Serre-Tate Theorems

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Serre-Tate Theorems

Serre-Tate "general" Theorem

In positive characteristic, there is an equivalence of categories relating

$$\left\{ \begin{array}{c} \text{deformations of an} \\ \text{abelian variety} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{deformations of its} \\ p\text{-divisible group} \end{array} \right\}$$

Serre-Tate Coordinates Theorem

For an *ordinary* abelian variety over a field k,

$$\left\{ \begin{array}{c} \text{local deformations} \\ \text{of } A \text{ to } R \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{choice of } g^2 \text{ ``local} \\ \text{coordinates'' in } 1 + \mathbf{m}_R \end{array} \right\}$$

Outline

Deformations of Abelian Schemes

Let A/S be an abelian scheme and let $S \to T$ be a morphism of schemes.

Definition

Then a *deformation of* A/S *to* T is an abelian scheme $\mathcal{A}/T$ together with an isomorphism over S:

$$F : \mathcal{A} \times_T S \cong A.$$

In other words, the following diagram commutes:

$$\begin{tikzcd}
F : \mathcal{A} \times_T S \cong A \arrow[r, dashed] \arrow[d, dashed] & \mathcal{A} \arrow[d] \\
S \arrow[r] & T.
\end{tikzcd}$$

We'll be interested in cases where

- $S = \mathrm{Spec}(R_0)$
- $T = \mathrm{Spec}(R)$
- $S \to T$ induced by $R \twoheadrightarrow R_0$

Definition

Such a deformation is also called a *lift* of A from R_0 to R. Even more, if R is local artinian and $R_0 = k$ is a field, then a deformation $\mathcal{A}/R$ is called an *infinitessimal (or local) deformation (or lift)*.

$$\begin{tikzcd} F : \mathcal{A} \otimes_R R_0 \cong A \arrow[r, dashed] \arrow[d, dashed] & \mathcal{A} \arrow[d] \\ \mathrm{Spec}(R_0) \arrow[r] & \mathrm{Spec}(R). \end{tikzcd}$$

Other deformation objects

Let G be an object defined over $\mathrm{Spec}(R_0)$, then we can also talk about deformations of G to R.

$$
\begin{tikzcd}
F:\ ? \otimes_R R_0 \cong G \arrow[r, dashed] \arrow[d, dashed] & ? \arrow[d] \\
\mathrm{Spec}(R_0) \arrow[r] & \mathrm{Spec}(R).
\end{tikzcd}
$$

- G is a p-divisible group
- G is an abelian scheme with additional structure (polarization, endomorphism...)

The category $\mathcal{C}_k$

Let k be a field and let

$$W := \begin{cases} k & \mathrm{char}(k) = 0 \\ W(k) & \mathrm{char}(k) > 0 \end{cases}$$

Objects

Pairs (R, ϕ)

- R is local artinian W-algebras
- $\phi : k \cong R/\mathbf{m}_R$ is a fixed isomorphism such that the following diagram commutes

$$\begin{tikzcd} W \arrow[r] \arrow[d] & R \arrow[d] \\ k \arrow[r, "\phi"] & R/\mathbf{m}_R \end{tikzcd}$$

The category $\mathcal{C}_k$

Let k be a field and let

$$W := \begin{cases} k & \mathrm{char}(k) = 0 \\ W(k) & \mathrm{char}(k) > 0 \end{cases}$$

Objects

Pairs (R, ϕ)

Morphisms

local W-algebra homomorphisms

Examples

- $k[\varepsilon]/(\varepsilon^n)$ when k has characteristic 0
- $W(k)/(p^n)$ when k has positive characteristic

Local Deformation Functor

Fix an abelian variety A/k.

$$\widehat{\mathcal{M}}_{A/k} : \mathcal{C}_k \longrightarrow \mathsf{Set}$$

$$R \longmapsto \{\text{deformations of } A \text{ to } R\}$$

Definition

$\widehat{\mathcal{M}}_{A/k}$ is called the *local (or infinitessimal) deformation functor* associated with A/k.

Representability of $\widehat{\mathcal{M}}_{A/k}$

Theorem (Grothendieck)

Let A/k be an abelian variety and $\widehat{\mathcal{M}}_{A/k}$ its associated local deformation functor. Then $\widehat{\mathcal{M}}_{A/k}$ is pro-representable by the complete, local noetherian ring

$$\mathcal{O} = W[[t_{11}, \ldots, t_{gg}]]$$

where g is the dimension of A.

