$

Let

$$E := \underline{\omega}_{\mathcal{X}/\mathcal{A}_g^\dagger} := e^* \Omega^1_{\mathcal{X}/\mathcal{A}_g^\dagger} \qquad \text{(rank } g \text{ vector-bundle on } \mathcal{A}_g^\dagger\text{)}$$

⌜Modular forms of wt. k defined over R: $H^0(\mathcal{A}_g^\dagger \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,R, (\det E)^{\otimes k})$.⌟

Let

$$\lambda_i = i\text{-th Chern class of } E, \quad \lambda_i \in CH^i(\mathcal{A}_g^\dagger).$$

Van der Geer proved that $\overline{EO}_\psi \in \mathbb{Q}[\lambda_1, \ldots, \lambda_g]$. The formulas are complicated, but, for example:

$$\text{Non-ordinary locus} = \overline{EO}_{(1,2,\ldots,g-1,g-1)} = (p-1)\lambda_1.$$

$$\text{Super Special locus} = \overline{EO}_{(0,0,\ldots\ldots,0)} = \prod_{i=1}^{g} (p^i + (-1)^i) \cdot \lambda_1 \lambda_2 \cdots \lambda_g.$$

§ Going deeper: p^n-torsion.

Fact: $\exists\ \infty$ many isomorphism classes among the groups $A_x[p^2]$, $x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$, when $g > 2$.

Let $\mathcal{G}$ be a p-divisible group over $k = \bar{k}$, $\mathrm{char}(k) = p$. Consider its Newton polygon as a function

$$\nu : [0, \dim(\mathcal{G}) + \dim(\mathcal{G}^t)] \longrightarrow \mathbb{R}$$

Following Lau-Nicole-Vasiu define:

(i) The isomorphism number $n_{\mathcal{G}}$: the minimal integer n such that if $\mathcal{G}[p^n] \cong \mathcal{H}[p^n]$ where $\mathcal{H}$ is a p-div. gp, then $\mathcal{G} \cong \mathcal{H}$.

(ii) $b_{\mathcal{G}}$ (the "isogeny cutoff" number) is the minimal n s.t. $\mathcal{G}[p^n]$

determines the isogeny class of $\mathcal{G}$. Namely, first n such that

$$\mathcal{G}[p^n] \cong \mathcal{H}[p^n] \Rightarrow \mathcal{G} \sim \mathcal{H}.$$

It was implicit already in Manin's paper that $n_{\mathcal{G}}$ and $b_{\mathcal{G}}$ are finite.

<u>Theorem</u>: $b_{\mathcal{G}} \leq \begin{cases} \nu(\dim(\mathcal{G})) + 1 & \text{if } (\dim \mathcal{G}, \nu(\dim \mathcal{G})) \text{ is a breakpoint of } \nu. \\ \lceil \nu(\dim \mathcal{G}) \rceil & \text{Else.} \end{cases}$

<u>Theorem</u>: If $\mathcal{G}$ is not ordinary, $n_{\mathcal{G}} \leq \lfloor 2\nu(\dim \mathcal{G}) \rfloor$.
If $\mathcal{G}$ is ordinary, $n_{\mathcal{G}} = 1$.

§ Formal schemes and completions of moduli spaces.

Let $X = \operatorname{Spec} A$ be an affine scheme

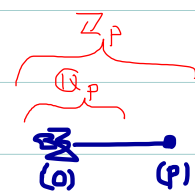
Let $Y \subseteq X$ be a closed subscheme defined by an ideal I.

$X^{\wedge Y}$ = formal completion of X along Y is

Y with the sheaf $\varprojlim \mathcal{O}_X / I^n$.

<u>Example</u>:

(i) $\mathbb{Z}_p$ as a scheme



$\mathbb{Z}_p$ as a formal scheme

(ii) $X = \mathbb{A}^1_{\mathbb{Z}_p}$

$Y = \mathbb{A}^1_{\mathbb{F}_p}$ $\quad I = (p)$ $\quad X^{\wedge Y} = \mathbb{A}^1_{\mathbb{F}_p}$ with sheaf

$$\varprojlim \mathbb{Z}_p[x]/(p^n) = \left\{ \sum_{i=0}^{\infty} a_i x^i : \underset{\mathbb{Z}_p}{\overset{\cap}{a_i}} \xrightarrow{i} 0 \right\} \subseteq \mathbb{Q}_p[[x]]$$

"a unit ball" in the space of functions converging on the closed unit disc.

$Y = 0 \in \mathbb{A}^1_{\mathbb{F}_p}$ $\quad I = (p, x)$ $\quad X^{\wedge Y}$ = single point with sheaf

$$\varprojlim \mathbb{Z}_p[x]/(p,x)^n = \left\{ \sum_{i=0}^{\infty} a_i x^i : a_i \in \mathbb{Z}_p \right\} \subseteq \mathbb{Q}_p[[x]]$$

functions converging on the open unit disc.

