# Hilbert Modular Varieties - an Introduction

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## The Algebraic Group

$L$ totally real, $[L : \mathbb{Q}] = g$,

$\mathbb{B} = \mathrm{Hom}(L, \mathbb{R}) = \{\sigma_1, \ldots, \sigma_g\}$.

$\mathbf{G} = \mathrm{Res}_{L/\mathbb{Q}} \mathrm{GL}_2$. For any $\mathbb{Q}$-algebra $R$:

$$\mathbf{G}(R) = \mathrm{GL}_2(L \otimes_{\mathbb{Q}} R).$$

In particular,

$$\mathbf{G}(\mathbb{Q}) = \mathrm{GL}_2(L), \qquad \mathbf{G}(\mathbb{R}) = \prod_{\sigma \in \mathbb{B}} \mathrm{GL}_2(\mathbb{R}).$$

Here,

$$\gamma \in \mathrm{GL}_2(\mathrm{L}) \mapsto (\sigma(\gamma))_{\sigma \in \mathbb{B}}.$$

Similarly, we have

$$\mathbf{G}' = \mathrm{Res}_{L/\mathbb{Q}} \mathrm{SL}_2.$$

# Subgroups from lattices

Let $\mathfrak{a}, \mathfrak{b}$, be fractional ideals of $L$. Define the subgroup $\mathrm{GL}(\mathfrak{a} \oplus \mathfrak{b})$ of $\mathbf{G}(\mathbb{Q})$ as the matrices

$$\left\{ \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) : a, d, \in \mathcal{O}_L, b \in \mathfrak{a}^{-1}\mathfrak{b}, c \in \mathfrak{a}\mathfrak{b}^{-1}, ad - bc \in \mathcal{O}_L^\times \right\}.$$

This group stabilizes the lattice $\mathfrak{a} \oplus \mathfrak{b}$ in $L \oplus L$ under right multiplication. Similarly, for $\mathbf{G}'$.

Let $\mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. We have

$$h_0 : \mathbb{S} \to \mathbf{G}, \quad x + iy \mapsto \left( \left( \begin{smallmatrix} x & -y \\ y & x \end{smallmatrix} \right), \dots, \left( \begin{smallmatrix} x & -y \\ y & x \end{smallmatrix} \right) \right).$$

The stabilizer of $h_0$ under conjugation by $\mathbf{G}$ is the maximal compact group

$$K_\infty = \left\{ \left( \left( \begin{smallmatrix} x_\sigma & -y_\sigma \\ y_\sigma & x_\sigma \end{smallmatrix} \right) \right)_\sigma : x_\sigma^2 + y_\sigma^2 \neq 0, \forall \sigma \right\}.$$

## The symmetric space

$\mathbf{G}(\mathbb{R}) \cong \prod_{\sigma \in \mathbb{B}} \mathrm{GL}_2(\mathbb{R})$ acts on the symmetric space $(\mathfrak{H}^{\pm})^g$, where $\mathfrak{H}^{\pm} = \{z \in \mathbb{C} : \pm \mathrm{Im}(z) > 0\}$. A component of which is

$$\mathfrak{H}^g = \{(\tau_1, \ldots, \tau_g) : \mathrm{Im}(\tau_i) > 0\}.$$

The action of $\mathbf{G}(\mathbb{Q})$ is by

$$\gamma * (\tau_1, \ldots, \tau_g) = (\sigma_1(\gamma)\tau_1, \ldots, \sigma_g(\gamma)\tau_g)$$

(in each component it is the usual action by fractional linear transformations). Note:

$$\mathrm{Stab}_{\mathbf{G}(\mathbb{R})}(i, \ldots, i) = K_\infty, \qquad \mathbf{G}(\mathbb{R})/K_\infty \cong (\mathfrak{H}^{\pm})^g.$$

We can view this space also as the space of conjugates of $h_0$ under $\mathbf{G}(\mathbb{R})$; $\mathbf{G}$ is a reductive connected algebraic group over $\mathbb{Q}$. This is Deligne's perspective.