Remark

We can also construct local deformation functors that carry "extra structure", say for (A, λ) with λ a principal polarization. In general, the local deformation functor is pro-representable by a $\mathcal{O}/\mathfrak{a}$ where $\mathfrak{a}$ is a suitably defined ideal depending on the extra structure. For example $\widehat{\mathcal{M}}_{(A,\lambda)}$ is pro-representable by

$$\mathcal{O} = W[[t_{11}, \ldots, t_{gg}]]/(\ldots, t_{ij} - t_{ji}, \ldots).$$

Serre-Tate "general" Theorem

In positive characteristic, there is an equivalence of categories relating

$$\left\{ \begin{array}{c} \text{deformations of an} \\ \text{abelian variety} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{deformations of its} \\ p\text{-divisible group} \end{array} \right\}$$

$\mathrm{Ab_R}$

Let R be a ring. Then we let $\mathrm{Ab_R}$ denote the category abelian schemes over R:

Objects

Abelian schemes over R; that is, smooth, proper group schemes $\pi : A \to R$ whose geometric fibres are connected.

Morphisms

Group homomorphisms over R

$\mathrm{Def}(R, R_0)$

Let R be a ring where p is nilpotent, and let $R \twoheadrightarrow R_0$ be a fixed surjective ring homomorphism. The category $\mathrm{Def}(R, R_0)$ is given by

Objects

Triples (A_0, G, ϵ) where

- A_0 is an abelian scheme over R_0
- G is a p-divisible group over R
- $\epsilon : G_0 \to A_0(p^\infty)$ an ismorphism of p-divisible groups over R_0

In other words, (G, ϵ) is a lift of the p-divisible group of A_0 to R.

Def(R, R$_0$)

Objects

Triples (A_0, G, ϵ)

Morphisms

A morphism $(A_0, G, \epsilon_A) \to (B_0, H, \epsilon_B)$ is a pair (f_0, g) where

- $f_0 : A_0 \to B_0$ morphism of abelian schemes$/R_0$
- $g : G \to H$ morphisms of p-divisible groups $/R$

such that $f_0(p^\infty) = g_0$; that is,

$$\begin{tikzcd} G_0 \arrow[r, "g_0"] \arrow[d, "\epsilon_A"'] & H_0 \arrow[d, "\epsilon_B"] \\ A_0(p^\infty) \arrow[r, "f_0(p^\infty)"'] & B_0(p^\infty) \end{tikzcd}$$

Serre-Tate Theorem

Theorem

Let R be a ring in which p is nilpotent, with $I \subset R$ a nilpotent ideal. Write $R_0 = R/I$. Then, there is an equivalence of categories given by

$$\mathrm{Ab}_\mathrm{R} \longrightarrow \mathrm{Def}(\mathrm{R}, \mathrm{R}_0)$$

$$A \longmapsto (A_0 = A \otimes_R R_0, A(p^\infty)).$$

Key facts

Let R be a ring in which p is nilpotent, with $I \subset R$ a nilpotent ideal. Write $R_0 = R/I$.
Suppose $I^{\nu+1} = 0$ and write $q = p^n$ for some n such that $p^n = 0$.

If G is an abelian scheme or p-divisible group over R:

- G is q-divisible as a group
- G_I is killed by q^ν where

$$G_I(S) := \ker(G(S) \to G(S/IS))$$

Lemma

If G, H are an abelian schemes or p-divisible groups over R:

1. $\mathrm{Hom}_R(G, H)$ and $\mathrm{Hom}_{R_0}(G_0, H_0)$ have no q-torsion

Proof

G and G_0 and q-divisible

Lemma

If G, H are an abelian schemes or p-divisible groups over R:

1. $\mathrm{Hom}_R(G, H)$ and $\mathrm{Hom}_{R_0}(G_0, H_0)$ have no q-torsion
2. Reduction modulo I is injective on hom sets:

$$\mathrm{Hom}\,(G, H) \hookrightarrow \mathrm{Hom}\,(G_0, H_0)$$

Proof

The kernel of this map is $\mathrm{Hom}\,(G, H_I)$, applying the "key facts" immediately shows this is trivial.

