Introduction to theta liftings
(following D. Prasad)

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Highbrow motivation:

\[ F : = \text{number field} \]
\[ H, G : = \text{reductive } F \text{ groups} \]

Suppose given
\[ L_H \rightarrow L_G. \]

Langlands functoriality:

There exists a corresponding transfer of automorphic forms.

Imply, e.g. relationship between \( L \)-functions.

Theta liftings give explicit examples.
The Weil representation:

\[ k := \text{field of characteristic } \neq 2 \]
\[ W := \text{finite dimensional } k\text{-vector space} \]

Nondegenerate symplectic pairing

\[ \langle \cdot, \cdot \rangle : W \times W \to k. \]

Heisenberg group: \( k\)-group \( H(W) \) such that for \( k\)-algebras \( R \)

\[ H(W)(R) := \{(w, t) : w \in W \otimes_k R, t \in R \} \]

where

\[ (w_1, t_1) \cdot (w_2, t_2) = (w_1 + w_2, t_1 + t_2 + \frac{1}{2}\langle w_1, w_2 \rangle). \]

Have nonsplit central extension

\[ 0 \to R \to H(W)(R) \to W \otimes_k R \to 0 \]
Let $\rho$ be a representation of $H(W)(R)$.

$\rho|_R$ yields $\psi : R^+ \to \mathbb{C}^\times$.

**Theorem (Stone-von Neumann):**

For any nontrivial character $\psi : F_v \to \mathbb{C}^\times$ there exists an irreducible representation

$$\rho_{\psi} : H(W)(F_v) \longrightarrow \text{Aut}(V_{\psi})$$

(unique up to equivalence) on which $F_v$ acts via $\psi$. 
Realization:

\[ k \in \{ F, F_v \} \]

\[ R \in \{ F_v, \mathbb{A}_F^S \} \]

\[ W = X \oplus Y, \ X, Y \text{ isotropic.} \]

Have smooth representation

\[ \rho_\psi : H(W)(R) \longrightarrow V_\psi := \text{Aut}(S(X(R))) \]

where

\[ \rho_\psi(w_1)f(x) = f(x + w_1) \]
\[ \rho_\psi(w_2)f(x) = \psi(\langle x, w_2 \rangle)f(x) \]
\[ \rho_\psi(t)f(x) = \psi(t)f(x) \]

for \( x, w_1 \in X(R), \ w_2 \in Y(R) \) and \( t \in R. \)
Weil representation:

Have action

\[ \text{Sp}(W)(R) \times H(W)(R) \rightarrow H(W)(R) \]
\[ (g, (w, t)) \mapsto (gw, t). \]

By uniqueness of \( \rho_\psi \), \( \text{Sp}(W)(R) \) acts on \( V_{\rho_\psi} \): there exists \( \omega_\psi(g) \) (unique up to scaling) s.t.

\[ \rho_\psi(gw, t) = \omega_\psi(g) \rho_\psi(w, t) \omega_\psi(g)^{-1} \]

Obtain

\[ \text{Sp}(W)(R) \rightarrow \text{PGL}(V_{\rho_\psi}) \]

and hence

\[ \text{Mp}(W) \rightarrow \text{GL}(V_{\rho_\psi}). \]

Note \( \text{Mp}(W) \) is \textbf{not} algebraic.
More details

Recall the equation

$$\rho_\psi(gw, t) = \omega_\psi(g) \rho_\psi(w, t) \omega_\psi(g)^{-1} \quad (1)$$

Define

$$\widehat{Sp}_\psi(W) := \{(g, \omega_\psi(g)) \in Sp(W) \times \mathbb{C}^\times : (1) \text{ holds}\}$$

Have exact sequence

$$1 \rightarrow \mathbb{C}^\times \rightarrow \widehat{Sp}_\psi(W) \rightarrow Sp(W) \rightarrow 1.$$  

**Theorem:** Restriction to the commutator subgroup yields an exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow [\widehat{Sp}_\psi(W), \widehat{Sp}_\psi(W)] \rightarrow Sp(W) \rightarrow 1$$

and

$$[\widehat{Sp}_\psi(W), \widehat{Sp}_\psi(W)] \cong Mp(W).$$
Realization:

\text{Sp}(W) \rightarrow \text{PGL}(V_\psi) \text{ given by}

\begin{align*}
\begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix} f(X) &= |\det(A)|^{\frac{1}{2}} f(A^t X) \\
\begin{pmatrix} I & B \\ 0 & I \end{pmatrix} f(X) &= \psi \left( \frac{X^t B X}{2} \right) f(X) \\
\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} f(X) &= \gamma \hat{f}(X)
\end{align*}

where \( \gamma \) is an 8th root of unity and \( \hat{\cdot} \) denotes Fourier transform.
**Dual reductive pairs:**

Let $G_1, G_2 \leq \text{Sp}(W)$ be reductive subgroups.

