Canonical models of Shimura varieties

Brian D. Smithling

University of Toronto

Montreal-Toronto Workshop in Number Theory
September 4, 2010

Motivation

Let V be a variety defined over $\mathbb{C}$.

Motivation

Let V be a variety defined over $\mathbb{C}$.

To study V in an arithmetic setting, it's reasonable to ask:

Motivation

Let V be a variety defined over $\mathbb{C}$.

To study V in an arithmetic setting, it's reasonable to ask:

1. Does V admit a model over a number field?

Motivation

Let V be a variety defined over $\mathbb{C}$.

To study V in an arithmetic setting, it's reasonable to ask:

1. Does V admit a model over a number field? (Not necessarily!)

Motivation

Let V be a variety defined over $\mathbb{C}$.

To study V in an arithmetic setting, it's reasonable to ask:

1. Does V admit a model over a number field? (Not necessarily!)
2. If so, how many?

Motivation

Let V be a variety defined over $\mathbb{C}$.

To study V in an arithmetic setting, it's reasonable to ask:

1. Does V admit a model over a number field? (Not necessarily!)
2. If so, how many? (Possibly lots!)

Motivation

Let V be a variety defined over $\mathbb{C}$.

To study V in an arithmetic setting, it's reasonable to ask:

1. Does V admit a model over a number field? (Not necessarily!)
2. If so, how many? (Possibly lots!)

The aim of this talk is to formulate a notion for a model of a Shimura variety to be *canonical*, and to state that every Shimura variety has a canonical model over its *reflex field* $E(G, X)$.

References

- P. Deligne, *Travaux de Shimura,* in Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, Springer, Berlin, 1971, pp. 123–165. Lecture Notes in Math., Vol. 244.
- ______ , *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques,* in Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 247–289.
- U. Görtz, *Mini-Kurs Shimura-Varietäten* (notes in German), available at `http://www.uni-due.de/~hx0050/bn/shimura.shtml`.
- J. Milne, *Introduction to Shimura varieties,* in Harmonic analysis, the trace formula, and Shimura varieties, Clay Math. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 2005, pp. 265–378. Extended version available at `http://www.jmilne.org/math`.

The cocharacter μ_h

Let (G, X) be a Shimura datum.

The cocharacter μ_h

Let (G, X) be a Shimura datum. Recall that X is a $G(\mathbb{R})$-conjugacy class of maps $h\colon \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$, and that for each such h we define the cocharacter over $\mathbb{C}$

$$\begin{aligned} \mu_h\colon \ \mathbb{G}_m &\longrightarrow \mathbb{G}_m \times \mathbb{G}_m \xrightarrow[\sim]{c} (\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m)_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}} \ ; \\ z &\longmapsto (z, 1) \end{aligned}$$

The cocharacter μ_h

Let (G, X) be a Shimura datum. Recall that X is a $G(\mathbb{R})$-conjugacy class of maps $h\colon \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$, and that for each such h we define the cocharacter over $\mathbb{C}$

$$\begin{aligned} \mu_h\colon\ \mathbb{G}_m &\longrightarrow \mathbb{G}_m \times \mathbb{G}_m \xrightarrow[\sim]{c} (\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m)_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}}\ ; \\ z &\longmapsto (z, 1) \end{aligned}$$

here c is the usual isomorphism whose inverse is induced by $R \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} R \times R$, $r \otimes z \mapsto (rz, r\overline{z})$, for any $\mathbb{C}$-algebra R.

The cocharacter μ_h

Let (G, X) be a Shimura datum. Recall that X is a $G(\mathbb{R})$-conjugacy class of maps $h\colon \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m \to G_{\mathbb{R}}$, and that for each such h we define the cocharacter over $\mathbb{C}$

$$\begin{aligned}\mu_h\colon\ \mathbb{G}_m &\longrightarrow \mathbb{G}_m \times \mathbb{G}_m \xrightarrow[\sim]{c} (\operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m)_{\mathbb{C}} \xrightarrow{h} G_{\mathbb{C}}\ ;\\ z &\longmapsto (z, 1)\end{aligned}$$

here c is the usual isomorphism whose inverse is induced by $R \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} R \times R$, $r \otimes z \mapsto (rz, r\overline{z})$, for any $\mathbb{C}$-algebra R.