Lemma

If G, H are an abelian schemes or p-divisible groups over R:

1. $\mathrm{Hom}_R(G, H)$ and $\mathrm{Hom}_{R_0}(G_0, H_0)$ have no q-torsion
2. Reduction modulo I is injective on hom sets:

$$\mathrm{Hom}\,(G, H) \hookrightarrow \mathrm{Hom}\,(G_0, H_0)$$

3. For any $f_0 \in \mathrm{Hom}\,(G_0, H_0)$, $q^\nu f_0$ lifts uniquely to some $\widetilde{q^\nu f_0} \in \mathrm{Hom}\,(G, H)$

Proof

Let L be any lift $H(S/IS) \to H(S)$:

$$\begin{tikzcd} G(S) \arrow[rrr, dashed, "\widetilde{q^\nu f_0}"] \arrow[dr, "\mathrm{mod}\ I"'] & & & H(S) \\ & G(S/IS) \arrow[r, "f_0"] & H(S/IS) \arrow[ur, "q^\nu \times L"'] & \end{tikzcd}$$

Lemma

If G, H are an abelian schemes or p-divisible groups over R:

1. $\mathrm{Hom}_R(G, H)$ and $\mathrm{Hom}_{R_0}(G_0, H_0)$ have no q-torsion
2. Reduction modulo I is injective on hom sets:

$$\mathrm{Hom}\,(G, H) \hookrightarrow \mathrm{Hom}\,(G_0, H_0)$$

3. For any $f_0 \in \mathrm{Hom}\,(G_0, H_0)$, $q^\nu f_0$ lifts uniquely to some $\widetilde{q^\nu f_0} \in \mathrm{Hom}\,(G, H)$
4. $f_0 \in \mathrm{Hom}\,(G_0, H_0)$ lifts uniquely to some $f \in \mathrm{Hom}\,(G, H)$ if and only if $\widetilde{q^\nu f_0}$ annihilates $G[q^\nu]$.

Lemma

If G, H are an abelian schemes or p-divisible groups over R:

- $f_0 \in \mathrm{Hom}\,(G_0, H_0)$ lifts uniquely to some $f \in \mathrm{Hom}\,(G, H)$ if and only if $\widetilde{q^\nu f_0}$ annihilates $G[q^\nu]$

Proof

- Necessary: since f lifts f_0, then $q^\nu f = \widetilde{q^\nu f_0}$ by Lemma part 2
- Sufficient: since $\widetilde{q^\nu f_0}$ annihilates $G[q^\nu]$, construct F such that

$$\begin{tikzcd} & G \arrow[dl, dashed, "F"'] \arrow[d, "\widetilde{q^\nu f_0}"] & \\ H \arrow[r, "q^\nu"'] & H \arrow[r] & 0 \end{tikzcd}$$

$q^\nu F_0 = q^\nu f_0$ implies $F_0 = f_0$ since $\mathrm{Hom}\,(G_0, H_0)$ has no q-torsion.

Equivalence of categories

$$A \to \mathsf{Def}(A) = (A_0, A(p^\infty), \varepsilon_A)$$

Let $A, B \in \mathsf{Ab}_\mathsf{R}$ and $(f_0, f(p^\infty)) : \mathsf{Def}(A) \to \mathsf{Def}(B)$ in $\mathsf{Def}(\mathsf{R}, \mathsf{R}_0)$.

Faithful

$\mathsf{Hom}\,(A, B) \hookrightarrow \mathsf{Hom}\,(A_0, B_0)$ so any lift of f_0 is unique

Full

$$A \to \mathrm{Def}(A) = (A_0, A(p^\infty), \varepsilon_A)$$

Let $A, B \in \mathrm{Ab}_{\mathrm{R}}$ and $(f_0, f(p^\infty)) : \mathrm{Def}(A) \to \mathrm{Def}(B)$ in $\mathrm{Def}(\mathrm{R}, \mathrm{R}_0)$.

- A lift $F : A \to B$ of f_0 exists if and only if $\widetilde{q^\nu f_0}$ kills $A[q^\nu]$
- Necessarily, $\widetilde{q^\nu f_0}(p^\infty)$ lifts $q^\nu(f_0(p^\infty))$
- Uniqueness of lifts implies $\widetilde{q^\nu f_0}(p^\infty) = q^\nu f(p^\infty)$
- $\widetilde{q^\nu f_0}(p^\infty)$ kills $A(p^\infty)[q^\nu] = A[q^\nu]$
- For the lift $F : A \to B$ of f_0, $F(p^\infty)$ lifts $f_0(p^\infty)$ so $F(p^\infty) = f(p^\infty)$ (uniqueness of lifts)

Essentially Surjective

Take $(A_0, G, \varepsilon) \in \mathsf{Def}(\mathsf{R}, \mathsf{R}_0)$
Let B/R be a lift of A_0 via an isomorphism

$$\alpha_0 : B_0 = B \otimes_R R_0 \to A_0.$$

This induces an isomorphism of p-divisible groups,

$$\alpha_0(p^\infty) : B_0(p^\infty) \to A_0(p^\infty),$$

and, let

$$q^\nu \widetilde{\alpha_0(p^\infty)} : B(p^\infty) \to G$$

be the unique lift of $q^\nu \alpha_0(p^\infty)$.

