A *Brief* Introduction to Hodge Structures

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Basic Definitions

Let V be a **real**, finite dimensional vector space. We define complex conj. on $V(\mathbb{C}) = \mathbb{C} \otimes V$ as

$$\overline{z \otimes v} = \overline{z} \otimes v.$$

Definitions

- A Hodge Decomposition of V: $V(\mathbb{C}) = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$, such that $V^{p,q} = \overline{V^{q,p}}$
- A (real) Hodge Structure: A real vector space + Hodge Decomposition
- Type of Hodge Structure: Set of 2-tuples such that $V^{p,q} \neq \{0\}$
- Weight Decomposition: $V = \bigoplus V_n$, where $V_n = \left(\bigoplus_{p+q=n} V^{p,q}\right) \cap V$
- An integral (rational) Hodge Structure: A free $\mathbb{Z}$ (or $\mathbb{Q}$) module W + a Hodge Structure on $W(\mathbb{R})$ such that the $W = \bigoplus(V_n \cap W)$

Basic Examples

We will usually be interested in the case when $V_n = \{0\}$ except for one fixed $n \in \mathbb{Z}$. In such a case we say the Hodge structure on V has **weight** n.

- $\mathbb{Q}(m)$ is the unique rational Hodge Structure of weight $-2m$ on the vector space $\mathbb{Q}$, ie. $(\mathbb{Q}(m))(\mathbb{C}) = \mathbb{Q}(m)^{-m,-m}$.

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- Suppose V is a real vector space with complex structure J ($J^2 = -Id$). $V(\mathbb{C}) = V^{1,0} \bigoplus V^{0,1}$ corresponds to the decomposition into $\pm i$ eigenspaces. We will see later that every real Hodge Structure of type $\{(1,0),(0,1)\}$ corresponds to a real vector space with a complex structure.

Hodge Structures as representations of $\mathbb{S}$

Consider $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$ as an algebraic torus over $\mathbb{R}$.

$$\mathbb{S}(\mathbb{R}) \cong \mathbb{C}^\times \quad \text{and} \quad \mathbb{S}(\mathbb{C}) \cong \mathbb{C}^\times \times \mathbb{C}^\times$$

Complex conj. acts on $\mathbb{S}(\mathbb{C})$ via $\overline{(z_1, z_2)} = (\overline{z_2}, \overline{z_1})$. We conclude $X^*(\mathbb{S}) = \mathbb{Z} \times \mathbb{Z}$ w/ Galois action sending (p, q) to (q, p).

Theorem

Let T be a torus over a field k, and K a Galois extension of k splitting T. To give a representation h of T on a k-vector space V amounts to giving a $X^*(T)$-grading $V \otimes K = V(K) = \bigoplus_{\chi \in X^*(T)} V_\chi$ with the property that

$$\sigma(V_\chi) = V_{^\sigma\chi}, \quad \text{for all } \sigma \in Gal(K/k), \quad \chi \in X^*(T).$$

The V_χ will be subspaces of simultaneous eigenvectors on which T acts through χ:

$$h(t)v = \chi(t) \cdot v, \quad \text{for all } v \in V_\chi, \quad t \in T(K).$$

Hodge Structures as representations of $\mathbb{S}$ (cntd.)

- By the theorem, if V is a real vector space, then a representation of algebraic groups $h : \mathbb{S} \to \mathrm{GL}(V)$ leads to a decomposition of $V(\mathbb{C})$ into simultaneous eigenspaces $V^{p,q}$ such that $\overline{V^{p,q}} = V^{q,p}$, ie. a Hodge decomposition.
- $v \in V^{p,q}$ iff $h(z)v = z^p \overline{z}^q v$ for all z (Discrepancy in notation between sources).
- Converse is also true. See (van Geeman) for construction of representation given Hodge decomposition.
- Weight decomposition corresponds to space

$$V_n = \{v \in V | h(r)v = r^n v, \ \forall r \in \mathbb{R}^* \subset \mathbb{S}(\mathbb{R})\}.$$

Basic Examples (Again!)

- The corresponding representation for $\mathbb{Q}(m)$ is the map $\mathbb{C}^\times \to \mathbb{R}$ sending z to multiplication by $(z\overline{z})^{-m}$.

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- For a Hodge structure arising from complex structure J as above, $h(a+bi) = a \cdot Id + b \cdot J$ gives the correct representation. Conversely, given any Hodge structure of type $\{(1,0),(0,1)\}$, the map $h(i)$ is a linear operator on $V(\mathbb{R})$ such that $h(i)^2 = -Id$.

More Definitions

Suppose (V, h_V) and (W, h_W) are Hodge structures of weight k_V and k_W. Then their tensor product induces a Hodge structure $(V \otimes W, h_V \otimes h_W)$ of weight $k_V + k_W$ where

$$(h_V \otimes h_W(z))v \otimes w = h_V(z)v \otimes h_W(z)w.$$

The dual space of a rational or real Hodge structure (V, h_V) also has an induced structure of weight $-k_V$,

$$h_V^* : \mathbb{C}^\times \to GL(V_\mathbb{R}^*), \quad (h_V^*(z)f)(v) = f(h_V(z)^{-1}v).$$

Combining these two results allows one to induce a Hodge structure on $\mathrm{Hom}(V, W) \cong V^* \otimes W$ when V and W are Hodge structures of vector spaces.