Of course, the conjugacy class C of μ_h is independent of h.

The reflex field

Let T be a maximal torus in G and $W := N_G T(\overline{\mathbb{Q}})/Z_G T(\overline{\mathbb{Q}})$ its absolute Weyl group.

The reflex field

Let T be a maximal torus in G and $W := N_G T(\overline{\mathbb{Q}})/Z_G T(\overline{\mathbb{Q}})$ its absolute Weyl group. Then the natural maps

$$\left\{\begin{matrix} W\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, T_{\overline{\mathbb{Q}}}) \end{matrix}\right\} \to \left\{\begin{matrix} G(\overline{\mathbb{Q}})\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, G_{\overline{\mathbb{Q}}}) \end{matrix}\right\} \to \left\{\begin{matrix} G(\mathbb{C})\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, G_{\mathbb{C}}) \end{matrix}\right\}$$

are all isomorphisms. Hence we may view C as an element of the leftmost set.

The reflex field

Let T be a maximal torus in G and $W := N_G T(\overline{\mathbb{Q}})/Z_G T(\overline{\mathbb{Q}})$ its absolute Weyl group. Then the natural maps

$$\left\{\begin{array}{c} W\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, T_{\overline{\mathbb{Q}}}) \end{array}\right\} \to \left\{\begin{array}{c} G(\overline{\mathbb{Q}})\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, G_{\overline{\mathbb{Q}}}) \end{array}\right\} \to \left\{\begin{array}{c} G(\mathbb{C})\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, G_{\mathbb{C}}) \end{array}\right\}$$

are all isomorphisms. Hence we may view C as an element of the leftmost set.

Definition

The *reflex field* $E(G,X)$ *of* (G,X) is the field of definition of C, that is, the fixed field inside $\overline{\mathbb{Q}}$ of the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing C.

The reflex field

Let T be a maximal torus in G and $W := N_G T(\overline{\mathbb{Q}})/Z_G T(\overline{\mathbb{Q}})$ its absolute Weyl group. Then the natural maps

$$\left\{\begin{array}{c} W\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, T_{\overline{\mathbb{Q}}}) \end{array}\right\} \to \left\{\begin{array}{c} G(\overline{\mathbb{Q}})\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, G_{\overline{\mathbb{Q}}}) \end{array}\right\} \to \left\{\begin{array}{c} G(\mathbb{C})\text{-conjugacy} \\ \text{classes in} \\ \mathrm{Hom}(\mathbb{G}_m, G_{\mathbb{C}}) \end{array}\right\}$$

are all isomorphisms. Hence we may view C as an element of the leftmost set.

Definition

The *reflex field* $E(G, X)$ *of* (G, X) is the field of definition of C, that is, the fixed field inside $\overline{\mathbb{Q}}$ of the subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixing C.

If there is a representative of C defined over $E \subset \overline{\mathbb{Q}}$ — in particular if E splits T — then $E \supset E(G, X)$. Hence $E(G, X)$ is a number field.

Towards giving meaning to “canonical”

The notion of canonical that we shall formulate for Shimura varieties will be in terms of *special points* and how the Galois group acts on them.

Towards giving meaning to “canonical”

The notion of canonical that we shall formulate for Shimura varieties will be in terms of *special points* and how the Galois group acts on them.

We’ll need some preparation.

Global class field theory

Let E be a number field, and let E^{ab} denote its maximal abelian extension inside a fixed algebraic closure $\overline{E}$.

Global class field theory

Let E be a number field, and let E^{ab} denote its maximal abelian extension inside a fixed algebraic closure $\overline{E}$.

Global class field theory asserts the existence of a canonical continuous surjective map, called the *reciprocity map*,

$$\mathrm{rec}_E \colon E^\times \backslash \mathbb{A}_E^\times \longrightarrow \mathrm{Gal}(E^{\mathrm{ab}}/E),$$

Global class field theory

Let E be a number field, and let E^{ab} denote its maximal abelian extension inside a fixed algebraic closure $\overline{E}$.