Essentially surjective

$$\widetilde{q^\nu \alpha_0(p^\infty)} : B(p^\infty) \to G$$

is an isogeny, and it has an "inverse" in the sense that there exists a lift $\widetilde{q^\nu \alpha_0(p^\infty)}^{-1}$ of $q^\nu \alpha_0(p^\infty)^{-1}$ such that

$$\widetilde{q^\nu \alpha_0(p^\infty)}^{-1} \circ \widetilde{q^\nu \alpha_0(p^\infty)} = q^{2\nu}$$

and

$$\widetilde{q^\nu \alpha_0(p^\infty)} \circ \widetilde{q^\nu \alpha_0(p^\infty)}^{-1} = q^{2\nu}$$

Therefore, setting $K = \ker(\widetilde{q^\nu \alpha_0(p^\infty)})$, we obtain a finite, flat subgroup of $B[q^{2\nu}]$.

Essentially surjective

Taking $A = B/K$ gives Def$(A) = (A_0, G, \varepsilon)$:
By construction, K lifts $B_0[q^\nu]$, so A is a lift of

$$B_0/B_0[q^\nu] \cong B_0 \cong A_0.$$

Furthermore,

$$A(p^\infty) \cong B(p^\infty)/K \cong G.$$

Serre-Tate Local Coordinates Theorem

For an *ordinary* abelian variety over a field $k = \bar{k}$ and $R \in \mathcal{C}_k$,

$$\left\{ \begin{array}{c} \text{local deformations} \\ \text{of } A \text{ to } R \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{choice of } g^2 \text{ ``local} \\ \text{coordinates'' in } 1 + \mathbf{m}_R \end{array} \right\}$$

Ordinary Abelian Varieties

Definition

Let k be an algebraically closed field of characteristic $p > 0$. Then an abelian variety A of dimension g over k is *ordinary* if the following equivalent conditions are satisfied:

- $\# A[p](k) = p^g$
- The largest étale quotient of $A[p]$ has rank g.
- $A(p^\infty) \cong \hat{\mathbb{G}}_m^g \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^g$

Serre-Tate Coordinates

Theorem (Part 1)

Let A be an ordinary abelian variety over $k = \bar{k}$ of characteristic p, and let $T_pA = \varprojlim A[p^n]$ be its Tate module. Then

- *there is an isomorphism of functors on the category $\mathcal{C}_k$:*

$$\widehat{\mathcal{M}}_{A/k}(-) \cong \mathit{Hom}_{\mathbb{Z}_p}(T_pA(k) \otimes_{\mathbb{Z}_p} T_pA^\vee(k), \hat{\mathbb{G}}_m(-)).$$

Interpreting Part 1

What does this mean when we apply the isomorphism to a particular $R \in \mathcal{C}_k$?
We can show that $\hat{\mathbb{G}}_m(R) = 1 + \mathbf{m}_R$, and picking a basis for $T_pA(k)$ and $T_pA^\vee(k)$ as free $\mathbb{Z}_p$-modules:

$$\widehat{\mathcal{M}}_{A/k}(R) = \left\{ \begin{array}{c} \text{deformations} \\ \text{of } A \text{ to } R \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \mathbb{Z}_p\text{-bilinear maps} \\ q : T_pA(k) \times T_pA^\vee(k) \to 1 + \mathbf{m}_R \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \text{choice of } g^2 \text{ ``local} \\ \text{coordinates'' in } 1 + \mathbf{m}_R \end{array} \right\}$$

$$\mathcal{A}/R \longleftrightarrow q_{\mathcal{A}} : T_pA(k) \times T_pA^\vee(k) \to 1 + \mathbf{m}_R$$

Theorem (Part 2)

Let A *be an ordinary abelian over* $k = \bar{k}$ *of characteristic* p, *and let* $T_p(A) = \varprojlim A[p^n]$ *be its Tate module. Then*

- *For any lift* $\mathcal{A}/R$ *of* A/k, *its dual* $\mathcal{A}^\vee/R$ *is a lift of* $A^\vee/k$. *If* A *admits a principal polarization, we get the symmetry condition:*

  $$q_{\mathcal{A}}(x, y) = q_{\mathcal{A}^\vee}(y, x)$$

  for all $x \in T_pA(k), y \in T_pA^\vee(k)$.