# Adelic points

From Deligne's perspective we care about

$$\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K_\infty K_f \cong \mathbf{G}(\mathbb{Q})\backslash(\mathfrak{H}^\pm)^g \times \mathbf{G}(\mathbb{A}_f)/K_f.$$

One can show:

$$\mathbf{G}(\mathbb{Q})\backslash\mathbf{G}(\mathbb{A})/K_\infty K_f \cong \cup_{j=1}^m \Gamma_j\backslash\mathfrak{H}^g,$$

where, for suitable $g_j \in \mathbf{G}(\mathbb{A}_f)$,

$$\mathbf{G}(\mathbb{A}) = \cup_{j=1}^m \mathbf{G}(\mathbb{Q})g_j\mathbf{G}(\mathbb{R})^0 K_f, \quad \Gamma_j = g_j\mathbf{G}(\mathbb{R})^0 K_f g_j^{-1} \cap \mathbf{G}(\mathbb{Q}).$$

For example, for $K_f = \mathbf{G}(\hat{\mathbb{Z}})$, we get

$$\bigcup_{[\mathfrak{a}]\in CL^+(L)} \mathrm{GL}(\mathcal{O}_L \oplus \mathfrak{a})^+\backslash\mathfrak{H}^g.$$

# Abelian varieties with RM

$(A, \iota, \lambda)$

- $\diamond$ $A$ $g$-dim'l AV over a base $k$ (field, ring, scheme)
- $\diamond$ $\iota : \mathcal{O}_L \hookrightarrow \mathrm{End}_k(A)$ $\quad (\rightsquigarrow \iota^\vee : \mathcal{O}_L \hookrightarrow \mathrm{End}_k(A^\vee))$
- $\diamond$ $\lambda : A \longrightarrow A^\vee$ an $\mathcal{O}_L$-equiv. polarization: $\lambda \circ \iota(a) = \iota^\vee(a) \circ \lambda$; equivalently, the Rosati involution acts trivially on $\mathcal{O}_L$.

The adelic description suggests a different definition. Fix a fractional ideal $\mathfrak{a}$ of $L$. Look at

$$(A, \iota, \lambda),$$

where

$$\lambda : \mathrm{Hom}_{\mathcal{O}_L, k}(A, A^\vee)^{symm} \xrightarrow{\cong} \mathfrak{a},$$

is an isomorphism of $\mathcal{O}_L$-modules with a notion of positivity (namely, identifies the cone of polarizations with $\mathfrak{a}^+$).

# Analytic families of abelian varieties with RM

Let $\mathfrak{a}, \mathfrak{b}$ be fractional ideals of $L$. For $z \in \mathfrak{H}^g$ embed $\mathfrak{a} \oplus \mathfrak{b}$ in $\mathbb{C}^g$ as a lattice:

$$\Lambda_z = \mathfrak{a} \cdot z + \mathfrak{b} \cdot 1 = \{(\sigma_i(a)z_i + \sigma_i(b))_i : a \in \mathfrak{a}, b \in \mathfrak{b}\}.$$

A polarization on $A_z = \mathbb{C}^g / \Lambda_z$ can be described by an alternating pairing on $\mathfrak{a} \oplus \mathfrak{b}$:

$$E_r((x_1, y_1), (x_2, y_2)) = \mathrm{Tr}_{L/\mathbb{Q}}(r(x_1 y_2 - x_2 y_1)), \quad r \in (\mathcal{D}_{L/\mathbb{Q}} \mathfrak{a}\mathfrak{b})^{-1}.$$

Have

$$\mathrm{Hom}_{\mathcal{O}_L}(A_z, A_z^\vee)^{symm} = (\mathcal{D}_{L/\mathbb{Q}} \mathfrak{a}\mathfrak{b})^{-1}, \ \{\text{Polarizations}\} \leftrightarrow (\mathcal{D}_{L/\mathbb{Q}} \mathfrak{a}\mathfrak{b})^{-1,+}.$$

# Analytic families (cont'd)

**Theorem**

*$\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b}) \backslash \mathfrak{H}^g$ parameterizes isomorphism classes of $(A, \iota, \lambda)$ with $\lambda : \mathrm{Hom}_{\mathcal{O}_L}(A_z, A_z^\vee)^{symm} \to (\mathcal{D}_{L/\mathbb{Q}} \mathfrak{a}\mathfrak{b})^{-1}$ an isomorphism, taking the polarizations to the totally positive elements.*

*$\mathrm{GL}(\mathfrak{a} \oplus \mathfrak{b})^+ \backslash \mathfrak{H}^g$ parameterizes isomorphism classes of $(A, \iota)$ such that there exists an isomorphism $\lambda : \mathrm{Hom}_{\mathcal{O}_L}(A_z, A_z^\vee)^{symm} \to (\mathcal{D}_{L/\mathbb{Q}} \mathfrak{a}\mathfrak{b})^{-1}$, taking the polarizations to the totally positive elements.*

**From now on**

We consider moduli of $(A, \iota, \lambda)$: abelian varieties with RM and a principal $\mathcal{O}_L$-linear polarization $\lambda$. (Corresponds over $\mathbb{C}$ to $\mathrm{SL}(\mathcal{D}_{L/\mathbb{Q}}^{-1} \oplus \mathcal{O}_L)$.)