Global class field theory asserts the existence of a canonical continuous surjective map, called the *reciprocity map*,

$$\mathrm{rec}_E \colon E^\times \backslash \mathbb{A}_E^\times \longrightarrow \mathrm{Gal}(E^{\mathrm{ab}}/E),$$

which induces a commutative diagram

$$\begin{tikzcd}
E^\times \backslash \mathbb{A}_E^\times \arrow[r, "\mathrm{rec}_E"] \arrow[d] & \mathrm{Gal}(E^{\mathrm{ab}}/E) \arrow[d] \\
E^\times \backslash \mathbb{A}_E^\times / \mathrm{Nm}_{L/E} \mathbb{A}_L^\times \arrow[r, "\mathrm{rec}_{L/E}", "\sim"'] & \mathrm{Gal}(L/E)
\end{tikzcd}$$

for every finite abelian Galois extension L/E.

Global class field theory

Let E be a number field, and let E^{ab} denote its maximal abelian extension inside a fixed algebraic closure $\overline{E}$.

Global class field theory asserts the existence of a canonical continuous surjective map, called the *reciprocity map*,

$$\mathrm{rec}_E \colon E^\times \backslash \mathbb{A}_E^\times \longrightarrow \mathrm{Gal}(E^{\mathrm{ab}}/E),$$

which induces a commutative diagram

$$\begin{tikzcd}
E^\times \backslash \mathbb{A}_E^\times \arrow[r, "\mathrm{rec}_E"] \arrow[d] & \mathrm{Gal}(E^{\mathrm{ab}}/E) \arrow[d] \\
E^\times \backslash \mathbb{A}_E^\times / \mathrm{Nm}_{L/E} \mathbb{A}_L^\times \arrow[r, "\mathrm{rec}_{L/E}", "\sim"'] & \mathrm{Gal}(L/E)
\end{tikzcd}$$

for every finite abelian Galois extension L/E.

We put

$$\mathrm{art}_E := \mathrm{rec}_E^{-1}, \qquad \mathrm{art}_{L/E} := \mathrm{rec}_{L/E}^{-1}$$

Special points

Let (G, X) be a Shimura datum.

Special points

Let (G, X) be a Shimura datum.

Definition

A point $h \in X$ is *special* if there exists a $\mathbb{Q}$-torus $T \subset G$ such that h factors through $T_{\mathbb{R}}$. For such h and T we call (T, h) a *special pair*.

Special points

Let (G, X) be a Shimura datum.

Definition

A point $h \in X$ is *special* if there exists a $\mathbb{Q}$-torus $T \subset G$ such that h factors through $T_{\mathbb{R}}$. For such h and T we call (T, h) a *special pair*.

Let (T, h) be a special pair and $E := E(\mu_h)$ the field of definition of μ_h.

Special points

Let (G, X) be a Shimura datum.

Definition

A point $h \in X$ is *special* if there exists a $\mathbb{Q}$-torus $T \subset G$ such that h factors through $T_{\mathbb{R}}$. For such h and T we call (T, h) a *special pair*.

Let (T, h) be a special pair and $E := E(\mu_h)$ the field of definition of μ_h. We get a composite map

$$\operatorname{Res}_{E/\mathbb{Q}} \mathbb{G}_m \xrightarrow{\operatorname{Res}_{E/\mathbb{Q}} \mu_h} \operatorname{Res}_{E/\mathbb{Q}} T_E \xrightarrow{\operatorname{Nm}_{E/\mathbb{Q}}} T,$$

Special points

Let (G, X) be a Shimura datum.

Definition

A point $h \in X$ is *special* if there exists a $\mathbb{Q}$-torus $T \subset G$ such that h factors through $T_{\mathbb{R}}$. For such h and T we call (T, h) a *special pair*.

Let (T, h) be a special pair and $E := E(\mu_h)$ the field of definition of μ_h. We get a composite map

$$\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m \xrightarrow{\mathrm{Res}_{E/\mathbb{Q}} \mu_h} \mathrm{Res}_{E/\mathbb{Q}} T_E \xrightarrow{\mathrm{Nm}_{E/\mathbb{Q}}} T,$$

where on R-points the norm map $\mathrm{Nm}_{E/\mathbb{Q}}$ sends $t \in T(R \otimes_{\mathbb{Q}} E)$ to $\prod_{\varphi : E \to \overline{\mathbb{Q}}} \varphi_*(t) \in T(R \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$, which lies in $T(R)$.