More Definitions (cntd.)

A **morphism of Hodge structures** is a map of representation spaces $f : (V, h_V) \to (W, h_W \otimes h_{\mathbb{Q}(-n)})$ for some fixed $n \in \mathbb{Z}$, ie.

$$f(h_V(z)v) = (z\bar{z})^n h_W(z) f(v).$$

A morphism is called strict when $n = 0$.

When V is a rational Hodge structure of weight $k = 2n$, we call the $\mathbb{Q}$ subspace

$$B(V) := V \cap V^{n,n}$$

the space of **Hodge classes**.

If $f : V \to W$ is a strict morphism of rational Hodge structures, we can consider $f \in \mathrm{Hom}_{\mathbb{Q}}(V, W) = V^* \otimes W$ with its natural Hodge structure. Then $(h_V^*(z) \otimes h_W(z) f)(v) = h_W(z) f(h_V(z^{-1})v) = f(v)$ and so f is of type $(0, 0)$ and therefore a Hodge class. It is not hard to see that in fact $B(V^* \otimes W)$ is equal to the set of all strict Hodge morphisms.

A Motivation for Hodge Classes

Let X be a smooth, complex, projective variety. Then the k^{th} Betti cohomology admits a rational Hodge structure of weight k:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

Here we identify $H^k(X, \mathbb{C})$ with harmonic differential forms and the subspaces $H^{p,q}(X)$ consist of the harmonic forms of type (p, q).

Now suppose Z is any irreducible subvariety of codimension p in X. Then there is a natural map sending Z into $H^{2p}(X, \mathbb{Q})$ and this extends to a linear map on the group of codimension-p cycles, $\sum a_i Z_i \to \sum a_i [Z_i]$. The image of this map is contained in $B(H^{2p}(X, \mathbb{Q}))$ and the Hodge Conjecture asserts that every Hodge class lies in the image.

Polarizations

Let (V, h) be a Hodge structure of weight k over R. A **polarization** on V is a bilinear map:

$$\Psi : V \times V \to R$$

such that for $v, w \in V(\mathbb{R})$ we have:

$$\Psi_{\mathbb{R}}(h(z)v, h(z)w) = (z\bar{z})^k \Psi_{\mathbb{R}}(v, w)$$

and the map $\Psi_{\mathbb{R}}(v, h(i)w)$ is symmetric and positive definite. A Hodge structure is called polarized if such a map exists. Any polarization induces a strict morphism from $V \otimes V$ to $R(-k)$.

Let $\Lambda \subset \mathbb{C}^n$ be a full lattice and $M = \mathbb{C}^n/\Lambda$ a complex torus. Λ has an integral Hodge structure of type $\{(1,0),(0,1)\}$ induced from that fact that $\Lambda \otimes \mathbb{R} = \mathbb{C}^n$.

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Question: When is M projective?

Hodge Structures & Complex Abelian Varieties

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Question: When is M projective?

- Appell-Humbert $\Rightarrow$ M is projective precisely when there exists a positive definite **Riemann form** on $\mathbb{C}^n$ relative to Λ

Riemann form relative to Λ: A Hermitian form H on $\mathbb{C}^n$ whose imaginary part $E(x,y) = \text{Im } H(x,y)$ restricted to $\Lambda \times \Lambda$ has image in $\mathbb{Z}$. E is necessarily alternating and satisfies $E(ix,iy) = E(x,y)$. Conversely, any map $E : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ with these properties induces a Riemann form $H = E(ix,y) + iE(x,y)$.

It is easy to check that given a polarization Ψ of the Hodge structure on Λ, $-\Psi$ satisfies the criteria above for the imaginary part E of a Riemann form. Conversely the negative of the imaginary part of any Riemann form induces a polarization.

Conclusion: A complex torus is polarized (projective) iff the natural Hodge structure on its lattice is.

Theorem (Riemann's Theorem)

The functor $A \to H_1(A, \mathbb{Z})$ is an equivalence of the category AV of polarized abelian varieties over $\mathbb{C}$ to the category of polarizable integral Hodge structures of type $\{(1,0),(0,1)\}$ with strict morphisms.

Principally Polarized Hodge Structures

We can take the results of Riemann's Theorem even further and talk about *principally* polarized complex manifolds and the corresponding Hodge structures.

$\Psi : \Lambda \times \Lambda \to \mathbb{Z}$ is a principal polarization if there exists a Frobenius basis for Ψ such that the corresponding matrix representation is $\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$.

- Riemann's Theorem $\Rightarrow$ Category of n-dim. principally polarized abelian varieties is equivalent to $2n$-dim. principally polarized integral Hodge structures of type $\{(1,0),(0,1)\}$.

This space is parameterized by $Sp_{2n}(\mathbb{Z})\backslash\mathfrak{h}_n$ where

$$\begin{aligned} \mathfrak{h}_n &= \{N \in M_n(\mathbb{C}) : N = N^t, \mathrm{Im}(N) \gg 0\} \\ &\cong Sp_{2n}(\mathbb{R})/K, \quad \text{where } K \text{ is a maximal compact subgroup.} \end{aligned}$$

Given $N \in \mathfrak{h}_n$, one can recover the corresponding lattice as $\mathrm{Sp}\{(Id_n, N)\}$ and the polarization wrt. this lattice is $\begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$.

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