# Cusps and the compact dual

The groups $\mathrm{GL}_2(L)^+$ acts on $\mathbb{P}^1(L)$ and the orbits of $\mathrm{GL}(\mathfrak{a} \oplus \mathfrak{b})^+$ (or $\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})$) are in bijection with $Cl(L)$. To a point $(\alpha : \beta) \in \mathbb{P}^1(L)$ we associate the ideal class $(\mathfrak{a}, \mathfrak{b}) \cdot^t (\alpha, \beta) = \alpha\mathfrak{a} \oplus \beta\mathfrak{b}$.

The compact dual depends only of $\mathbf{G}'(\mathbb{R}) = \prod_\sigma \mathrm{SL}_2(\mathbb{R})$. It is thus equal to $\mathbb{P}^1(\mathbb{C})^g$. The boundary of $\mathfrak{H}^g$ is thus

$$\cup_{i=1}^{g} \mathbb{P}^1(\mathbb{C})^i \times \mathbb{P}^1(\mathbb{R}) \times \times \mathbb{P}^1(\mathbb{C})^{g-i-1},$$

but the rational boundary components for $\mathbf{G}'$ are precisely

$$\mathbb{P}^1(L) \hookrightarrow \mathbb{P}^1(\mathbb{Q})^g.$$

The minimal (or Bailey-Borel-Satake) compactification of $\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})\backslash\mathfrak{H}^g$ is, set-theoretically,

$$\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})\backslash \left(\mathfrak{H}^g \cup \mathbb{P}^1(L)\right).$$

# Factors of automorphy

For each $\mathbf{k} = (k_1, \ldots, k_g) \in \mathbb{Z}^g$, $\gamma \in \mathrm{SL}_2(L)$ and $z = (z_1, \ldots, z_g) \in \mathfrak{H}^g$,

$$j_{\mathbf{k}}(\gamma, z) := \prod_{i=1}^{g} j(\sigma_i(\gamma), z_i)^{k_i}.$$

For $f : \mathfrak{H}^g \to \mathbb{C}$ holomorphic, let

$$f|_{\mathbf{k}}\gamma = j_{\mathbf{k}}(\gamma, z)^{-1} f(\gamma z).$$

Let $\Gamma \subset \mathrm{SL}_2(L)$ be a congruence subgroup. We say $f$ is a weight $\mathbf{k}$ modular form of level $\Gamma$ if

$$f|_{\mathbf{k}}\gamma = f, \qquad \forall \gamma \in \Gamma.$$

(If $g > 1$ there is no need to require it is holomorphic at infinity (Koecher's principle).)

## Factors of automorphy (2)

The vector valued factor of automorphy

$$\mathrm{diag}(j(\sigma_1(\gamma), z_1), \ldots, j(\sigma_g(\gamma), z_g)),$$

defines a vector bundle over $\Gamma\backslash\mathfrak{H}^g$. It is easy to see from our construction of analytic families $\pi : (A^u, \iota^u, \lambda^u) \to \Gamma\backslash\mathfrak{H}^g$ that it is the relative cotangent space at the identity (the Hodge bundle):

$$\mathbb{E} = \pi_*(\Omega^1_{(A^u,\iota^u,\lambda^u)\to\Gamma\backslash\mathfrak{H}^g}).$$

We have

$$\mathbb{E} = \oplus_i \mathbb{L}_i,$$

where $\mathbb{L}_i$ is defined by the factor of automorphy $j_{e_i}(\gamma, z)$. Bailey-Borel proved that their compactification is an algebraic variety given as $\mathbf{Proj}(\sum_{k=0}^{\infty} \Gamma(\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})\backslash\mathfrak{H}^g, \det(\mathbb{E})^k))$.

# Fourier expansions

The group $\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})$ contains the subgroup

$$\left\{ \begin{pmatrix} \epsilon & b \\ 0 & \epsilon^{-1} \end{pmatrix} : \epsilon \in \mathcal{O}_L^\times, b \in \mathfrak{a}^{-1}\mathfrak{b} \right\}.$$

A modular form $f$ of level $\mathrm{SL}(\mathfrak{a} \oplus \mathfrak{b})$ has then a Fourier expansion

$$f(z_1, \dots, z_g) = \sum_{\nu \in (\mathfrak{a}^{-1}\mathfrak{b})^\vee} a(\nu) q^\nu,$$

$$(\mathfrak{a}^{-1}\mathfrak{b})^\vee = \mathcal{D}_{L/\mathbb{Q}}^{-1}\mathfrak{a}\mathfrak{b}^{-1}, \ \ q^\nu = \exp(2\pi i(\sigma_1(\nu)z_1 + \cdots + \sigma_g(\nu)z_g)).$$