Special points

Let (G, X) be a Shimura datum.

Definition

A point $h \in X$ is *special* if there exists a $\mathbb{Q}$-torus $T \subset G$ such that h factors through $T_{\mathbb{R}}$. For such h and T we call (T, h) a *special pair*.

Let (T, h) be a special pair and $E := E(\mu_h)$ the field of definition of μ_h. We get a composite map

$$\operatorname{Res}_{E/\mathbb{Q}} \mathbb{G}_m \xrightarrow{\operatorname{Res}_{E/\mathbb{Q}} \mu_h} \operatorname{Res}_{E/\mathbb{Q}} T_E \xrightarrow{\operatorname{Nm}_{E/\mathbb{Q}}} T,$$

where on R-points the norm map $\operatorname{Nm}_{E/\mathbb{Q}}$ sends $t \in T(R \otimes_{\mathbb{Q}} E)$ to $\prod_{\varphi : E \to \overline{\mathbb{Q}}} \varphi_*(t) \in T(R \otimes_{\mathbb{Q}} \overline{\mathbb{Q}})$, which lies in $T(R)$. Taking $\mathbb{A}_{\mathbb{Q}}$-points and projecting down to $\mathbb{A}_{f,\mathbb{Q}}$-points, we get the composite, which we denote r_h,

$$\mathbb{A}_E^\times \xrightarrow{\operatorname{Res}_{E/\mathbb{Q}} \mu_h} T(\mathbb{A}_E) \xrightarrow{\operatorname{Nm}_{E/\mathbb{Q}}} T(\mathbb{A}_{\mathbb{Q}}) \longrightarrow T(\mathbb{A}_{f,\mathbb{Q}}).$$

Canonical models

Let $\mathrm{Sh}_K(G, X)$ denote the Shimura variety attached to (G, X) and the compact open subgroup $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$.

Canonical models

Let $\mathrm{Sh}_K(G, X)$ denote the Shimura variety attached to (G, X) and the compact open subgroup $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$. For $h \in X$ and $a \in G(\mathbb{A}_{f,\mathbb{Q}})$, let $[h, a]_K$ denote the class of (h, a) in $\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{f,\mathbb{Q}})/K$.

Canonical models

Let $\mathrm{Sh}_K(G, X)$ denote the Shimura variety attached to (G, X) and the compact open subgroup $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$. For $h \in X$ and $a \in G(\mathbb{A}_{f,\mathbb{Q}})$, let $[h, a]_K$ denote the class of (h, a) in $\mathrm{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{f,\mathbb{Q}})/K$.

For (T, h) a special pair in (G, X), recall that we have maps

$$\mathrm{art}_{E(\mu_h)} \colon \mathbb{A}^\times_{E(\mu_h)} \twoheadrightarrow \mathrm{Gal}\big(E(\mu_h)^{\mathrm{ab}}/E(\mu_h)\big),$$
$$r_h \colon \mathbb{A}^\times_{E(\mu_h)} \longrightarrow T(\mathbb{A}_{f,\mathbb{Q}}).$$

Canonical models

Let $\mathrm{Sh}_K(G,X)$ denote the Shimura variety attached to (G,X) and the compact open subgroup $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$. For $h \in X$ and $a \in G(\mathbb{A}_{f,\mathbb{Q}})$, let $[h,a]_K$ denote the class of (h,a) in $\mathrm{Sh}_K(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{f,\mathbb{Q}})/K$.

For (T,h) a special pair in (G,X), recall that we have maps

$$\mathrm{art}_{E(\mu_h)} \colon \mathbb{A}_{E(\mu_h)}^{\times} \twoheadrightarrow \mathrm{Gal}\big(E(\mu_h)^{\mathrm{ab}}/E(\mu_h)\big),$$
$$r_h \colon \mathbb{A}_{E(\mu_h)}^{\times} \longrightarrow T(\mathbb{A}_{f,\mathbb{Q}}).$$

Definition

A model $M_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ defined over $E(G,X)$ is *canonical* if for every special pair (T,h) in (G,X) and every $a \in G(\mathbb{A}_{f,\mathbb{Q}})$,

Canonical models

Let $\mathrm{Sh}_K(G,X)$ denote the Shimura variety attached to (G,X) and the compact open subgroup $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$. For $h \in X$ and $a \in G(\mathbb{A}_{f,\mathbb{Q}})$, let $[h,a]_K$ denote the class of (h,a) in $\mathrm{Sh}_K(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{f,\mathbb{Q}})/K$.

