Classification of Dieudonné Modules up to Isogeny

Andrew Fiori

McGill University

April 2013

The Motivation

Why up to Isogeny?

- Easier problem might shed light on the harder problem.
- The theory might actually be nicer.
- Fits in well with a different perspective on Shimura varieties.
- In the end we obtain stratifications of the special fibers of Shimura varieties.

Isogenies of Dieudonné Modules

Definition

We say two Dieudonné modules M and N are isogenous if:

$$M \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq N \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

Basically we are just inverting p. But what does this have to do with isogenies?

Theorem

Let $f : G \to H$ be a morphism of p-divisible groups. The following are equivalent:

- *f is an isogeny.*
- *$M(f) : M(H) \to M(G)$ is injective.*
- *The induced map $M(H) \otimes \mathbb{Q}_p \to M(G) \otimes \mathbb{Q}_p$ is an isomorphism.*

Isomorphism classes of Isogenous Dieudonné Modules

The functor from isogeny classes of p-divisible groups to modules over $A = W(k)[1/p]\langle F \rangle$ taking:

$$G \mapsto M(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is an anti-equivalence between p-divisible groups up to isogeny and such modules where there exists an F, pF^{-1} stable $W(k)$-lattice M.

Theorem

The isomorphism classes of p-divisible groups isogenous to a fixed p-divisible group G are in bijection with the isomorphism classes of F, pF^{-1} stable $W(k)$-lattices contained in

$$M(G) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

Category of A modules

The classification of p-divisible groups up to isogeny is thus highly related to the classification of Dieudonné modules up to isogeny which is in turn related to:

Question

What is the structure of the category of finitely generated modules over

$$W(k)[1/p]\langle F\rangle?$$

It should be emphasized that not all finitely generated $W(k)[1/p]\langle F\rangle$ modules actually come from an isogeny class of p-divisible groups.

Some Observations

Regardless of the field k over which we work every finitely generated $W(k)[1/p]\langle F\rangle$-module admits a grading by irreducible $W(k)[1/p]\langle F\rangle$-modules.

We thus really only need to understand the irreducible $W(k)[1/p]\langle F\rangle$-modules and their extension classes.

If k is a finite field then by restriction every finitely generated A-module becomes a finitely generated $\mathbb{Q}_p[F]$ module. We will not actually use this observation, but it gives a reason to believe a good structure exists.

Semi-simplicity (k-algebraically closed)

The important structure theorem is the following:

Theorem (Manin)

Suppose k is algebraically closed. The category of finitely generated $W(k)[1/p]\langle F\rangle$-modules is semi-simple. The simple objects being the modules:

$$E_{s/r} = W(k)[1/p] \otimes_{\mathbb{Z}_p\langle F\rangle} \mathbb{Z}_p[F]/(F^r - p^s)$$

where r, s are coprime integers, $r > 0$.

Note that

$$E_{s/r} \otimes E_{s'/r'} = E_{(sr'+rs')/rr'}^{gcd(r,r')}$$

which motivates the convention used in some sources:

$$E_{(ns/nr)} = E_{s/r}^{n}.$$

Main Steps of Proof

The main steps of the proof are to prove each of the following statements.

1. There are no non-trivial morphisms $E_\lambda \to E_{\lambda'}$ for $\lambda \neq \lambda'$.
2. Any non-trivial morphism $E_\lambda \to E_\lambda$ is an isomorphism.
3. There are no non-trivial extensions of E_λ by $E_{\lambda'}$.
   (This uses that k is algebraically closed.)
4. Every A-module has a quotient isomorphic to E_λ for some λ.
   (This uses that k is algebraically closed.)
5. The modules E_λ are simple.

Some comments

It is not surprising that $\mathbb{Q}_p[F]/(F^r - p^s)$ is simple since it is a 'field'. It is also not surprising it remains simple after unramified base extensions. Note that in this context the resulting object is a 'skew-field' and not a 'field'.

The statements about morphisms between these is then also not so surprising. Though the actual proof is a direct computation showing no element of $E_{s/r}$ is acted upon by $F^{r'}$ as $p^{s'}$ unless $s/r = s'/r'$.

