Geometric generating series, base change, and distinction

Jayce Getz
McGill University
Funded by NSERC.

(joint work with Mark Goresky).

Notation for Hirzebruch-Zagier:

Let $p \equiv 1 \pmod 4$ be a prime.

- $Y_p := \mathrm{SL}_2(\mathcal{O}_{\mathbb{Q}(\sqrt{p})})\backslash\mathfrak{H}^2$

- $X_p :=$ Satake-Baily-Borel compactification of Y_p

- $X_p^T :=$ a particular (smooth) toroidal compactification of Y_p.

$$Z_m^\circ : \;=\; \pi\left(\left\{(z_1, z_2) \in \mathfrak{H}^2 : \begin{array}{c} z_2 = \frac{\gamma' z_1 - b\sqrt{p}}{a\sqrt{p} z_1 + \gamma} \\ a, b \in \mathbb{Z}, \gamma \in \mathcal{O}_{\mathbb{Q}(\sqrt{p})} \\ \gamma\gamma' + abp = m \end{array}\right\}\right)$$

where

$$\pi : \mathfrak{H}^2 \to Y_p$$

is the canonical projection.

Let

$$\Phi := \sum_{n=0}^{\infty} [Z_n] q^n \qquad (q := e^{2\pi i z}).$$

Theorem (Hirzebruch-Zagier Invent. '76)

For each $\xi \in H_2(X_p^T)$ we have that

$$\langle \xi, \Phi \rangle_H := \sum_{n \geq 0} \langle \xi, [Z_n] \rangle_H \ q^n$$

is a weight 2 modular form with nebentypus.

Theorem (Zagier, LNM '76):

$$\begin{aligned}\langle [Z_m], \Phi \rangle_H &= c(m) E_{2,p}^{+}(z) \\ &+ r \sum_{n=1}^{\infty} \left(\sum_f \frac{\left(\int_{Z_1} \omega(\widehat{f}) \right)^2}{(\widehat{f}, \widehat{f})} a_f(m) a_f(n) \right) q^n.\end{aligned}$$

. . . .

- $\sum_f$ is over a certain basis of $S_2^{+}\left(\Gamma_0(p), \left(\frac{p}{\cdot}\right)\right)$.

- $E_{2,p}^{+}$:= an Eisenstein series for this space.
  r := constant
  $c(m)$:= constant depending on m.

- $(,)$:= Petersson inner product.

- $\widehat{f}$ is the Naganuma lift of f
  $\eta_{\widehat{f}}$:= $(1,1)$-form attached to $\widehat{f}$.

Salient points:

- Subspace of $H_2(X_p^T)$ spanned by algebraic cycles.

- Geometric generating series:

$$\Phi \in H_2(X_p^T) \otimes M_2^+(\Gamma_0(p)).$$

- Explicit description of

$$\mathrm{Im}\left(\langle \cdot, \Phi \rangle : H_2(X_p^T)^\vee \longrightarrow M_2^+(\Gamma_0(p))\right).$$

All three points are consequences of

- Intersection homology.
- Langlands functoriality.
- Distinction.

Notation:

- $L/E :=$ quadratic extension of totally real number fields with Hecke character η.

- $\Sigma(L) :=$ set of embeddings $\sigma : L \hookrightarrow \mathbb{R}$.

- $\mathcal{O}_L :=$ ring of integers of L.

- $\mathfrak{c} :=$ ideal of $\mathcal{O}_L$.

- $\mathfrak{c}_E :=$ ideal of $\mathcal{O}_E$.

- $G := G_L := \mathrm{Res}_{L/\mathbb{Q}}(\mathrm{GL}_2)$.

Hilbert modular varieties

$$\begin{aligned} Y_0(\mathfrak{c}) : &= G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty K_0(\mathfrak{c}). \\ X_0(\mathfrak{c}) : &= \text{Satake-Baily-Borel of } Y_0(\mathfrak{c}). \end{aligned}$$

- K_∞ is the stabilizer of

$$\begin{array}{ccc} \mathbb{C}^\times & \to & G(\mathbb{R}) \\ x+iy & \mapsto & \left(\begin{pmatrix} x & y \\ -y & x \end{pmatrix}, \ldots, \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \right). \end{array}$$

- $K_0(\mathfrak{c})$ is the compact open

$$\left\{ \gamma \in \mathrm{GL}_2(\widehat{\mathcal{O}_L}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{\mathfrak{c}} \right\}.$$

Analytic realization:

$$Y_0(\mathfrak{c}) = \coprod_j \Gamma_j \backslash \mathfrak{H}^{\Sigma(L)}.$$

Base change:

$\mathbb{T}_{\mathfrak{c}}$:=Hecke algebra over L
$\mathbb{T}_{\mathfrak{c}_E}$:=Hecke algebra over E

$$b : \mathbb{T}_{\mathfrak{c}} \longrightarrow \mathbb{T}_{\mathfrak{c}_E}$$

. . . .

$$\{\text{Newforms on } E\} \longrightarrow \{\text{Newforms on } L\}^{\mathrm{Gal}(L/E)}$$
$$f \longmapsto \widehat{f}$$

$$\lambda_{\widehat{f}}(\mathfrak{n}) = \lambda_f(b(T(\mathfrak{n})))$$

Notation:

Define $S^{+}(\mathfrak{c}_E, \eta)$ to be

$$\left\{ g \in S(K_0(\mathfrak{c}_E), \eta) : \begin{array}{c} a(\mathfrak{m}, g) = 0 \text{ if } \eta(\mathfrak{m}) = -1 \\ \text{or } \mathfrak{m} + \mathfrak{c}_E \neq \mathcal{O}_E \end{array} \right\}.$$

$$IH^{E}(X_0(\mathfrak{c})) := \bigoplus_{\substack{f \in S^{\mathrm{new}}(K_0(\mathfrak{c})): \\ \lambda_f(\mathfrak{m}^\sigma) = \lambda_f(\mathfrak{m}) \\ \forall \sigma \in \mathrm{Gal}(L/E)}} IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))(f).$$

$$IH^{\eta}(X_0(\mathfrak{c})) := \bigoplus_{\substack{\text{nebentypus}(g) = \eta \\ \widehat{g} \in S(K_0(\mathfrak{c}))}} IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))(\widehat{g}).$$

The Hecke operators $\widehat{T}(\mathfrak{m})$:

For $\mathfrak{m} \subset \mathcal{O}_L$, define a Hecke operator

$$\widehat{T}(\mathsf{N}_{L/E}(\mathfrak{m})) \in \mathbb{T}_{\mathfrak{c}} \otimes \mathbb{C}.$$

For example, $\widehat{T}(\mathsf{N}_{L/E}(\mathfrak{P})^r)$ is

$$\begin{cases} \frac{1}{2}\left(T(\mathfrak{P}^r) + T(\overline{\mathfrak{P}}^r)\right) & \text{if } \mathfrak{p} \text{ splits} \\ T(\mathfrak{P}^r) + \eta(\mathfrak{p})\mathsf{N}_{E/\mathbb{Q}}(\mathfrak{p})\widehat{T}(\mathfrak{p}^{2r-2}) & \text{if } \mathfrak{p} \text{ is inert} \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\begin{aligned} \mathbb{T}_{\mathfrak{c}_E} &\longrightarrow \mathbb{T}_{\mathfrak{c}} \\ T(\mathfrak{n}) &\longmapsto \widehat{T}(\mathfrak{n}) \end{aligned}$$

is a section of b:

$$\lambda_{\widehat{f}}(\widehat{T}(\mathfrak{n})) = \lambda_f(\mathfrak{n})$$

Definition of $\gamma(\mathfrak{m}')$:

Let

$$Q : IH^E(X_0(\mathfrak{c})) \longrightarrow IH^\eta(X_0(\mathfrak{c}))$$

be the projection.

For each $\gamma \in IH^E(X_0(\mathfrak{c}))$, define

$$\gamma(\mathfrak{m}') := \begin{cases} \widehat{T}(\mathsf{N}_{L/E}(\mathfrak{m}))_* Q\gamma & \text{if } \mathfrak{m}' = \mathsf{N}_{L/E}(\mathfrak{m}), \\ 0 & \text{otherwise.} \end{cases}$$

Results:

Theorem 1 (G.-Goresky)**.**

Let L/E *be quadratic and let* $\gamma \in IH^E(X_0(\mathfrak{c}))$.

We then have that

$$\Phi_\gamma\left(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\right) := |y|_{\mathbb{A}_E} \sum_{\substack{\xi \in E^\times \\ 0 \ll \xi}} \gamma(\xi y \mathcal{D}_{E/\mathbb{Q}}) q(\xi x, \xi y)$$

is an element of

$$IH^E(X_0(\mathfrak{c})) \otimes S^+(\mathcal{N}(\mathfrak{c}), \eta).$$

For $t \in \mathbb{T}_{\mathfrak{c}}$ *one has*

$$\langle \psi, \Phi_\gamma \rangle$$

Theorem 2 (G.-Goresky).

