p-Divisible Groups: Definitions and Examples

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Connected vs. étale

Suppose $S = \operatorname{Spec} R$ is the spectrum of a complete, local, noetherian ring. Let k be the residue field at the closed point of R. We have canonical exact sequence

$$0 \to G^0 \to G \to G^{\text{et}} \to 0$$

where G^0 is connected and G^{et} is étale. The functors associating G with G^0 or G^{et} are exact.

- G is connected iff $G = G^0$. In such case, the order of G is a power of $p = \text{char}(k)$ (or 1 if $p = 0$).
- G is étale iff $G = G^{\text{et}}$. In such case, G becomes a constant group scheme after base change. The functor $G \to G(\overline{k})$ is an equivalence of categories between finite, commutative étale R-groups and finite π-modules on which π acts continuously ($\pi = \text{Gal}(\overline{k}/k)$).
- When $R = k$ is a perfect field, the exact sequence splits so $G \cong G^0 \times G^{\text{et}}$.

Cartier dual

Let $G = \mathrm{Spec}(A)$ over R. Define $A' = \mathrm{Hom}_R(A, R)$.

$$
\begin{array}{rcl}
\begin{array}{c}
m^{\#} : A \to A \otimes A \\
\mu : A \otimes A \to A \\
\pi^{\#} : R \to A \\
e^{\#} : R \to A \\
\iota^{\#} : A \to A
\end{array}
& \Rightarrow &
\begin{array}{c}
\mu' : A' \to A' \otimes A' \\
m'^{\#} : A' \otimes A' \to A' \\
e'^{\#} : A' \to R \\
\pi'^{\#} : A' \to R \\
\iota'^{\#} : A \to A
\end{array}
\end{array}
$$

These dual maps turn A' into a ring and the scheme $G' = \mathrm{Spec}(A')$ into a finite commutative group scheme with the same order as G. G' is called the *Cartier Dual* of G.

When we are again in the case where $R = k$ is a perfect field, this gives a further splitting of the category of finite commutative group schemes:

$$G \cong G^{0-0} \times G^{0-\mathrm{et}} \times G^{\mathrm{et}-0} \times G^{\mathrm{et}-\mathrm{et}}.$$

- If k is characteristic 0, then all but the étale - étale category are trivial
- If k has characteristic p and is algebraically closed, then the only non-trivial simple objects for each category are

$$\begin{array}{rll} 0-0: & \alpha_p & \text{(Self dual)} \\ 0-\text{et}: & \mu_p & \text{(Dual to } \underline{\mathbb{Z}/p\mathbb{Z}}) \\ \text{et}-0: & \underline{\mathbb{Z}/p\mathbb{Z}} & \text{(Dual to } \mu_p) \\ \text{et}-\text{et}: & \underline{\mathbb{Z}/l\mathbb{Z}} & (l \neq p \text{ a prime.}) \end{array}$$

Because the étale - étale subcategory can be understood very well through Galois action and the étale fundamental group, we want to study the groups arising in the other three subcategories. All such groups must have p-power rank and torsion, in the sense that the map $[p^t] : G \to G$ is equal to $e \circ \pi$ for some sufficiently large power t.

p-torsion groups

When k has positive characteristic, we have useful tools to study each subcategory of the finite p-torsion commutative group schemes.

Canonical Group Homomorphisms

- Relative Frobenius $F : G \to G^{(p)}$
- Verschiebung $V : G^{(p)} \to G$
  Where V is the dual map of $F : G' \to G'^{(p)} \cong (G^{(p)})'$
- $V \circ F = [p] : G \to G, \qquad F \circ V = [p] : G^{(p)} \to G^{(p)}$

Examples

- V is the identity on $G = G^{(p)} = \mu_p$ and F is the zero map. By duality, F is the identity and V is trivial on $\underline{\mathbb{Z}/p\mathbb{Z}}$. More generally $F = [p]$ and $V = Id$ on μ_{p^r}.
- F (and hence V) is trivial on α_p. More generally for α_{p^n}, $\mathrm{Ker}(F) \cong \alpha_p$ and $\mathrm{Ker}(V) = \alpha_{p^n}$.