The statement about extensions being trivial reduces to showing that the operator $F^{r'} - p^{s'}$ is surjective on $E_{r/s}$. This is done by proving you can solve a certain system of equations using that k is algebraically closed.

For an irreducible $W(k)[1/p]\langle F\rangle$-module M of dimension r over $W(k)[1/p]$ then for all $0 \neq x \in M$ the set:

$$\{x, Fx, F^2x, \dots F^{r-1}x\}$$

is a $W(k)[1/p]$ basis for M and thus we can 'represent' F by the matrix:

$$\begin{pmatrix} 0 & 0 & \dots & a_0 \\ 1 & 0 & \dots & a_1 \\ & \ddots & & \vdots \\ & & 1 & a_n \end{pmatrix}$$

When k is algebraically closed we are claiming that there exists

$$x' = \sum b_i F^i x$$

with $F^{r'}x' - p^{s'}x' = 0$. These equations can be solved over an algebraically closed field once one picks appropriate values r', s'.

Slopes

We define the slope of the simple Dieudonné module E_λ to be λ. This slope entirely determines the isomorphism class for simple modules over an algebraically closed k.

One should notice that when we express:

$$\lambda = \frac{s}{r}$$

then:

- r is the dimension of the Dieudonné module which if this came from a p-divisible group would be that groups rank.
- If this were coming from a p-divisible group then s would be the dimension of the tangent space of the group.

Newton Polygons

As slopes determine the simple modules over an algebraically closed k. A collection of slopes given in increasing order and with multiplicity:

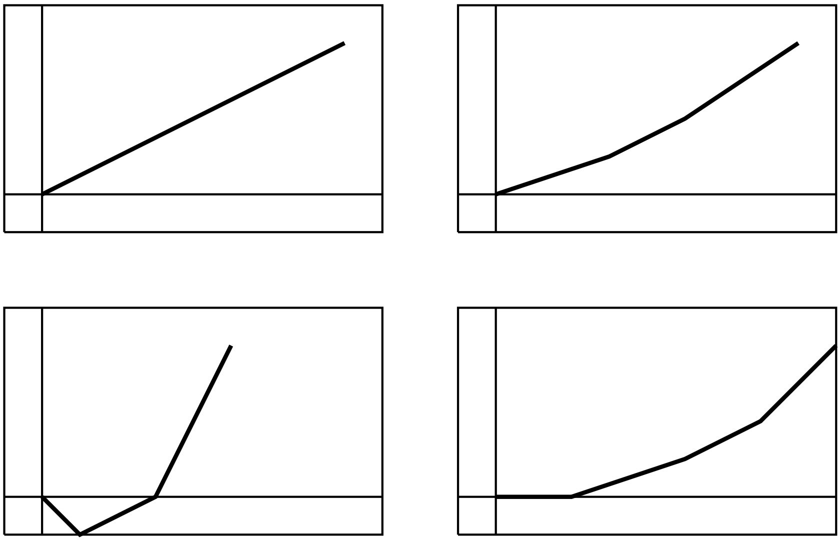
$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n$$

determines an isomorphism class of A-module when k is algebraically closed. Concretely it determines the A-module:

$$E_{\lambda_1} \oplus E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_{n-1}} \oplus E_{\lambda_n}.$$

One can arrange this collection of 'slopes' into pictures which we call Newton polygons. To the slope s/r we draw a line segment of slope s/r of horizontal length r.

Example Polygons



Question

Given an A-module how would you go about computing the Newton polygon?

Suppose k has p^a elements. Then for any $W(k)[1/p]\langle F\rangle$-module we have that F^a is linear.

Theorem (Manin)

Let M be a Dieudonné module over $W(k)[1/p]\langle F\rangle$ where k has p^a elements. Let $P(X) = det(F^a - X\,id) = \prod(\tau_i - X)$ be the characteristic polynomial of F^a. Then the slopes of M are:

$$v_{W(k)}(\tau_i)$$

counted with multiplicities.