Suppose that Z is a subanalytic cycle on $X_0(\mathfrak{c})$ admitting a class $[Z] \in IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))$. If

$$\mathfrak{n} + d_{L/E}(\mathfrak{c} \cap \mathcal{O}_E) = \mathfrak{m} + d_{L/E}(\mathfrak{c} \cap \mathcal{O}_E) = \mathcal{O}_E$$

and $\mathfrak{n}, \mathfrak{m}$ are norms from $\mathcal{O}_L$, then the $\mathfrak{n}$th Fourier coefficient of

$$\langle [Z](\mathfrak{m}), \Phi_{[Z]} \rangle_{IH}$$

is

$$\frac{1}{4} \sum_{J \subset \Sigma(E)} \sum_{f} \frac{\int_Z \overline{\omega_J(\widehat{f})} \int_Z \omega_J(\widehat{f})}{\int_{Y_0(\mathfrak{c})} \omega_J(\widehat{f}) \wedge \overline{\omega_J(\widehat{f})}} \lambda_f(\mathfrak{m}) \lambda_f(\mathfrak{n})$$

where f ranges over newforms of nebentypus η with $\widehat{f} \in S(K_0(\mathfrak{c}))$.

Otherwise, the $\mathfrak{n}$th Fourier coefficient is zero.

Déjà vu:

Theorem (Zagier, LNM '76):

$$\begin{aligned}\langle [Z_m], \Phi \rangle_H &= c(m) E_{2,p}(z) \\ &+ r \sum_{n=1}^{\infty} \left(\sum_f \frac{\left(\int_{Z_1} \eta_{\widehat{f}} \right)^2}{(\widehat{f}, \widehat{f})} a_f(m) a_f(n) \right) q^n.\end{aligned}$$

Remark:

Have:

- Geometric generating series:

$$\Phi_\gamma \in IH^\eta(X_0(\mathfrak{c})) \otimes S^+(\mathcal{N}(\mathfrak{c}), \eta)).$$

- Explicit description of

  $\text{Im}\left(\langle \cdot, \Phi_\gamma \rangle : IH^\eta(X_0(\mathfrak{c}))^\vee \longrightarrow S^+(\mathcal{N}(\mathfrak{c}), \eta)\right).$

Still need:

- Subspace of $IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))$ spanned by algebraic cycles.

Examples of cycles admitting classes in IH:

- Have
$$\iota : G_E \hookrightarrow G = G_L.$$
- This gives, for every compact open $K \leq G(\mathbb{A}_f)$, **Shimura subvarieties**
$$Y_{\iota^{-1}(K_0(\mathfrak{c}))\cap G_E(\mathbb{A}_f)} \hookrightarrow Y_0(\mathfrak{c})$$
$$X_{\iota^{-1}(K_0(\mathfrak{c}))\cap G_E(\mathbb{A}_f)} \hookrightarrow X_0(\mathfrak{c}).$$
where
$$Y_{K_E} := G_E(\mathbb{Q})\backslash G_E(\mathbb{A})/K_{E,\infty}K_E.$$

The associated cycles intersect the cusps nontrivially.

Theorem 3 (G.-Goresky).

A Shimura subvariety $Z \subset X_0(\mathfrak{c})$ *as above admits a canonical class*

$$[Z] \in IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c})).$$

Moreover

$$\langle [Z], [\omega(f)] \rangle_{IH} = \int_Z \omega(f).$$

Let Z be a Shimura subvariety as above. Let

$$P_{\text{new}} : IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c})) \longrightarrow IH^{\text{new}}_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))$$

be the projection.

Theorem 4 (G.-Goresky)**.**

The class $P_{\text{new}}([Z])$ is an element of $IH^{\eta}(X_0(\mathfrak{c}))$. Moreover, if $g \in S(K_0(\mathfrak{c}))$ is the base change of a form f of nebentypus η, then

$$\begin{aligned} \langle [Z], [\omega(g)] \rangle_{IH} &= \int_Z \omega(g) \\ &= L(Ad(f) \otimes \eta, 1). \end{aligned}$$

Otherwise,

$$\langle [Z], [\omega(g)] \rangle_{IH} = 0.$$

This provides

- Subspace of $IH^{\eta}(X_0(\mathfrak{c})) \leq IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))$ spanned by algebraic cycles.