p-divisible groups

Let $h \in \mathbb{N}$. A p-divisible (*Barsotti-Tate*) group G over base S of height h is an inductive system

$$G = (G_v, i_v), \qquad v \geq 0,$$

where

(i) G_v is a finite flat group scheme over S of order p^{vh},

(ii) for each $v \geq 0$,

$$0 \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{[p^v]} G_{v+1}$$

is exact.

An immediate result of (ii) is the exact sequences

$$(*) \qquad 0 \longrightarrow G_v \xrightarrow{i} G_{v+u} \xrightarrow{[p^v]} G_u \longrightarrow 0.$$

Observe that if the G_v are étale, then when $S = \operatorname{Spec} k$ is the spectrum of a field,

$$G_v(\overline{k}) \cong (\mathbb{Z}/p^v\mathbb{Z})^h.$$

A *homorphism* of p-divisible groups $f : G \to H$ is a system of maps $(f_v : G_v \to H_v)$ that commute inclusion maps i_v. Such a map is an *isogeny* if it is an epimorphism with finite kernel. The composition of two isogenies is an isogeny.

Every Barsotti-Tate group comes equipped with the family of isogenies $\{[p^v]\}$ with kernel G_v. A map f is called a *quasi-isogeny* if $[p^n] \circ f$ is an isogeny for some n.

By considering each G_v as an fppf sheaf over S, we may form the limit

$$G := \varinjlim G_v$$

in the category of fppf sheaves of abelian groups. Each $G_v = G_v[p^v]$. This is actually an equivalence between BT-groups and a full subcategory of fppf sheaves of abelian groups over S.
Example: $\lim \mathbb{Z}/p^v\mathbb{Z} = \lim \frac{1}{p^v}\mathbb{Z} = \mathbb{Q}_p/\mathbb{Z}_p$ as fppf sheaves.

Serre dual

Each p-divisible group has a dual group induced from the dual diagram of $(*)$:

$$0 \longrightarrow G'_u \xrightarrow{[p^v]'} G'_{v+u} \xrightarrow{i'} G'_v \longrightarrow 0.$$

In particular we see that $\{(G'_v, [p]' : G'_v \to G'_{v+1})\}$ gives an inductive system of finite flat group schemes. One can show that the system also satisfies (ii) making it a p-divisible group, called the *Serre Dual* of G.

<u>Example</u> The Cartier duals of $\underline{\mathbb{Z}/p^n\mathbb{Z}}$ are μ_{p^n}. $[p]' : \mu_{p^n} \to \mu_{p^{n+1}}$ corresponds to map on rings $x \in R[x]/(x-1)^{p^{n+1}} \to x \in R[x](x-1)^{p^n}$. As maps of sheafs this corresponds to $(\mathbb{Q}_p/\mathbb{Z}_p)' = \mu_{p^\infty}$.

Abelian varieties and p-divisible groups

Suppose A is an abelian variety over perfect field k of dimension g. Let $A(p)$ denote the p-divisible group of p-torsion points of A. $A(p)$ is of height $2g$.

When $\mathrm{char}(k) \neq p$:

- $A(p)$ is étale.
- The p-adic *Tate module* $T_p(A) = A(p)(\overline{k})$ gives a Galois representation $\mathrm{Gal}(\overline{k}/k) \to GL_{2g}(\mathbb{Z}_p)$, $(T_p(A))^\vee = H^1_{et}(A, \mathbb{Z}_p)$.
- Galois action on $T_p(A)$ gives local p-component of the Hasse-Weil zeta function when k is a number field.