- Holomorphic $\Rightarrow$ $\nu \gg 0$ or $\nu = 0$.
- Action of $\mathcal{O}_L^\times$ $\Rightarrow$

$$a(\nu) = \left( \prod \sigma_i(\epsilon)^{-k_i} \right) \cdot a(\epsilon^2 \nu), \quad \epsilon \in \mathcal{O}_L^\times.$$

(So if $k_1 = \cdots = k_g$ and are even $a(\nu) = a(\epsilon^2\nu)$. Then, if the strict class number of $L$ is 1, $a(\nu)$ depends only on the ideal $\nu$.)

# Moduli problems

Let $\mathfrak{n}$ be an integral ideal of $\mathcal{O}_L$; $(A, \iota)$ an abelian variety with RM. A $\Gamma({}_1\mathfrak{n})$-level structure is a closed immersion $\alpha : \mathcal{O}_L/\mathfrak{n} \hookrightarrow A$.

Consider the moduli of $\underline{A} = (A, \iota, \lambda, \alpha)$, a principally polarized abelian variety $(A, \lambda)$ with RM $\iota$, and a $\Gamma({}_1\mathfrak{n})$-level structure $\alpha$. If $\mathfrak{n}$ is large enough, this is representable by a scheme $\mathscr{M}({}_1\mathfrak{n})$ that is irreducible and smooth over $\mathbb{Z}[\mathrm{Norm}(\mathfrak{n})^{-1}]$. We have

$$\mathscr{M}({}_1\mathfrak{n})(\mathbb{C}) \cong \Gamma({}_1\mathfrak{n})\backslash\mathfrak{H}^g,$$

where $\Gamma({}_1\mathfrak{n})$ are the matrices in $\mathrm{SL}(\mathcal{D}_{L/\mathbb{Q}}^{-1} \oplus \mathcal{O}_L)$ that are upper unipotent modulo $\mathfrak{n}$.

# Modular forms over any base

Let $\pi : \underline{A}^u = (A^u, \iota^u, \lambda^u, \alpha^u) \to \mathrm{Spec}(\mathbb{Z}[\mathrm{Norm}(\mathfrak{n})^{-1}])$ be the universal object. Define the Hodge bundle

$$\mathbb{E} = \pi_*(\Omega^1_{\underline{A}^u/\mathscr{M}({}_1\mathfrak{n})}).$$

This is a rank $g$ vector bundle over $\mathscr{M}({}_1\mathfrak{n})$, which is a rank 1 $\mathcal{O}_L \otimes \mathcal{O}_{\mathscr{M}({}_1\mathfrak{n})}$ vector bundle. And (after base change to $\mathcal{O}_M[\mathrm{Nm}(\mathfrak{n})^{-1}]$, $M$ a normal closure of $L$ in $\mathbb{C}$)

$$\mathbb{E} = \oplus_{i=1}^{g} \mathbb{L}_i,$$

a sum of line bundles such that over $\mathbb{C}$ the line bundle $\mathbb{L}_i$ is defined by the factor of automorphy $j_{e_i}(\gamma, z)$. A modular form of weight $\mathbf{k} = (k_1, \dots, k_g)$ and level $\mathscr{M}({}_1\mathfrak{n})$, defined over a base $S$ (an $\mathcal{O}_M[\mathrm{Nm}(\mathfrak{n})^{-1}]$-scheme), is an element in

$$\Gamma(\mathscr{M}({}_1\mathfrak{n}) \otimes_{\mathcal{O}_M[\mathrm{Nm}(\mathfrak{n})^{-1}]} S, \mathbb{L}_1^{k_1} \otimes \cdots \otimes \mathbb{L}_g^{k_g}).$$

# Modular forms over any base (2)

The $q$-expansion can be defined in this generality. If $S$ is the spectrum of a ring $R$ then a modular form over $S$ has $q$-expansion in

$$\mathbb{Z}[[q^\nu : \nu \in (\mathfrak{a}^{-1}\mathfrak{b})^\vee]] \otimes R.$$

The converse is also true. Further, a modular form is zero if and only if its $q$-expansion is zero; the Galois action can be described via the action on the Fourier coefficients.