For (T,h) a special pair in (G,X), recall that we have maps

$$\mathrm{art}_{E(\mu_h)} \colon \mathbb{A}^{\times}_{E(\mu_h)} \twoheadrightarrow \mathrm{Gal}\big(E(\mu_h)^{\mathrm{ab}}/E(\mu_h)\big),$$
$$r_h \colon \mathbb{A}^{\times}_{E(\mu_h)} \longrightarrow T(\mathbb{A}_{f,\mathbb{Q}}).$$

Definition

A model $M_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ defined over $E(G,X)$ is *canonical* if for every special pair (T,h) in (G,X) and every $a \in G(\mathbb{A}_{f,\mathbb{Q}})$,

1. $[h,a]_K \in M_K(G,X)(\mathbb{C})$ is defined over $E(\mu_h)^{\mathrm{ab}}$;

Canonical models

Let $\mathrm{Sh}_K(G,X)$ denote the Shimura variety attached to (G,X) and the compact open subgroup $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$. For $h \in X$ and $a \in G(\mathbb{A}_{f,\mathbb{Q}})$, let $[h,a]_K$ denote the class of (h,a) in $\mathrm{Sh}_K(G,X)(\mathbb{C}) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_{f,\mathbb{Q}})/K$.

For (T,h) a special pair in (G,X), recall that we have maps

$$\mathrm{art}_{E(\mu_h)} \colon \mathbb{A}^{\times}_{E(\mu_h)} \twoheadrightarrow \mathrm{Gal}\big(E(\mu_h)^{\mathrm{ab}}/E(\mu_h)\big),$$
$$r_h \colon \mathbb{A}^{\times}_{E(\mu_h)} \longrightarrow T(\mathbb{A}_{f,\mathbb{Q}}).$$

Definition

A model $M_K(G,X)$ of $\mathrm{Sh}_K(G,X)$ defined over $E(G,X)$ is *canonical* if for every special pair (T,h) in (G,X) and every $a \in G(\mathbb{A}_{f,\mathbb{Q}})$,

1. $[h,a]_K \in M_K(G,X)(\mathbb{C})$ is defined over $E(\mu_h)^{\mathrm{ab}}$; and
2. for all $s \in \mathbb{A}^{\times}_{E(\mu_h)}$, we have

$$\mathrm{art}_{E(\mu_h)}(s) \cdot [h,a]_K = [h, r_h(s)a]_K.$$

Recall that $\mathrm{Sh}(G, X)$ is the limit of the inverse system $\big(\mathrm{Sh}_K(G, X)\big)_K$, and that $G(\mathbb{A}_{f,\mathbb{Q}})$ acts on the right via the rule

$$[h, a]_K \cdot g = [h, ag]_{g^{-1}Kg} \quad \text{for} \quad g \in G(\mathbb{A}_{f,\mathbb{Q}}).$$

Recall that $\mathrm{Sh}(G, X)$ is the limit of the inverse system $\big(\mathrm{Sh}_K(G, X)\big)_K$, and that $G(\mathbb{A}_{f,\mathbb{Q}})$ acts on the right via the rule

$$[h, a]_K \cdot g = [h, ag]_{g^{-1}Kg} \quad \text{for} \quad g \in G(\mathbb{A}_{f,\mathbb{Q}}).$$

Definition

A *model* of $Sh(G, X)$ over the field $E \subset \mathbb{C}$ is an inverse system of E-schemes $\big(M_K(G, X)\big)_K$ equipped with a right $G(\mathbb{A}_{f,\mathbb{Q}})$-action and a $G(\mathbb{A}_{f,\mathbb{Q}})$-equivariant isomorphism

$$\big(M_K(G, X)_{\mathbb{C}}\big)_K \xrightarrow{\sim} \big(\mathrm{Sh}_K(G, X)\big)_K.$$