Tate Conjecture (Tate, Faltings)

Suppose k is finitely generated over its prime field, and $\mathrm{char}(k) \neq p$. If A and B are two abelian varieties over k then

$$\mathrm{Hom}_k(A, B) \otimes \mathbb{Z}_p \cong \mathrm{Hom}_{\mathrm{Gal}(k)}(T_p(A), T_p(B)).$$

Classification through $A(p)$

Suppose $\mathrm{char}(k) = p$.

- $A[p]$ killed by $[p] = V \circ F$. $p^g = \deg(F) = \deg(V)$ so $p^f = |A[p](\overline{k})| \leq p^g$

$$A(p)_{\overline{k}} \cong (\mu_{p^\infty})^e \times (\mathbb{Q}_p/\mathbb{Z}_p)^f \times G^{0,0}.$$

- $A^t(p) = (A(p))'$, so $A(p)$ is isogenous to $(A(p))'$. Thus $e = f$.
- a-number: $a := \dim_k \mathrm{Hom}(\alpha_p, A) = \dim_k \mathrm{Hom}(\alpha_p, G^{0,0}[p])$. $a \leq g$ because $\mathrm{Ker}(F)$ has rank p^g.

Can stratify the moduli space $\mathcal{A}_g \otimes k$ via the isogeny class of p-divisible group (Newton polygons) or isomorphism class of p-torsion (EO strata). Leads to interesting cycles in $Ch^*(\tilde{\mathcal{A}}_g)$ (ex. $[V_f]$ a multiple of tautological class λ_{g-f}, recover Deuring Mass Formula when $g = 1, f = 0$).

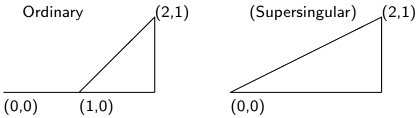
Low dimension examples

$\underline{g = 1:}$

- A is an ordinary elliptic curve $A[p]_{\overline{k}} \cong \mu_p \oplus (\mathbb{Z}/p\mathbb{Z})$.
- A is a supersingular elliptic curve, $A[p]_{\overline{k}}$ sits in non-split exact sequence

$$0 \to \alpha_p \to A[p] \to \alpha_p \to 0.$$

The embedded subgroup α_p is unique since $a = 1$. Call $A[p] =: M$ in this case.

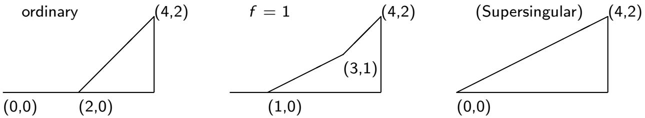


$\underline{g=2}$:

- A is again ordinary: $A[p]_{\overline{k}} \cong (\mu_p)^2 \oplus (\mathbb{Z}/p\mathbb{Z})^2$.
- A has étale part of order p, so $A[p]_{\overline{k}} \cong M \oplus \mu_p \oplus (\mathbb{Z}/p\mathbb{Z})$.
- $A[p]$ has no étale part. If A is superspecial, then $A[p] \cong M \oplus M$.
- Otherwise there is a filtration $G_1 \subset G_2 \subset G_3 \subset A[p]$ with

$$G_1 \cong \alpha_p, \quad F(G_2) = 0, \quad V(G_3) = V(G_2) = G_1, \quad V(A[p]) = G_2.$$

In particular $A[p]$ has $a = 1$ so A is distinct from the superspecial case.



In general there are 2^g EO strata, given by a *final sequence* $\psi : \{1, \ldots 2g\} \to \{1, \ldots, g\}$.