In particular, if two modular forms over $R$, of the same weight, have the same Fourier expansion then they are equal. In characteristic zero something stronger is true: if two modular forms over $\mathbb{C}$ have the same Fourier expansion then they are equal. This is because the Fourier expansion converges and describes the form on an open neighborhood of the cusp (and not just in its completed local ring). This is not the case over an arbitrary ring and one arrives at the notion of **congruence**.

# Another view on modular forms and weights

One can also think about a Hilbert modular form $f$ defined over $R$ of weight $\mathbf{k}$ and level $\Gamma({}_1\mathfrak{n})$ as a rule,

$$(\underline{A};\omega) = (A,\iota,\lambda,\alpha;\omega)/R_1 \mapsto f(\underline{A},\omega) \in R_1,$$

for every $R_1$-algebra (where $\omega$ is a generator over $\mathcal{O}_L \otimes_{\mathbb{Z}} R_1$ of $Lie(A)$), that depends only on the isomorphism class of $(\underline{A},\omega)$, commutes with base-change, and satisfies

$$f(\underline{A},r\omega) = (\chi_1^{k_1}\cdots\chi_g^{k_g})(r)\cdot f(\underline{A},r\omega), \qquad r \in (\mathcal{O}_L \otimes R_1)^\times$$

where $\chi_1,\dots,\chi_g$ are the characters of the torus $\mathbf{Res}_{\mathcal{O}_L/\mathbb{Z}}\mathbb{G}_m$.

## Congruences

Let $p$ be a prime that is unramified in $L$ (for simplicity). Then,

$$\mathbb{B} = \cup_{\mathfrak{p}|p} \mathbb{B}_{\mathfrak{p}}, \quad \mathbb{B}_{\mathfrak{p}} = \mathrm{Hom}(L, L_{\mathfrak{p}}) \circlearrowleft \phi, \quad \phi = \mathrm{Frobenius}.$$

Consider modular forms over $\overline{\mathbb{F}}_p$. Given such an abelian variety over $\overline{\mathbb{F}}_p$ there is a duality between $H^0(A, \Omega^1_{A/\overline{\mathbb{F}}_p})$ and $H^1(A, \mathcal{O}_A)$. A basis $\omega$ gives a basis over $\overline{\mathbb{F}}_p$: $\{\omega_\sigma : \sigma \in \mathbb{B}\}$, and a dual basis $\{\eta_\sigma : \sigma \in \mathbb{B}\}$ for $H^1(A, \mathcal{O}_A)$. We define a partial Hasse invariant

$$h_\sigma(\underline{A}, \omega) = \phi(\eta_{\phi^{-1}\circ\sigma})/\eta_\sigma.$$

It is a modular form of weight $\chi^p_{\phi^{-1}\circ\sigma}\chi^{-1}_\sigma$ and has $q$-expansion 1.

**Theorem**

*The kernel of the $q$-expansion map on the graded ring $\oplus_{\mathbf{k}}\Gamma(\mathscr{M}({}_1\mathfrak{n}) \otimes \overline{\mathbb{F}}_p, \mathbb{L}^{\mathbf{k}})$ is the ideal $\langle h_\sigma - 1 : \sigma \in \mathbb{B}\rangle$.*

# $p$-adic Hilbert modular forms

One can develop a theory of $p$-adic Hilbert modular forms as a uniform limit of $q$-expansions of Hilbert modular forms in the $p$-adic metric.

For example, for $g = 1$, if $f \equiv f' \pmod{p^r}$ (integral, normalized) then $k \equiv k' \pmod{(p-1)p^{r-1}}$ and so a $p$-adic limit has weight in the completion of the characters $\mathbb{Z}$ of $\mathbb{G}_m$ at the subgroups $(p-1)p^r\mathbb{Z}$, which is $\mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p$.

Similarly, for Hilbert modular forms, if $f \equiv f' \pmod{p^r}$ (integral, normalized) then $k \equiv k' \pmod{W(r)}$, where $W(r)$ are the characters trivial on $\mathbf{Res}_{\mathcal{O}_L/\mathbb{Z}}\mathbb{G}_m(\mathbb{Z}/p^r)$, and so a $p$-adic limit has weight in the completion $\hat{W}$ of the characters $X$ of $\mathbf{Res}_{L/\mathbb{Q}}\mathbb{G}_m$ at the subgroups $W(r)$. (Example: if $p$ is inert then $\hat{W} = \mathbb{Z}/(p^g - 1) \times \mathbb{Z}_p^g$).

# Another view of $p$-adic modular forms

Let $\mathfrak{X}$ be the rigid $p$-adic analytic space associated to $\mathscr{M}({}_1\mathfrak{n})$, $X$ its mod $p$ reduction ($X = \mathscr{M}({}_1\mathfrak{n}) \otimes \overline{\mathbb{F}}_p$) and $\mathfrak{X}^{ord}$ the ordinary locus - points corresponding to abelian varieties with ordinary reduction.