Morphisms of formal schemes are morphisms of locally ringed spaces.

<u>Note:</u> Schemes $\subseteq$ Formal schemes.

Back to general setting: X, Y, I.

$X \longleftrightarrow$ functor of points $\mathcal{Y}_X$: $\mathcal{Y}_X(S) = Mor_{Sch}(S, X)$

$X^{\wedge Y} \longleftrightarrow$ functor of points $\hat{\mathcal{Y}}_{X^{\wedge Y}}$: $\hat{\mathcal{Y}}_{X^{\wedge Y}}(S) = Mor_{FSch.}(S, X^{\wedge Y})$.

But $\hat{\mathcal{Y}}$ is determined by its value on "artinian points". For example,

if $y_0 \in Y$, R a local artinian ring s.t. $R/m_R = k(y_0)$, we have the artinian points

in $Mor_{Sch}(\mathrm{Spec} R, X)$ whose images are y_0. But $\mathrm{Spec} R$ is canonically a formal scheme because $R = \varprojlim_n R/\mathfrak{m}_R^n$, and so these points can also be viewed as in the set $\mathrm{Mor}_{FSch.}(\mathrm{Spf}\, R, X^{\wedge Y})$.

That is, the functor $\hat{\mathcal{Y}}$ is at least <u>approximated</u> by known values.

§7. The Drinfeld case.

Let $K \supseteq \mathbb{Q}_p$ be a complete discretely valued field. π a uniformizer. Let

$$\mathcal{C} \xrightarrow{f} \operatorname{Spec} \mathcal{O}_K$$

a relative curve: f is proper, flat, with geometry connected reduced fibers of dimension 1. And assume that $\mathcal{C} \bmod \pi$ is a union of irreducible components crossing transversely, each isomorphic to $\mathbb{P}^1$. Then, there exist $\Gamma \subseteq PGL_2(K)$ such that, for $\Omega^2 := \mathbb{P}^2(\hat{\mathbb{C}}_p) \setminus \mathbb{P}^2(K)$ (Drinfeld upper ½ space)

$$\mathcal{C}^{an} \cong \Gamma \backslash \Omega^2.$$

Let Δ be a quaternion algebra over $\mathbb{Q}$, T its set of ramified primes.

Let L be a maximal order. Let $*$ be the canonical involution and

$\delta = \prod_{p \in T} p$ the discriminant. There's an element $a \in L$ such that $a^2 = -\delta$.

Let $\sigma(x) = a x^* a^{-1}$ (a positive involution on Δ).

The moduli problem of abelian surfaces with L action and pr. pol'n inducing

σ, and with full level $n \geq 3$ (and satisfying a Kottwitz condition)

is representable by a curve

$$S_n \longrightarrow \operatorname{Spec} \mathbb{Z}[1/n].$$

Čerednik proved that $S_n^{\wedge p} = S_n(\hat{\mathbb{C}}_p)$ has p-adic uniformization for $p \in T$.

In particular, that $S_n \otimes \mathbb{F}_p$ is a transversal union of $\mathbb{P}^1$'s.

Moreover, let Δ' be the quaternion algebra over $\mathbb{Q}$ with ramification

$$T \cup \{\infty\} - \{p\}.$$

If $x \in S_n(\overline{\mathbb{F}}_p)$ then A_x is supersingular and

$$\Delta' \cong \mathrm{End}_L(A_x) \otimes \mathbb{Q}.$$

Fix isomorphisms

$\Delta' \otimes \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$, $\Delta' \otimes \mathbb{A}^{f,p} \cong \Delta \otimes \mathbb{A}^{f,p}$ by means of which $\prod_{\ell \neq p} (L \otimes \mathbb{Z}_\ell) \subseteq \Delta'(\mathbb{A}^{f,p})$.

Let $U_n \subset \prod_{\ell \neq p} L \otimes \mathbb{Z}_\ell$ be the principal level n subgroup. Then

$$S_n^{\wedge p} \cong GL_2(\mathbb{Q}_p) \backslash (\Omega^2 \times X_n),$$

where,

$$X_n = U_n \backslash \Delta'(\mathbb{A}^f)^\times / \Delta'^\times.$$

A key point in Drinfeld's arguments is to show that every point $x \in S_n(\overline{\mathbb{F}}_p)$ is supersingular and $S_n(\overline{\mathbb{F}}_p)$ is a single isogeny class (even with pol'n and L-action). Using Serre-Tate the infinitesimal nbhds of the special fiber can be studied via p-div. groups and those via formal $\mathcal{O}$-modules, where $\mathcal{O} = L \otimes \mathbb{Z}_p$. These are modules over the Cartier ring; in char. p they are just Dieudonné modules.