Deformation theory

For both abelian schemes and p-divisible groups, we have the notion of deformations. If $S \to T$ is a morphism, a deformation of A_S is an abelian scheme A'_T together with isomorphisms $A'_T \times_T S \cong A_S$. Similar for p-divisible groups. A deformation of A defines a deformation of $A(p)$. The Serre-Tate theorem provides a converse in the case of infinitesimal deformations when S has characteristic p:

Theorem (Serre-Tate)

Let R be a ring in which p is nilpotent, $I \subset R$ a nilpotent ideal, $R_0 = R/I$. Then the functor $A \to (A_0, A[p^\infty], \epsilon)$ from the category of abelian schemes over R to the category of triples:

$i)A_0$ an abelian scheme over R_0
$ii)\Gamma_{R_0}$ a p-divisible group over R
$iii)\epsilon : \Gamma \to A_0(p)$ an isomorphism

is an equivalence of categories.

- Serre-Tate allows one to solve deformation problems of abelian varieties by appealing to the deformation theory of p-divisible groups (Dieudonne modules and semi-linear algebra).
- Many alternative formulations and generalizations of the above theorem, including various endomorphism and polarization structure (Replace R_0 with a field k, R with artinian $W(k)$-algebra). For example, when L is a totally real field

$$\left\{ \begin{array}{c} \text{Iso. classes of deformations} \\ \text{of } (A, \iota, \lambda, \alpha)/k \text{ to } R \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Iso. classes of deformations} \\ \text{of } (A(p), \iota(p))/k \text{ to } R \end{array} \right\}$$

where $\iota(p)$ is a homomorphism $\mathcal{O}_L \to \mathrm{End}(A(p))$.

p-divisible groups and Formal groups

Let R be a complete, noetherian, local ring with residue field k of characteristic $p > 0$. Recall that an n-dimensional commutative *formal group* $\Gamma = \mathrm{Spf}(R[[x_1, \ldots, x_n]])$ over R is

$$F = (F_i \in R[[z_1, \ldots, z_n, y_1, \ldots, y_n]])_{i=1}^{n}$$

satisfying

(i) $X = F(X, 0) = F(0, X)$,

(ii) $F(X, F(Y, Z)) = F(F(X, Y), Z)$,

(iii) $F(X, Y) = F(Y, X)$.

Write $\psi(X) = F(X, F(X, \ldots, F(X, X)) \ldots)$ (p X's) and $[p]$ the corresponding map on Γ. We say Γ is p-divisible if $[p] : \Gamma \to \Gamma$ is an isogeny ($R[[x_1, \ldots, x_n]]$ is free of finite rank over itself with respect to ψ).

If Γ is p-divisible, then for $v \in \mathbb{N}$, the scheme

$$G_v = \mathrm{Spec}(R[[x_1, \ldots, x_n]]/\psi^v(< x_1, \ldots, x_n >))$$

is a connected group scheme over R. We get a p-divisible group $\Gamma(p) = (G_v, i_v)$.

Theorem (Tate)

Given R as above, the map $\Gamma \to \Gamma(p)$ is an equivalence of categories between p-divisible formal groups over R and connected p-divisible groups over R.

This identification allows us to study p-divisible groups by their associated formal groups and vice-versa.
For example, every p-divisible group G has an invariant dimension d defined by the dimension of its formal group (of its connected component).
The dimension satisfies: $d + d' = h$ (d' the dimension of G').

Formal group applications

Local CFT:

Fix a local field K and its ring of integers $\mathcal{O}$ with residue field $\mathbb{F}_q$. For $\pi \in K$ prime, let $f = \pi.X + X^q$. There exists a unique 1-dimensional *Lubin-Tate* commutative formal group (Γ, F_f) over $\mathcal{O}$ such that f is an endomorphism. (Γ, F_f) is in fact a formal $\mathcal{O}$-module, with π acting by f.