Recall that $\mathrm{Sh}(G,X)$ is the limit of the inverse system $\big(\mathrm{Sh}_K(G,X)\big)_K$, and that $G(\mathbb{A}_{f,\mathbb{Q}})$ acts on the right via the rule

$$[h,a]_K \cdot g = [h,ag]_{g^{-1}Kg} \quad \text{for} \quad g \in G(\mathbb{A}_{f,\mathbb{Q}}).$$

Definition

A *model* of $Sh(G,X)$ over the field $E \subset \mathbb{C}$ is an inverse system of E-schemes $\big(M_K(G,X)\big)_K$ equipped with a right $G(\mathbb{A}_{f,\mathbb{Q}})$-action and a $G(\mathbb{A}_{f,\mathbb{Q}})$-equivariant isomorphism

$$\big(M_K(G,X)_{\mathbb{C}}\big)_K \xrightarrow{\sim} \big(\mathrm{Sh}_K(G,X)\big)_K. \qquad (*)$$

A model $\big(M_K(G,X)\big)_K$ of $Sh(G,X)$ is *canonical* if it is defined over $E(G,X)$ and the arrow $(*)$ makes each $M_K(G,X)$ a canonical model of $\mathrm{Sh}_K(G,X)$.

Existence and uniqueness

Theorem

For any Shimura datum (G, X), $\mathrm{Sh}(G, X)$ has a canonical model, and the canonical model is unique up to canonical isomorphism.

Example: the Siegel modular variety

Let (V, ψ) be a nondegenerate symplectic space over $\mathbb{Q}$, $G := GSp(\psi)$, and (G, X) the usual Shimura datum.

Example: the Siegel modular variety

Let (V, ψ) be a nondegenerate symplectic space over $\mathbb{Q}$, $G := GSp(\psi)$, and (G, X) the usual Shimura datum. Since G is split, $E(G, X) = \mathbb{Q}$.

Example: the Siegel modular variety

Let (V, ψ) be a nondegenerate symplectic space over $\mathbb{Q}$, $G := GSp(\psi)$, and (G, X) the usual Shimura datum. Since G is split, $E(G, X) = \mathbb{Q}$.

Let $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$ be a compact open subset.

Example: the Siegel modular variety

Let (V, ψ) be a nondegenerate symplectic space over $\mathbb{Q}$, $G := GSp(\psi)$, and (G, X) the usual Shimura datum. Since G is split, $E(G, X) = \mathbb{Q}$.

Let $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$ be a compact open subset. $\mathrm{Sh}_K(G, X)(\mathbb{C})$ can be identified with a moduli space of abelian varieties, and we'd like to understand the special points in terms of this.

Example: the Siegel modular variety

Let (V, ψ) be a nondegenerate symplectic space over $\mathbb{Q}$, $G := GSp(\psi)$, and (G, X) the usual Shimura datum. Since G is split, $E(G, X) = \mathbb{Q}$.

Let $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$ be a compact open subset. $\mathrm{Sh}_K(G, X)(\mathbb{C})$ can be identified with a moduli space of abelian varieties, and we'd like to understand the special points in terms of this.

More precisely, recall that $\mathrm{Sh}_K(G, X)(\mathbb{C}) \cong \mathscr{M}_K/\sim$, where $\mathscr{M}_K$ is the space of triples $(A, s, \eta K)$ such that

Example: the Siegel modular variety

Let (V, ψ) be a nondegenerate symplectic space over $\mathbb{Q}$, $G := GSp(\psi)$, and (G, X) the usual Shimura datum. Since G is split, $E(G, X) = \mathbb{Q}$.

Let $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$ be a compact open subset. $\mathrm{Sh}_K(G, X)(\mathbb{C})$ can be identified with a moduli space of abelian varieties, and we'd like to understand the special points in terms of this.

More precisely, recall that $\mathrm{Sh}_K(G, X)(\mathbb{C}) \cong \mathscr{M}_K/\sim$, where $\mathscr{M}_K$ is the space of triples $(A, s, \eta K)$ such that

- A is an abelian variety over $\mathbb{C}$ of dimension $\frac{1}{2}\dim_{\mathbb{Q}} V$;

Example: the Siegel modular variety

Let (V, ψ) be a nondegenerate symplectic space over $\mathbb{Q}$, $G := GSp(\psi)$, and (G, X) the usual Shimura datum. Since G is split, $E(G, X) = \mathbb{Q}$.