Proofs:

Proof of Theorem 1:

Consequence of the fact that

$$\begin{aligned} \mathbb{T}_{\mathfrak{c}_E} &\longrightarrow \mathbb{T}_{\mathfrak{c}} \\ T(\mathfrak{n}) &\longmapsto \widehat{T}(\mathfrak{n}) \end{aligned}$$

is a section of

$$b : \mathbb{T}_{\mathfrak{c}} \longrightarrow \mathbb{T}_{\mathfrak{c}_E}$$

Thus

$$\lambda_{\widehat{f}}(\widehat{T}(\mathfrak{n})) = \lambda_f(\mathfrak{n}).$$

This is where we use

- Langlands functoriality.

Proof of Theorem 2:

Step 1: Prove Theorem 2 up to nonzero constants:

$$\langle [Z](\mathfrak{m}), \Phi_\gamma \rangle_{IH} = \sum_f c_f \lambda_f(\mathfrak{m}) f^{(\mathfrak{c} \cap \mathcal{O}_E) d_{L/E}},$$

where $c_f \in \mathbb{C}^\times$.

Step 1 is a consequence of:

1. The intersection pairing $\langle , \rangle_{IH}$ is nondegenerate.

2. There is an isomorphism of Hecke modules
$$\mathcal{Z} : H_{(2)}^{[L:\mathbb{Q}]}(X_0(\mathfrak{c})) \xrightarrow{\sim} IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c}))$$
(the Zucker conjecture, proven by Saper-Stern and Looijenga).

3. The L_2-cohomology has a nice description, due to work of Harder (rephrased by Hida):
$$H_{(2)}^{\bullet}(X_0(\mathfrak{c})) \cong \text{invariant forms} \oplus \bigoplus_{J \subset \Sigma(L)} \bigoplus_{f \in S(K_0(\mathfrak{c}))} \omega_J(f)$$

Step 2: Prove compatibilities between pairings.

E.g.

$$\begin{aligned}&\langle \mathcal{Z}[\omega_J(f)], \mathcal{Z}[\omega_{\Sigma(L)-J}(g)]\rangle_{IH}\\&= \int_{Y_0(\mathfrak{c})} \omega_J(f) \wedge \omega_{\Sigma(L)-J}(g)\\&= *\langle f, g\rangle_{Petersson}.\end{aligned}$$

and

$$\int_Z \omega_J(f) = \langle [Z], [\omega_J(f)]\rangle_K.$$

These statements would be classical if $X_0(\mathfrak{c})$ was a manifold, but it is not:

- $Y_0(\mathfrak{c})$ is only an orbifold.

- $X_0(\mathfrak{c})$ has isolated singularities.

A proof of these statements involves

- Subanalytic triangulations of spaces with isolated singularities, and how differential forms and chains behave with respect to such triangulations.

- Deligne's characterization of the intersection cohomology sheaf on a pseudomanifold X as an element of $D^b(X)$.

Clearly using

- Intersection homology.

Proof of Theorem 3:

We have to show that a Shimura subvariety $Z \subset X_0(\mathfrak{c})$ admits a canonical class in intersection homology.

The key input:

Theorem 5 (Saper). *The quotient map* $Y_0(\mathfrak{c})^{RBS} \to X_0(\mathfrak{c})$ *induces an isomorphism*

$$IH_\bullet(Y_0(\mathfrak{c})^{RBS}) \xrightarrow{\sim} IH_\bullet(X_0(\mathfrak{c})).$$

Proof of Theorem 4:

Use a Rankin-Selberg convolution.

Leads to **distinction**:

$H' \leq H :=$ reductive F-groups.

An automorphic representation π of $H(\mathbb{A}_F)$ is H'-**distinguished** if some form in the space of π has a nonzero period over

$$H'(F)\backslash H'(\mathbb{A}_F) \cap H(\mathbb{A}_F)^1.$$

Summary:

- We produced a Hilbert modular form with coefficients in intersection homology:

$$\Phi_\gamma \in IH_{[L:\mathbb{Q}]}(X_0(\mathfrak{c})) \otimes S(K_0(\mathfrak{c})).$$

- The Fourier coefficients of

$$\langle [Z], \Phi_{[Z]} \rangle_{IH}$$

were then computed in terms of period integrals for nice cycles Z.

- Constructed cycles in $IH_{L:\mathbb{Q}}(X_0(\mathfrak{c}))$.

Remark 6. *In the book, we allow arbitrary weight and character.*

Where to go from here:

General theory of distinction.

- Nonvanishing of cohomological periods.

Questions in geometry.

- Integration formulae of the type
$$\langle [Z], \omega(f) \rangle_{IH} = \int_Z \omega(f)$$
for locally symmetric spaces of higher $\mathbb{Q}$-rank (non-isolated singularities)?

- Construction of canonical classes (also in étale setting).

Modularity

- Relationship between models of representations and Hecke algebras.