- $\mathfrak{m}^s = \{\alpha \in K^s | \quad |\alpha| < 1\}$ becomes an $\mathcal{O}$-module with

$$\alpha + \beta = F_f(\alpha, \beta), \qquad a.\alpha = [a]_f(\alpha).$$

- The submodule λ_n killed by π^n is isomorphic to $\mathcal{O}/(\pi^n)$

Theorem

For each n let $K_{\pi,n} = K[\lambda_n]$

(a) *$K_{\pi,n}$ is totally ramified of degree $(q-1)q^{n-1}$.*

(b) *$\mathcal{O}$ acting on λ_n defines an isomorphism $(\mathcal{O}/(\pi)^n)^\times \to \mathrm{Gal}(K_n/K)$.*

(c) *$\bigcup K_n$ is a maximal totally ramified extension of K. Hence $\mathrm{Gal}(K^{ab}/K) \cong \mathcal{O}^\times \times \widehat{\mathbb{Z}}$.*

Algebraic Geometry:

$f : X \to \operatorname{Spec}(R)$ a proper flat variety, $\widehat{\operatorname{Pic}}(X)$. If A an artinian local R-algebra with residue field k, then:

$$0 \to \widehat{\operatorname{Pic}}(X)(\operatorname{Spec} A) \to H^1(X \times A, \mathbb{G}_m) \to H^1(X, \mathbb{G}_m).$$

- $\widehat{\operatorname{Pic}}(X)$ is pro-representable by a formal group scheme. (Schlessinger)
- Define functors $\Phi^r(X) : \mathrm{Art}/R \to \mathrm{Ab}$ that fit into similar exact sequence for $H^r(X, \mathbb{G}_m)$. $\Phi^q(X)$ is pro-representable when $R^{q-1}f_*(\mathbb{G}_m)$ is *formally smooth*. $\Phi^2(X) = \hat{\mathrm{Br}}(X)$ is the *formal Brauer group*. (Artin-Mazur)

When $k = R$:

- $\dim \Phi^r(X) = h^{0,r} = \dim_k H^r(X, \mathcal{O}_X)$.
- $D\Phi^r(X) = H^r(X, \mathcal{W})$ ($\mathcal{W}$ the Witt vector sheaf).
- If $\hat{\mathrm{Br}}(X)$ is representable by a formal group of finite height h (+hypotheses) then

  $$\mathrm{rk}(NS(X)) \leq b_2 - 2h \qquad (b_2 \text{ crystalline Betti number}).$$

  This is stronger than the characteristic 0 equivalent since $h \geq h^{0,2}$.
- One can stratify moduli spaces through height of the formal Brauer groups to get interesting cycles (ex. Katsura & van der Geer: $K3$ surfaces and abelian surfaces).

Topology:

Let h^* be a *complex-oriented* generalized cohomology theory ($h^*(\mathbb{CP}^\infty) = S[[x]], \frac{1}{2} \in S$, ex. Betti cohomology, complex cobordism). If γ is the universal line bundle, there exists

$$m : \mathbb{CP}^\infty \times \mathbb{CP}^\infty \to \mathbb{CP}^\infty$$

such that $m^*(\gamma) = \gamma \boxtimes \gamma$. m (unique up to homotopy) along with i (complex conjugation) and e (constant map) form an *abelian group law up to homotopy*. The dual map

$$m^* : h^*(\mathbb{CP}^\infty) \to h^*(\mathbb{CP}^\infty) \otimes_S h^*(\mathbb{CP}^\infty)$$

is a 1-dimensional formal group law.

- Associate ring map (complex genus) $\phi : \Omega_*^U \to S$

$$\log_\phi(x) = \sum_{n\geq 0} \frac{\phi([\mathbb{CP}^n])}{n+1} x^{n+1}.$$

- ϕ is *elliptic* if

$$\log_\phi(x) = \int_0^x \frac{1}{\sqrt{1-2\delta t^2+\epsilon t^4}} dt, \qquad \delta, \epsilon \in S.$$

- ϕ elliptic iff

$$F(x,y) = \frac{x\sqrt{1-2\delta y^2+\epsilon y^4} + y\sqrt{1-2\delta x^2+\epsilon x^4}}{1-\epsilon x^2 y^2}$$

  which is associated to elliptic curve $y^2 = 1-2\delta x^2+\epsilon x^4$.

We recover elliptic cohomology theory in this way.

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