Let $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$ be a compact open subset. $\mathrm{Sh}_K(G, X)(\mathbb{C})$ can be identified with a moduli space of abelian varieties, and we'd like to understand the special points in terms of this.

More precisely, recall that $\mathrm{Sh}_K(G, X)(\mathbb{C}) \cong \mathscr{M}_K/\sim$, where $\mathscr{M}_K$ is the space of triples $(A, s, \eta K)$ such that

- A is an abelian variety over $\mathbb{C}$ of dimension $\frac{1}{2} \dim_{\mathbb{Q}} V$;
- s or $-s$ is a polarization of $H_1(A, \mathbb{Q})$; and

Example: the Siegel modular variety

Let (V, ψ) be a nondegenerate symplectic space over $\mathbb{Q}$, $G := GSp(\psi)$, and (G, X) the usual Shimura datum. Since G is split, $E(G, X) = \mathbb{Q}$.

Let $K \subset G(\mathbb{A}_{f,\mathbb{Q}})$ be a compact open subset. $\mathrm{Sh}_K(G, X)(\mathbb{C})$ can be identified with a moduli space of abelian varieties, and we'd like to understand the special points in terms of this.

More precisely, recall that $\mathrm{Sh}_K(G, X)(\mathbb{C}) \cong \mathscr{M}_K/\sim$, where $\mathscr{M}_K$ is the space of triples $(A, s, \eta K)$ such that

- A is an abelian variety over $\mathbb{C}$ of dimension $\frac{1}{2}\dim_{\mathbb{Q}} V$;
- s or $-s$ is a polarization of $H_1(A, \mathbb{Q})$; and
- η is an isomorphism $V \otimes_{\mathbb{Q}} \mathbb{A}_{f,\mathbb{Q}} \xrightarrow{\sim} H_1(A, \mathbb{A}_{f,\mathbb{Q}})$ sending ψ to an $\mathbb{A}_{f,\mathbb{Q}}^{\times}$-multiple of s.

To get at the answer, recall that a *CM-field* is a totally imaginary quadratic extension of a totally real finite extension of $\mathbb{Q}$. A *CM-algebra* is a finite product of CM fields.

To get at the answer, recall that a *CM-field* is a totally imaginary quadratic extension of a totally real finite extension of $\mathbb{Q}$. A *CM-algebra* is a finite product of CM fields. We say that an abelian variety over $\mathbb{C}$ is *CM* if there exists a CM-algebra E and an embedding $E \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $H_1(A, \mathbb{Q})$ is a free E-module of rank 1.

To get at the answer, recall that a *CM-field* is a totally imaginary quadratic extension of a totally real finite extension of $\mathbb{Q}$. A *CM-algebra* is a finite product of CM fields. We say that an abelian variety over $\mathbb{C}$ is *CM* if there exists a CM-algebra E and an embedding $E \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $H_1(A, \mathbb{Q})$ is a free E-module of rank 1.

Theorem

The triple $(A, s, \eta K)$ corresponds to a point $[h, a]_K \in \mathrm{Sh}_K(G, X)(\mathbb{C})$ with h special $\iff$ A is a CM-abelian variety.

To get at the answer, recall that a *CM-field* is a totally imaginary quadratic extension of a totally real finite extension of $\mathbb{Q}$. A *CM-algebra* is a finite product of CM fields. We say that an abelian variety over $\mathbb{C}$ is *CM* if there exists a CM-algebra E and an embedding $E \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $H_1(A, \mathbb{Q})$ is a free E-module of rank 1.

Theorem

The triple $(A, s, \eta K)$ corresponds to a point $[h, a]_K \in \mathrm{Sh}_K(G, X)(\mathbb{C})$ with h special $\iff$ A is a CM-abelian variety.

Theorem

The moduli space $\mathscr{M}_K/\!\sim$ is defined in a natural way over $\mathbb{Q}$ and in this way is a canonical model for $\mathrm{Sh}_K(G, X)$.