One can view $p$-adic Hilbert modular forms as sections of line bundles defined over $\mathfrak{X}^{ord}$. One is interested in forms that are sections over a larger region than $\mathfrak{X}^{ord}$. These are called overconvergent modular forms. For example, any classical modular form is a section over the whole of $\mathfrak{X}$ (and vice-versa, if the weight is classical).

The definition of "regions" is via tubular neighborhoods of $X - X^{ord} = \cup_{\sigma \in \mathbb{B}} \mathrm{div}(h_\alpha)$. This divisor is a reduced, regularly crossing divisor. Moreover, at any point $x \in X$ the forms $\{h_\sigma : h_\sigma(x) = 0\}$ can be taken as a part of a system of local parameters.

# Classicality

A $\Gamma({}_0p)$-level structure on $\underline{A}$ (a principally polarized abelian variety with real multiplication) is a subgroup $H \subseteq A[p]$ that is $\mathcal{O}_L$ invariant and has rank $p^g$; it is automatically isotropic.

Call an overconvergent modular form of level $\Gamma({}_1\mathfrak{n}, {}_0p)$ – that is, both with a $\Gamma({}_1\mathfrak{n})$-level and a $\Gamma({}_0p)$-level – and weight $k \in W$ **classical** if it a modular form in the usual sense. If $f$ is also a generalized eigenform for the $U$ operator,

$$(U_p - \lambda)^n f = 0, \quad n \gg 0,$$

we say that $f$ has slope $\alpha = \mathrm{val}_p(\lambda)$. For $g = 1$, the classical forms have slope $0 \leq \alpha \leq k - 1$.

# Classicality (cont'd)

**Theorem**

*For modular curves, or Shimura curves, if $k > 0$, $0 \leq \alpha < k - 1$, then $f$ is classical. In particular, the space spanned by such forms is finite dimensional.*

The situation for Hilbert modular forms is still unknown in general. One can deal with primes that completely split. Very recently the case of $g = 2$, $p$ inert, was apparently settled.

**Theorem**

*For $g = 2$, $p$ inert, if $f$ is overconvergent and $\alpha < \min\{k_1, k_2\} - 2$ then $f$ is classical.*

(The 2 should be understood here as $[\mathcal{O}_L/p : \mathbb{Z}/p]$.)

# Galois representations

There is a notion of Hecke operators $T_{\mathfrak{n}}$ for Hilbert modular forms, associated to integral ideals. Assume for simplicity that $L$ has strict class number one. Then for a normalized cuspform

$$f = \sum_{\nu \in \mathcal{D}_L^{-1,+}} a(\nu) q^{\nu},$$

of parallel weight $k$, the coefficients $a(\nu)$ depend only on the integral ideal $(\nu \mathcal{D}_L^{-1})$ (and so we write $a((\nu \mathcal{D}_L^{-1}))$ instead of $a(\nu)$). We associate to $f$ the $L$-function

$$\sum_{\mathfrak{m}} a(\mathfrak{m}) \mathrm{Nm}(\mathfrak{m})^{-s} = \prod_{\mathfrak{p}} (1 - a(\mathfrak{p}) \mathrm{Nm}(\mathfrak{p})^{-s} + \mathrm{Nm}(\mathfrak{p})^{k-1-2s}).$$

(Finitely many Euler factors need to be modified as a function of the level.)

# Galois representations (2)

**Theorem**

*Let $f$ be a Hilbert newform of weight $k$ and level $\mathfrak{n}$. Let $K_f$ be the field of coefficients of $f$. Let $\ell$ be a prime, $\lambda$ a prime of $K_f$ that lies above $\ell$, and $K_{f,\lambda}$ the completion of $K_f$ at $\lambda$. There is an absolutely irreducible totally odd Galois representation*

$$\rho_{f,\lambda}\colon \mathrm{Gal}(\overline{L}/L) \to \mathrm{GL}_2(K_{f,\lambda}),$$

*unramified outside $\ell\mathfrak{n}$, such that for any prime $\mathfrak{p} \nmid \ell\mathfrak{n}$*

1. $\mathrm{tr}(\rho_{f,\lambda}(\mathrm{Frob}_{\mathfrak{p}})) = a(\mathfrak{p})$.
2. $\det((\rho_{f,\lambda}(\mathrm{Frob}_{\mathfrak{p}})) = \mathrm{Nm}(\mathfrak{p})^{k-1}$.

