Breuil-Kisin modules

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Let k be a perfect field of char $p > 0$, $W := W(k)$ ring of Witt vectors.
Take a *totally ramified extension*

$$\begin{tikzcd} K \arrow[r, dash] \arrow[d, dash, "e"'] & \mathcal{O}_K \ni \pi \arrow[d, dash] \\ K_0 := W[1/p] \arrow[r, dash] & W \end{tikzcd}$$

$E(u)$ min.pol. of π

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Let $\mathfrak{S} := W[[u]]$, with $\varphi_{\mathfrak{S}}$ extension of the Frobenius on W.

Definition

Define the category $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$ *of finite free* $\mathfrak{S}$*-modules* $\mathfrak{M}$, *with an injective semi-linear map*

$$\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M}$$

such that the cokernel of the linearization

$$\varphi^*(\mathfrak{M}) := \mathfrak{S} \otimes_{\mathfrak{S}, \varphi_{\mathfrak{S}}} \mathfrak{M} \xrightarrow{1 \otimes \varphi_{\mathfrak{M}}} \mathfrak{M}$$

is killed by $E(u)$.

Motivation

Giving a classification of Barsotti-Tate groups over $\mathcal{O}_K$ for any prime p and for any ramification index e.

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- Proof in Kisin's paper: "Crystalline representations and *F*-crystals".

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Giving a classification of Barsotti-Tate groups over $\mathcal{O}_K$ for any prime p and for any ramification index e.

- Conjecture by Breuil: relation between $\mathrm{BT}(\mathcal{O}_K)$ and $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$.
- Proof in Kisin's paper: "Crystalline representations and F-crystals".

Theorem

There is an exact contravariant functor

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{BT}(\mathcal{O}_K).$$

When $p > 2$ this is an equivalence of categories, when $p = 2$ it is an equivalence up to isogeny.

Consider $T \to \operatorname{Spec} W$, p loc. nilpotent. Let $G \in \mathrm{BT}(T)$.

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- If $T = \operatorname{Spec} k$, classical Dieudonné theory:

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- In general, we have crystalline theory: $\mathbb{D}(G) \in \mathrm{Cris}(T/W)$. For a PD-thickening $(T \hookrightarrow X) \in \mathrm{Cris}(T/W)$ and $H \in \mathrm{BT}(X)$ a lifting of $G \in \mathrm{BT}(T)$ we have

Theorem (Grothendieck-Messing)

There is an equivalence of categories

$$\begin{array}{ccc} \mathrm{BT}(X) & \xrightarrow{\mathbb{D}} & \mathscr{C} \\ H & \mapsto & (G := H \times_X T, \underline{V}(H) \hookrightarrow \underline{Lie}(E(H)) =: \mathbb{D}(G)(X)) \end{array}$$

- Consider $W[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1}$ (PD-envelope of $(W[u],(E(u)))$). There is a surjection

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- Let S be the p-adic completion of $W[u]\left[\frac{E(u)^i}{i!}\right]_{i\geq 1}$. The map above extends to a surjection

$$S \twoheadrightarrow \mathcal{O}_K.$$

Denote $\mathrm{Fil}^1 S := \mathrm{Ker}(S \twoheadrightarrow \mathcal{O}_K)$, $\varphi_S : S \to S$ extension of Frob on W.

Definition

Define the category $\mathrm{BT}^{\varphi}_{/S}$ *of couples* $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M})$ *where*

- $\mathcal{M}$ *is a finite free* S*-module;*
- $\mathrm{Fil}^1\mathcal{M}$ *is an* S*-submodule of* $\mathcal{M}$ *such that*
  - $\mathrm{Fil}^1 S \cdot \mathcal{M} \subset \mathrm{Fil}^1\mathcal{M}$,
  - $\mathcal{M}/\mathrm{Fil}^1\mathcal{M}$ *is a free* $\mathcal{O}_K$*-module,*
  - *there exists a* φ_S*-semilinear map* $\varphi_1 : \mathrm{Fil}^1\mathcal{M} \to \mathcal{M}$ *such that* $\varphi^*(\mathrm{Fil}^1\mathcal{M}) \to \mathcal{M}$ *is onto.*

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Theorem (Kisin)

There is an exact contravariant functor

$$\begin{array}{ccc} \mathrm{BT}(\mathcal{O}_K) & \longrightarrow & \mathrm{BT}^{\varphi}_{/S} \\ G & \longmapsto & (\mathbb{D}(G)(S), \mathrm{Fil}^1\mathbb{D}(G)(S)) \end{array}$$

it's an equivalence for $p > 2$*, up to isogeny for* $p = 2$*.*

Sketch of proof (Deformation argument)

Let $(\mathcal{M}, \mathrm{Fil}^1\mathcal{M}) \in \mathrm{BT}^{\varphi}_{/S}$.