To get at the answer, recall that a *CM-field* is a totally imaginary quadratic extension of a totally real finite extension of $\mathbb{Q}$. A *CM-algebra* is a finite product of CM fields. We say that an abelian variety over $\mathbb{C}$ is *CM* if there exists a CM-algebra E and an embedding $E \hookrightarrow \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $H_1(A, \mathbb{Q})$ is a free E-module of rank 1.

Theorem

The triple $(A, s, \eta K)$ corresponds to a point $[h, a]_K \in \mathrm{Sh}_K(G, X)(\mathbb{C})$ with h special $\iff$ A is a CM-abelian variety.

Theorem

The moduli space $\mathscr{M}_K/\!\sim$ is defined in a natural way over $\mathbb{Q}$ and in this way is a canonical model for $\mathrm{Sh}_K(G, X)$.

Of course, the proof makes essential use of the theory of complex multiplication for abelian varieties.

We note that the the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $(\mathscr{M}_K/\sim)(\mathbb{C})$ admits a transparent expression in terms of triples $(A, s, \eta K)$. In particular, if we write

$$\sigma \cdot (A, s, \eta K) = (\sigma A, {}^{\sigma}s, {}^{\sigma}\eta K),$$

then σA is the abelian variety obtained from A by applying σ to the coefficients of the equations defining A.

Example: $GSpin(n, 2)$

Let (V, q) be a $\mathbb{Q}$-quadratic space of signature $(n, 2)$, $G := GSpin(q)$, and $\mathrm{Sh}(G, X)$ the associated Shimura variety as discussed in the previous talk.

Example: $GSpin(n, 2)$

Let (V, q) be a $\mathbb{Q}$-quadratic space of signature $(n, 2)$, $G := GSpin(q)$, and $\mathrm{Sh}(G, X)$ the associated Shimura variety as discussed in the previous talk. The reflex field $E(G, X) = \mathbb{Q}$.

Example: $GSpin(n, 2)$

Let (V, q) be a $\mathbb{Q}$-quadratic space of signature $(n, 2)$, $G := GSpin(q)$, and $\mathrm{Sh}(G, X)$ the associated Shimura variety as discussed in the previous talk. The reflex field $E(G, X) = \mathbb{Q}$.

The Shimura variety $\mathrm{Sh}(G, X)$ has the important property of being of *Hodge type*, which means that there exists an embedding

$$(G, X) \hookrightarrow \big(GSp(\psi), X(\psi)\big)$$

into the the Shimura datum attached to a rational symplectic space.

Example: $GSpin(n, 2)$

Let (V, q) be a $\mathbb{Q}$-quadratic space of signature $(n, 2)$, $G := GSpin(q)$, and $\mathrm{Sh}(G, X)$ the associated Shimura variety as discussed in the previous talk. The reflex field $E(G, X) = \mathbb{Q}$.

The Shimura variety $\mathrm{Sh}(G, X)$ has the important property of being of *Hodge type*, which means that there exists an embedding

$$(G, X) \hookrightarrow \big(GSp(\psi), X(\psi)\big)$$

into the the Shimura datum attached to a rational symplectic space.

It is a general fact that the $\mathbb{C}$-points of Shimura varieties of Hodge type can be described as a moduli space of abelian varieties which is similar in spirit to, but more complicated than, the Siegel case.

Example: $GSpin(n, 2)$

Let (V, q) be a $\mathbb{Q}$-quadratic space of signature $(n, 2)$, $G := GSpin(q)$, and $\mathrm{Sh}(G, X)$ the associated Shimura variety as discussed in the previous talk. The reflex field $E(G, X) = \mathbb{Q}$.

The Shimura variety $\mathrm{Sh}(G, X)$ has the important property of being of *Hodge type*, which means that there exists an embedding

$$(G, X) \hookrightarrow \big(GSp(\psi), X(\psi)\big)$$

into the the Shimura datum attached to a rational symplectic space.

It is a general fact that the $\mathbb{C}$-points of Shimura varieties of Hodge type can be described as a moduli space of abelian varieties which is similar in spirit to, but more complicated than, the Siegel case. One again has the fact that special points correspond to CM-abelian varieties.