Given a Galois representation with the properties of the representation $\rho_{f,\lambda}$ one conjectures that it comes from a Hilbert modular newform. Strategy: prove first that there is a $p$-adic Hilbert modular form then prove it is in fact classical.

# The geometry of the $U_p$-operator

Let $(\mathfrak{p}, \mathfrak{n}) = 1$. For simplicity $p$ inert in $L$, $p\mathcal{O}_L = \mathfrak{p}$. Let $\mathfrak{X}$, $\mathfrak{Y}$, be the completions of $\mathscr{M}({}_1\mathfrak{n}) \otimes \mathbb{Q}_p$ and $\mathscr{M}({}_1\mathfrak{n}, {}_0p) \otimes \mathbb{Q}_p$, respectively, along their special fibres $X$, $Y$ (so $X = \mathscr{M}({}_1\mathfrak{n}) \otimes \mathbb{F}_p$, $Y = \mathscr{M}({}_1\mathfrak{n}, {}_0p) \otimes \mathbb{F}_p$). We have the following diagram

$$
\begin{tikzcd}
\mathfrak{Y}^{ord} \arrow[r, hook] \arrow[d, "\pi"'] & \mathfrak{Y}^{an} \arrow[r, leftrightsquigarrow] \arrow[d] & \mathfrak{Y} \arrow[d, "\pi"'] & Y \arrow[l, hook'] \arrow[d, "\pi"'] \\
\mathfrak{X}^{ord} \arrow[r, hook] \arrow[u, dotted, bend right, "\mathfrak{s}=\mathrm{Frob}"'] & \mathfrak{X}^{an} \arrow[r, leftrightsquigarrow] & \mathfrak{X} & X \arrow[l, hook'] \arrow[u, dotted, bend right, "\mathfrak{s}=\mathrm{Frob}"']
\end{tikzcd}
$$

Note that $\mathfrak{X}, \mathfrak{Y}$ are supported on $X$, $Y$, and so it makes sense that the geometry of $X$, $Y$ and the projection $\pi$ plays a key role. The Raynaud-Berthelot theory establishes that.

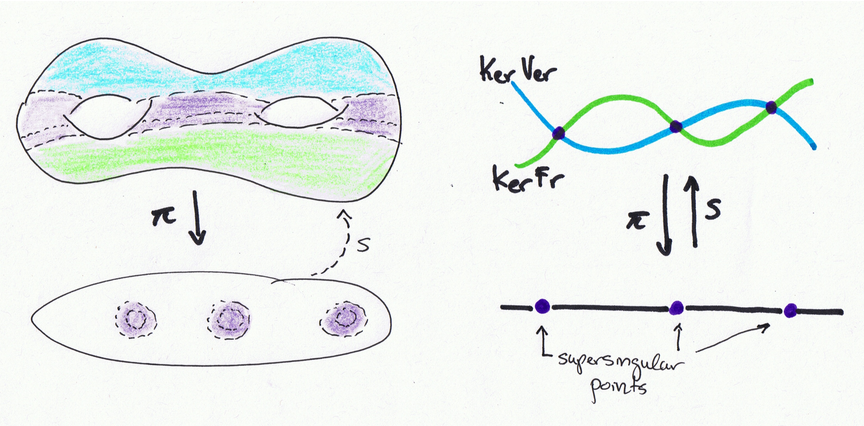
## The geometry of the $U_p$-operator (cont'd)

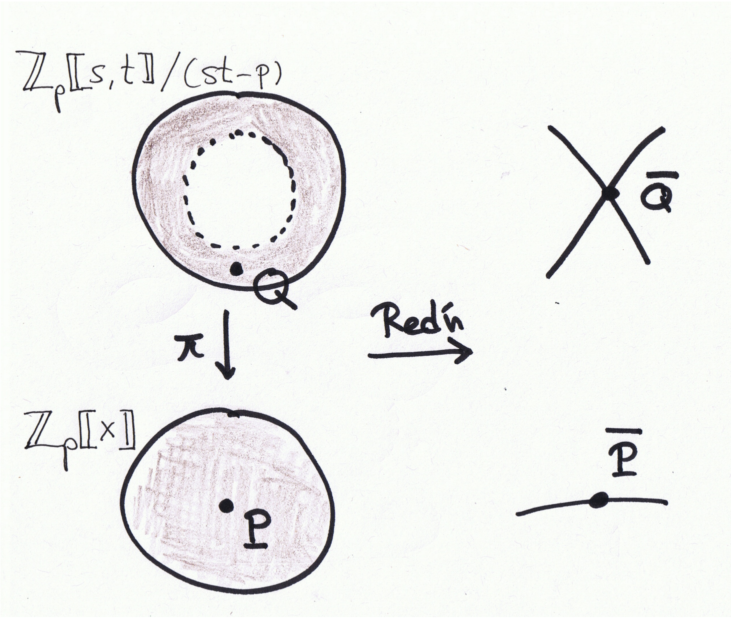
The $U$ operator is the trace of the operator sometimes called $\mathrm{Frob}$. It is defined on the ordinary locus $\mathfrak{X}^{ord}$ in terms of points:

$$x \leftrightarrow \underline{A}_x \mapsto \underline{A}_x/\mathfrak{s}(x) \leftrightarrow \mathrm{Frob}(x).$$

The study of the $U$-operator of overconvergent modular forms requires its extension outside the ordinary locus, which, in turn, requires the extension of the section $\mathfrak{s}$.

# The geometry of $X$, $Y$ $(g = 1)$




$\mathbb{Z}_p[[s,t]]/(st-p)$
Q
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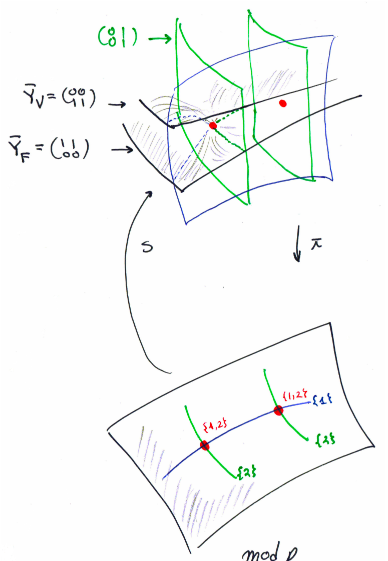
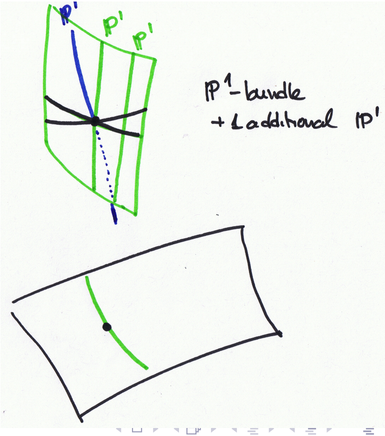
# The geometry of $X$, $Y$ ($g = 2$, $p$ inert)



## Humbert cycles

Let $\mathfrak{H}_g$ be the Siegel upper half space $\{\tau \in M_g(\mathbb{C}) : \tau^t = \tau, \mathrm{Im}(\tau) \gg 0\}$. There is a natural morphism

$$\mathrm{SL}(\mathcal{D}_L^{-1} \oplus \mathcal{O}_L)\backslash\mathfrak{H}^g \longrightarrow \mathrm{Sp}_{2g}(\mathbb{Z})\backslash\mathfrak{H}_g.$$

When $g = 1$ this is a triviality; when $g = 3$ this is (still) a mystery; when $g = 2$ the image is called the Humbert surface $H_\Delta$ where $\Delta$ is the discriminant of $L$. One can similarly define Humbert surfaces for every quadratic positive discriminant. For example, for $\Delta = 1$ this is the image of

$$\mathrm{SL}_2(\mathbb{Z} \oplus \mathbb{Z})\backslash\mathfrak{H}^2 \longrightarrow \mathrm{Sp}_4(\mathbb{Z})\backslash\mathfrak{H}_2.$$

**Theorem**

*Let $G_\Delta = \sum H_{\Delta/f^2}$. The series $\sum_{\Delta \geq 0}[G_\Delta]q^\Delta$ is an elliptic modular form of weight $5/2$ valued in the second intersection cohomology group of $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathfrak{H}_2$ and of level $\Gamma_0(4)$.*

# Hirzebruch-Zagier cycles

There is a similar definition of cycles on a Hilbert modular surface, the Hirzebruch-Zagier cycles. Like the Humbert cycles they are images of lower-dimensional Shimura varieties (thus, curves). The collection process into cycles is more sophisticated, though.

**Theorem**

*Let $T_n, n \geq 0$ be the Hirzebruch-Zagier cycles. Let $d_L$ is the discriminant of $L$. The series $\sum_{n\geq 0}[T_n]q^n$ is an elliptic modular form of weight 2, valued in the second intersection cohomology group of $\mathrm{SL}_2(\mathcal{O}_L)\backslash\mathfrak{H}^2$ and of level $\Gamma_0(d_L)$, and character $\chi_{d_L}$.*

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