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- *For* $i = 1, \ldots, e$ *define the* S*-algebra* $R_i := W[u]/u^i$ *and*

$$\mathcal{M}_i := \mathcal{M} \otimes_S R_i, \qquad \mathrm{Fil}^1\mathcal{M}_i := \mathrm{Im}(\mathcal{M} \to \mathcal{M}_i).$$

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$$\mathcal{M}_i := \mathcal{M} \otimes_S R_i, \qquad \mathrm{Fil}^1\mathcal{M}_i := \mathrm{Im}(\mathcal{M} \to \mathcal{M}_i).$$

- *For every* i *there is a surjective map*

$$f_i : R_i := W[u]/u^i \twoheadrightarrow \mathcal{O}_K/\pi^i,$$

with PD-kernel (pR_i)*, so*

$$(\mathrm{Spec}(\mathcal{O}_K/\pi^i) \hookrightarrow \mathrm{Spec}(R_i)) \in \mathrm{Cris}((\mathcal{O}_K/\pi^i)/W).$$

Strategy

Construct inductively p-divisible groups G_i over $\mathcal{O}_K/\pi^i$ for any $i = 1, \ldots, e$ such that

$$\mathbb{D}(G_i)(R_i) \simeq \mathcal{M}_i.$$

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$$R_1 = W[u]/u \simeq W, \qquad \mathcal{O}_K/\pi \simeq k.$$

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- *$i = 1$: Dieudonné theory*

$$R_1 = W[u]/u \simeq W, \qquad \mathcal{O}_K/\pi \simeq k.$$

- *For $i = 2, \ldots, e$ gradual lifting after Grothendieck-Messing:*

$$R_i \twoheadrightarrow \mathcal{O}_K/\pi^{i-1} \text{ with PD-kernel } (u^{i-1}, p),$$

and hence

$$(\mathrm{Spec}(\mathcal{O}_K/\pi^{i-1}) \hookrightarrow \mathrm{Spec}(R_i)) \in \mathrm{Cris}((\mathcal{O}_K/\pi^{i-1})/W).$$

From S to $\mathfrak{S}$

The map

$$\begin{array}{ccc} \mathfrak{S} & \xrightarrow{\varphi} & S \\ u & \longmapsto & u^p \end{array}$$

defines a functor

$$\begin{array}{ccc} \mathrm{BT}^{\varphi}_{/\mathfrak{S}} & \longrightarrow & \mathrm{BT}^{\varphi}_{/S} \\ \mathfrak{M} & \longmapsto & \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S \end{array}$$

(For $\mathcal{M} = \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S$ define

$$\mathrm{Fil}^1 \mathcal{M} := \{ m \in \mathcal{M} \mid (1 \otimes \varphi)(m) \in \mathfrak{M} \otimes_{\mathfrak{S},\varphi} \mathrm{Fil}^1 S \subset \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S \}).$$

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Hence we obtain a functor

$$\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \longrightarrow \mathrm{BT}(\mathcal{O}_K).$$

General strategy for the construction of an inverse

Definition

Denote by $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$ *the category of finite free* $\mathfrak{S}$*-modules* $\mathfrak{M}$ *equipped with a* $\varphi_{\mathfrak{S}}$*-semi-linear Frobenius*

$$\varphi_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{M},$$

such the cokernel of its linearization

$$\begin{array}{ccc} \varphi^*\mathfrak{M} & \to & \mathfrak{M} \\ \lambda \otimes m & \mapsto & \lambda\varphi_{\mathfrak{M}}(m) \end{array}$$

is killed by some power of $E(u)$ *(of finite* E*-height).* $\mathrm{BT}^{\varphi}_{/\mathfrak{S}} \subset \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$.

Categories of semi-linear algebra data

Definition (MF_K^{φ})

- *An étale φ-module over K_0 is a finite K_0-vector space D equipped with a semi-linear map $\varphi : D \to D$ such that its linearization $\varphi^*(D) \xrightarrow{\sim} D$ is an isomorphism.*

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Theorem (Kisin)

There is a fully faithful functor

$$MF_K^{\varphi, \mathrm{Fil}_{\geq 0}, ad} \longrightarrow \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p.$$

A geometric interpretation of $\mathrm{MF}_K^{\varphi,\mathrm{Fil}_{\geq 0}}$ and $\mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$

- Δ: open rigid analytic unit disk K_0;
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of finite E*-height* ($\mathrm{Coker}(\varphi^*(\mathcal{M}) \to \mathcal{M})$ *is killed by a power of* $E(u)$*).*

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We call of *Barsotti-Tate type* those $D \in \mathrm{MF}_K^{\varphi,\mathrm{Fil}_{\geq 0},ad}$ such that $\mathrm{Fil}^i D_K = 0$ when $i \notin \{0, 1\}$.