The proof consists in a careful analysis of which modules arise, and then that $GL_2(\mathbb{Q}_p)\backslash \Omega^2 \times X_n$ is a parameter space to those.

What enabled Drinfel'd to "pull that off" is – besides being Drinfeld – that the parameter space is one-dimensional and that the formal group behaves much like a 1-dim'l formal group. This cannot be expected in general and passing to higher dimensional settings one gives up an explicit description of the deformation of the p-divisble group, except over very special rings giving 1^{st} order deformations. Over these, the technique of local models allows comparison with incidence schemes.

§8 Rapoport-Zink spaces.

One considers a Shimura variety S of PEL type. Say $S / \mathrm{Spec}\, \mathcal{O}$, where $\mathcal{O}$ is the localization of a ring of integers of a # field K. We may assume S has good reduction everywhere. Let $\mathfrak{p}$ be a prime of $\mathcal{O}$.

S mod $\mathfrak{p}$ has a generalized Newton stratification. Let $Y \subseteq S$ mod $\mathfrak{p}$ be the basic stratum. Consider $S^{\wedge Y}$.

Example: $S = \mathcal{A}_{g,n}$ = moduli of ppav with full-level n structure.

Y = supersingular locus.

RZ provided a description of $S^{\wedge Y}$ as pro-representing the following functor. Let E be the reflex field and $E_{\mathfrak{p}}$ its completion at $\mathfrak{p}$.

<u>Theorem</u>: There is an isomorphism of formal schemes over $\mathrm{Spf}\, \mathcal{O}_{E_{\mathfrak{p}}}$,

$$I(\mathbb{Q}) \backslash [\mathcal{M} \times G(\mathbb{A}^{f,p}) / C^p] \cong S^{\wedge Y}$$

<u>Here:</u> $G/\mathbb{Q}$ the reductive group connected to S.

C^p the appropriate level structure away from p.

I an inner form of G, $I(\mathbb{Q})$ = q. isogenies of some fixed $(A_0, \lambda_0) \in S(\overline{\mathcal{O}_K/\mathfrak{p}})$.

<u>Note</u>: $I(\mathbb{Q}) \to G(\mathbb{A}^{f,p})$ by construction. We shall have $I(\mathbb{Q}) \to J(\mathbb{Q}_p) \circlearrowright \mathcal{M}$.

The formal scheme $\mathcal{M}$:

The fixed (albeit arbitrary) object $(A_0, \lambda_0) \in S(\overline{\mathbb{Q}}_{K_p})$ has a p-divisible group $\mathbb{X}$, endowed with a polarization and endomorphism data.

Let $\breve{E} = E_p^{ur}$ and let

$\mathrm{Nilp}_{O_{\breve{E}}}$ = category of locally noetherian $O_{\breve{E}}$ schemes in which p is locally nilpotent.

For a scheme $B \in \mathrm{Nilp}$ let $\bar{B} = B \bmod p$.

Consider the functor $B \in \mathrm{Nilp} \longmapsto$ isomorphism classes of

a p-div. gp X with Pol + endo' data and with $\rho : \mathbb{X} \times_{\mathbb{F}_p} \bar{B} \longrightarrow X \times_B \bar{B}$

(subject to certain regularity conditions).

This functor is pro-representable by $\mathcal{M}$.

In the theorem:

$$I(\mathbb{Q})\backslash[\mathcal{M} \times G(\mathbb{A}^{f,p})/C^p] \cong S^{\wedge Y}$$

The l.h.s. is in fact a finite disjoint union

$$\Gamma_i \backslash \mathcal{M}$$

where Γ_i are discrete co-compact-modulo-center subgroup of $J(\mathbb{Q}_p)$

where J is the automorphism group scheme of $\mathbb{X}$ with its add'l structure.

This is a great result, but what can we say about $\mathcal{M}$??

To study $\mathcal{M}$ one introduces a **period morphism**

$$\boxed{\check{\pi} : \mathcal{M}^{rig} \longrightarrow \mathcal{F}^{rig}}$$

where $\mathcal{M}^{rig}$ is the rigid analytic space associated to formal scheme $\mathcal{M}$ (morally: $\mathcal{M}^{rig} = \mathcal{M}(\hat{\mathbb{C}}_p)$) and $\mathcal{F}^{rig}$ is (in the case we discuss) the Grassmann variety for max'l isotropic spaces of "$\mathbb{Q}_p^{2g}$".