This follows based on a straight forward computation using the classification theorem.

Newton Polygons of p-divisible groups

Not all possible A-modules arise from p-divisible groups and thus not all Newton polygons arise from p-divisible groups. This leads to the natural question:

Question

Which A-modules arise from p-divisible groups?

To which we have the following answer:

Theorem

A Newton polygon arises from a p-divisible group if and only if all the slopes λ satisfy $0 \leq \lambda \leq 1$.

Set:

$$W = \underset{\rightarrow}{Lim}(Ker\ p^n : W_{\mathbb{F}_p} \rightarrow W_{\mathbb{F}_p})$$

Then the associated group G_λ is the kernel of the map $F^r - V^s$ acting on W.

Newton Polygons of Abelian Varieties

Given the existence of a map from abelian varieties to Newton polygons (by way of the associated p-divisible group and its Dieudonné module) a very natural question arises

Question

Which Newton polygons are in the image of this map?

Admissible Newton Polygons

The answer depends on what structures your abelian variety can have.

- Dimension g start $(0, 0)$ end at $(2g, *)$.
- A polarization implies it is 'symmetric', in the sense that λ and $1 - \lambda$ must appear with the same multiplicity.
- This in turn implies the end point is $(2g, g)$.

Theorem

Every Newton polygon satisfying the above arises from an abelian variety.

Kottwitz Formulation

It is sometimes useful to encode data about decompositions of vector spaces using the representation of a torus on the vector space. This is what is typically done with the Hodge filtration in the theory of Shimura varieties.
The same thing can be accomplished for slopes. Since we want rational slopes $\mathbb{G}_m$ is insufficient its character group is $\mathbb{Z}$. However, using that the character group of the pro-algebraic group:

$$\mathbb{D} = \underset{\leftarrow}{\mathrm{Lim}}\, \mathbb{G}_m$$

is $\mathbb{Q}$ the slope decomposition for M can be encoded in by a representation of $\mathbb{D}$ on M.

In the end one finds themselves interested in the *σ-conjugacy classes* of the representations of $\mathbb{D}$ satisfying axioms making them admissible. We will not go so deeply into the theory.

Basic Stratum

Theorem

The subset of abelian varieties with a given Newton polygon in the special fibre of a Shimura variety gives a stratification.

Using the interpretation of Kottwitz we define the following:

Definition

The *basic stratum* is the one for which the cocharacter of $\mathbb{D}$ factors through the centre of the corresponding reductive group.

In general the basic strata are the lowest dimensional (hence closed) strata, they correspond to the 'highest' possible Newton polygons.

Example

For the case of polarized abelian varieties the reductive group is the group of symplectic similitudes GSp_{2g} and its centre is the scalar multiples of the identity.
The only cocharacter which can factor through the centre have a single eigenvalue. Thus, the basic stratum is those whose Newton polygons have all slopes equal.
In this was we see that the basic stratum for the family of polarized abelian varieties consists of *supersingular* abelian varieties.

Additional Structures

The functoriality of the construction tells us that if an abelian variety has endomorphism algebra B then these must also act on the Dieudonné module. This leads to another natural question:

Question

What restrictions on the Newton polygon is imposed by the existence of extra endomorphisms?

This problem is important as it is highly linked to studying the special fibres of PEL Shimura varieties.

We won't give all the results, as there are too many cases, we just mention one example...

The case of real multiplication

Suppose we restrict attention to the case of principally polarized abelian varieties of dimension g with real multiplication by $\mathcal{O}_K$, here K is a totally real field in which p is inert of degree g over $\mathbb{Q}$.

Theorem (Goren-Oort)

The Dieudonné module of A an abelian variety as above is of the form either:

$$E_{\ell/g} \oplus E_{1-\ell/g}$$

for some $0 \le \ell \le g$ (we are not assuming ℓ, g coprime) or

$$E_{1/2}^{g}.$$

Their results actually say much more than this, they in fact give the dimension of the moduli space for the possible Newton polygons.

The End

Thank you.