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Corollary

The fully faithful functor

$$\mathrm{MF}_K^{\varphi,\mathrm{Fil}_{\geq 0},adm} \longrightarrow \mathrm{Mod}^{\varphi}_{/\mathfrak{S}}$$

restricts to an equivalence of categories

$$\{adm\ \varphi\text{-modules of BT-type}\} \xrightarrow{\sim} \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p.$$

Caveat!

Extra structure required:

$$\begin{tikzcd}
\mathrm{MF}_K^{\varphi,\mathrm{Fil}_{\geq 0}} \arrow[r, "\text{fully faithful}"] \arrow[d] & \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p \arrow[d] \\
\mathrm{Mod}^{\varphi}_{/\mathscr{O}} & \mathrm{Mod}^{\varphi}_{/\mathscr{O}}
\end{tikzcd}$$

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Extra structure required: *monodromy operators*

$$\begin{tikzcd}
\mathrm{MF}^{\varphi,\mathrm{Fil}_{\geq 0},N}_{K} \arrow[r, "\text{fully faithful}"] \arrow[d, "\sim"'] & \mathrm{Mod}^{\varphi,N}_{/\mathfrak{S}} \otimes \mathbb{Q}_p \arrow[d, "\sim"'] \\
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Corollary

There is an equivalence of categories

$$\{\textit{adm } \varphi\textit{-modules of BT-type}\} \xrightarrow{\simeq} \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p.$$

The case of abelian varieties

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with the natural Hodge filtration

$$\omega_A \subset H^1_{dR}(A/).$$

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General picture

We have obtained a functor

$$\begin{array}{ccccc} \mathrm{BT}^{\varphi}_{/\mathfrak{S}} & \longrightarrow & \mathrm{BT}^{\varphi}_{/S} & \xrightarrow{\simeq} & \mathrm{BT}(\mathcal{O}_K) \\ \mathfrak{M} & \longmapsto & \mathfrak{M} \otimes_{\mathfrak{S},\varphi} S & & \\ & & (\mathbb{D}(G), \mathrm{Fil}^1) & \leftarrow & G \end{array}$$

Relations

$$\mathrm{MF}_K^{\varphi,ad} \xrightarrow{\text{fully faithful}} \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p,$$

and by restriction

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Relations

$$Rep^{cris}_{\mathbb{Q}_p}(G_K) \xrightarrow[Colmez-Fontaine]{\simeq} \mathrm{MF}^{\varphi,ad}_K \xrightarrow{\text{fully faithful}} \mathrm{Mod}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p,$$

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$$Rep^{cris,(0,1)}_{\mathbb{Q}_p}(G_K) \xrightarrow{\simeq} \{\text{adm } \varphi\text{-mod BT-type}\} \xrightarrow{\simeq} \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p.$$

The construction of an inverse

Consider $G \in Bt(\mathscr{O}_K)$ and its Tate module

$$T_p(G) := \mathrm{Hom}(Q_p/Z_p, G \otimes_K \mathcal{O}_{\overline{K}}).$$

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we obtain a $\mathfrak{M} \in \mathrm{BT}^{\varphi}_{/\mathfrak{S}} \otimes \mathbb{Q}_p$.

Abelian varieties: the image of $V_p(A[p^\infty])$ in $\{\text{adm } \varphi\text{-mod BT-type}\}$ is precisely $H^1_{dR}(A)$ (Fontaine).

Classification of finite flat group schemes

As a corollary of the result, we obtain a classification of p-groups.

Definition

A p-group is a finite flat group scheme over $\mathcal{O}_K$, which is killed by a power of p. Denote by $(p - Gr)$ the category of such objects.

The kernel of an isogeny $G \xrightarrow{f} G' \to 0$ of p-divisible groups is a finite flat p-group scheme.

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Theorem (Oort)

Every finite flat p-group H is embedded in a p-divisible group G. The cokernel of the injection $0 \to H \to G$ is again a p-divisible group.

Denote by $(Mod/\mathfrak{S})$ the category of $\mathfrak{S}$-modules $\mathfrak{M}$ such that

- $\mathfrak{M}$ is equipped with an injective semi-linear map $\varphi_{\mathfrak{M}}$,
- $\mathfrak{M}$ is of finite E-height with respect to $\varphi_{\mathfrak{M}}$,
- $\mathfrak{M}$ has projective dimension 1 as an $\mathfrak{S}$-module and it is killed by a power of p.

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The objects in $(Mod/\mathfrak{S})$ are precisely the cokernels of injections in $\mathrm{BT}^{\varphi}_{/\mathfrak{S}}$.

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Theorem (Kisin)

There is an exact anti-equivalence of categories

$$(Mod/\mathfrak{S}) \xrightarrow{\simeq} (p-Gr/\mathcal{O}_K).$$

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Thank you for your attention.