The way $\check{\pi}$ is constructed is as follows: one proves

$$E(\mathbb{X}) \otimes_{W(\bar{\mathbb{F}}_p)} \mathcal{O}_{\mathcal{M}^{rig}} \cong (M_{X^{univ}})^{rig}$$

the tangent space to the "universal vectorial extension" (becomes de Rham cohomology when looking at abelian varieties)

Dieudonné module of our fixed p-div. group $\mathbb{X}$

sheaf of regular functions

the univ. deformation of $\mathbb{X}$

Then, the surjection

$$M_{X^{univ}} \longrightarrow \underline{\mathrm{Lie}}\, X^{univ}/\mathcal{M}$$

has a kernel that "varies along $\mathcal{M}$". Therefore, it is an $\mathcal{M}$-valued point

of a Grassmannian $\mathcal{F}$, that is, a morphism

$$\check{\pi} : \mathcal{M}^{rig} \longrightarrow \mathcal{F}^{rig}$$

RZ prove $\check{\pi}$ is <u>étale</u> (that means that locally formally the deformations

of p-div. gps are the same as deforming the Hodge filtration, and in fact, this

<u>theorem</u> of Grothendieck is <u>used</u> in the proofs).

They also determine the fibers of $\check{\pi}$.

The main problem is then to characterize the image of $\check{\pi}$.

<u>Conjecture</u> (Fontaine): the image is the (weakly) admissible points.

Without going into details now, this is a condition on which subspaces of K^{2g} ($K \supseteq \mathbb{Q}_p$) can arise via the Hodge filtration once we make K^{2g} into an isocrystal (the structure is induced from the Dieudonné isocrystal of $\mathbb{X}$). That is, it is a conjecture about which p-divisible groups with supersingular (in general, <u>basic</u>) reduction may arise over $\mathcal{O}_K$.

This is now known by works of Colmez, Fontaine, Faltings.

Last comment: This has a lot to do with Galois representations.

A p-divisible group over a char. 0 field K is <u>étale</u>. Thus,

To give a p-divisible group G of height h, is the same as to give

a representation

$$\mathrm{Gal}_K \longrightarrow \varprojlim_n GL_h(\mathbb{Z}/p^n\mathbb{Z}) = GL_h(\mathbb{Z}_p)$$

Thus, ultimately, determining the image of $\check{\chi}$ is an assertion about

<u>p-adic representations</u> of Gal_K. (see, in this context, Hartl)

§9. Why are Rapoport-Zink spaces important?

a (Obviously) they are a way to understand part of the mod p and p-adic geometry of moduli spaces.

b The formal scheme $\mathcal{M}^{rig}$ comes with an intrinsic tower of étale covers. These covers come from putting a level structure on X^{rig} (the generic fibre of the universal object).

Analogy(?)

$$\begin{array}{ccc} E^{univ} & & \Gamma(n)\backslash\mathfrak{h} =: \mathfrak{h}_{\Gamma(n)} \\ \downarrow & \swarrow & \\ \Gamma(1)\backslash\mathfrak{h} & & \end{array} \qquad \begin{array}{ccc} X^{rig} & & \mathcal{M}_T^{rig} \\ \downarrow & \swarrow & \\ \mathcal{M}^{rig} & & \end{array}$$

$$\varinjlim_n H^i_c(\overline{\mathfrak{h}}_{\Gamma(n)}, \mathbb{Q}_\ell) \qquad \varinjlim_T H^i_c(\overline{\mathcal{M}}_T^{rig}, \mathbb{Q}_\ell)$$

$\circlearrowleft Gal_{\mathbb{Q}}$ $\circlearrowleft GL_2(\mathbb{A}^f)$ $\circlearrowleft Gal_K$ $\circlearrowleft G(\mathbb{Q}_p)$ $\circlearrowleft J(\mathbb{Q}_p)$ $(K = E_{\mathfrak{p}})$

The cohomology $\varinjlim_{T} H^i_c(\bar{\mathcal{M}}^{rig}_T, \mathbb{Q}_\ell)$ carries an action of:

Gal_K, $J(\mathbb{Q}_p)$ and $G(\mathbb{Q}_p)$.

* The action of $G(\mathbb{Q}_p)$ is induced from its action on the tower $\{\mathcal{M}^{rig}_T\}_T$ as Hecke correspondences.

* The action of $J(\mathbb{Q}_p)$ is through its action on the functor of deformations:

$$\rho : \mathbb{X} \otimes_{W(k)} \bar{S} \longrightarrow X \times_S \bar{S} \quad \rightsquigarrow \quad \rho \circ \gamma : \mathbb{X} \otimes_{W(k)} \bar{S} \longrightarrow X \times_S \bar{S}.$$

* As $\bar{\mathcal{M}}^{rig}_T = \mathcal{M}^{rig}_T \otimes_K \bar{K}$, Gal_K acts as usual on étale cohomology.

Very broadly speaking, this cohomology groups are conjectured to realize the local Langlands correspondence between Gal_K and $G(\mathbb{Q}_p)$.