**Definition:**

$(G_1, G_2)$ is a **dual reductive pair** if

$$Z_{\text{Sp}(W)}(G_1) = G_2 \text{ and } Z_{\text{Sp}(W)}(G_2) = G_1.$$
Global theta liftings

Let $\psi : F\backslash\mathbb{A}_F \to \mathbb{C}$ be nontrivial. Have a Weil representation

$$\text{Sp}(W)(\mathbb{A}_F) \to S(X(\mathbb{A}_F))$$

**Theorem (Weil):** The cover

$$\text{Mp}(W)(\mathbb{A}_F) \to \text{Sp}(W)(\mathbb{A}_F)$$

splits over $\text{Sp}(W)(F)$.

Thus $\text{Sp}(W)(F)$ operates on $S(X(\mathbb{A}_F))$. 
Define a distribution

$$\theta : S(X(\mathbb{A}_F)) \longrightarrow \mathbb{C}$$

$$f \longmapsto \sum_{x \in X(F)} f(x)$$

and a function

$$\theta_f(x) := \theta(x \cdot f) : \text{Sp}(W)(F) \backslash \text{Mp}(W)(\mathbb{A}_F) \longrightarrow \mathbb{C}$$

This is the \textbf{theta function} attached to \(f\).
\((G_1, G_2) \leq \text{Sp}(W) := \text{dual reductive pair.}\)

\(\tilde{G}_i \leq \text{Mp}(W) := \text{the inverse images under } \text{Mp}(W) \to \text{Sp}(W)(F).\)

Suppose that \(\text{Mp}(W)\) splits over \(G_1 \times \tilde{G}_2\).
Have a map

$$\theta_f : \mathcal{A}(G_1(F) \backslash G_1(\mathbb{A}_F)) \rightarrow \mathcal{A}(\tilde{G}_2(F) \backslash \tilde{G}_2(\mathbb{A}_F))$$

where

$$\theta_f(\phi)(g_2) = \int_{G_1(F) \backslash G_1(\mathbb{A}_F)} \theta_\phi(g_1, g_2)f(g_1)dg_1.$$ 

The function $\theta_f(\phi)$ is the theta lift of $\phi$.

Of course, can switch roles of $G_1$ and $G_2$ to obtain inverse lifts.
Can restrict the domain to the $\pi_1$-isotypic subspace, where $\pi_1$ is an automorphic representation of $G_1$.

Basic question:

When is the space generated by $\theta_f(\phi)$ for $\phi \in \pi_1$ not identically zero?
**Shintani’s example**

$V/F :=$ orthogonal space $W/F :=$ symplectic space

So $V \otimes W$ is a symplectic space

$$(O(V), \text{Sp}(W)) \leq \text{Sp}(W)$$

a dual reductive pair.

Assume $\dim(W) = 2$, isotropic basis $e_1, e_2$.

$$X = V \otimes e_1, \quad Y = V \otimes e_2$$

For $f \in \mathcal{S}(V)$ obtain $\theta$-function

$$\theta_f(g) = \sum_{x \in X(F)} f(xg)$$

Since $\text{Sp}(W) \cong \text{SL}_2$, obtain a theta lifting

$$\theta_f : \mathcal{A}(O(F) \backslash O(\mathbb{A}_F)) \rightarrow \mathcal{A}(\text{SL}_2(F) \backslash H(\mathbb{A}_F))$$

where $H \in \{\text{SL}_2, \tilde{\text{SL}}_2\}$. 

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Concrete examples:

Let $F = \mathbb{Q}$, $q =$ quadratic form on $V$.

(1) $\dim(V) = 1$, $q = x^2$, $O(V) = \{\pm 1\}$.

Obtain classical theta functions.

(2) $\dim(V) = 2$, $q =$ norm form of $K/\mathbb{Q}$ ($[K : \mathbb{Q}] = 2$), $O(V) \cong \text{Res}_{K/\mathbb{Q}}\text{GL}_1/\text{GL}_1$.

Lifting constructs automorphic induction of Hecke characters to “CM” or dihedral automorphic representations on $\text{SL}_2$. 
dim(V) = 3, \( q = x^2 - yz \), \( SO(V) \cong PGL_2 \)

Inverse of Shimura correspondence

\[ \pi := \text{cuspidal representation of } PGL_2(\mathbb{A}_\mathbb{Q}) \]
\[ \sigma := \text{cuspidal representation of } \widetilde{SL}_2(\mathbb{A}_\mathbb{Q}). \]

**Theorem (Waldspurger):**

(1) \( \theta(\pi) \neq 0 \) iff \( L\left(\frac{1}{2}, \pi\right) \neq 0. \)
(2) \( \theta(\sigma) \neq 0 \) iff \( \sigma \) has a \( \psi \)-Whittaker model.