Remark

We canonically identify $A \cong (A^\vee)^\vee$ through the principal polarization in order to consider $q_{\mathcal{A}^\vee}$ as a bilinear form on $T_pA^\vee(k) \times T_pA(k)$ as opposed to $T_pA^\vee(k) \times T_p(A^\vee)^\vee(k)$.

Interpreting Part 2

What does this mean when we apply the isomorphism to a particular $R \in \mathcal{C}_k$?

$$\widehat{\mathcal{M}}_{(A,\lambda)}(R) = \left\{ \begin{array}{c} \text{deformations} \\ \text{of } (A,\lambda) \text{ to } R \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \text{symmetric } \mathbb{Z}_p\text{-bilinear maps} \\ q : T_pA(k) \times T_pA^\vee(k) \to 1 + \mathbf{m}_R \end{array} \right\}$$

$$\longleftrightarrow \left\{ \begin{array}{c} \text{choice of } g(g+1)/2 \text{ ``local} \\ \text{coordinates'' in } 1 + \mathbf{m}_R \end{array} \right\}$$

$$(\mathcal{A}, \lambda_{\mathcal{A}})/R \longleftrightarrow q_{\mathcal{A}} : T_pA(k) \times T_pA^\vee(k) \to 1 + \mathbf{m}_R$$

Theorem (Part 3)

Let A be an ordinary abelian over $k = \bar{k}$ of characteristic p, and let $T_p(A) = \varprojlim A[p^n]$ be its Tate module. Then

- *for $R \in \mathcal{C}_k$, and let $\mathcal{A}/R$, $\mathcal{B}/R$ be deformations of two ordinary abelian varieties A/k and B/k respectively with corresponding bilinear forms $q_{\mathcal{A}}$ and $q_{\mathcal{B}}$. A homomorphism $f : A \to B$ extends to a homomorphism $\mathfrak{f} : \mathcal{A} \to \mathcal{B}$ if and only if*

$$q_{\mathcal{A}}(x, f^\vee(y)) = q_{\mathcal{B}}(f(x), y)$$

  for all $x \in T_pA(k), y \in T_pB^\vee(k)$.

Canonical Lifts

Definition

A lift of A/k to R is called the *canonical lift* of A to R and denoted $\mathcal{A}^{can}/R$ if its corresponding bilinear form is the trivial one.

$$\widehat{\mathcal{M}}_{A/k}(R) \longleftrightarrow \left\{ \begin{array}{c} \mathbb{Z}_p\text{-bilinear maps} \\ q : T_pA(k) \times T_pA^\vee(k) \to 1 + \mathbf{m}_R \end{array} \right\}$$

$$\mathcal{A}/R \longleftrightarrow q_{\mathcal{A}}$$

$$\mathcal{A}^{can}/R \longleftrightarrow 1$$

Lifting from characteristic p to characteristic 0

Let $k = \bar{\mathbb{F}}_p$. Since $R_n = W(k)/p^n W(k) \in \mathcal{C}_k$ for all $n \in \mathbb{N}$, one can form a limit object

$$\varinjlim \mathcal{A}^{can}/R_n$$

giving rise to a canonical lift of A to $W(k)$, $\mathcal{A}^{can}/W(k)$. So this is a case where lifts from characteristic p (k) to characteristic zero $(W(k))$ exist!
Again,

$$\mathcal{A}/W(k) \longleftrightarrow q_{\mathcal{A}}$$
$$\mathcal{A}^{can}/W(k) \longleftrightarrow q_{\mathcal{A}} = 1$$

By descent arguments one gets similar results for k non-algebraically closed fields. This means we can get lifts from finite fields to p-adic fields.

Characterization by Endomorphisms

Q: When is a lift $\mathcal{A}/W(k)$ the canonical lift?
Recall:

$$f \in \operatorname{End}(A) \text{ lifts } \Leftrightarrow q_{\mathcal{A}}(x, f^{\vee}(y)) = q_{\mathcal{A}}(f(x), y)$$

Observation: every endomorphism of A/k lifts to $\mathcal{A}^{can}/W(k)$.

Theorem

Let $A/\mathbb{F}_q$ and $k = \bar{\mathbb{F}}_q$. Then $\mathcal{A}/W(k) = \mathcal{A}^{can}/W(k)$ if and only if all endomorphisms of A lift to $\mathcal{A}$ if and only if Fr_q lifts.

Characterization by Endomorphisms

Theorem

Let $A/\mathbb{F}_q$ and $k = \bar{\mathbb{F}}_q$. Then $\mathcal{A}/W(k) = \mathcal{A}^{can}/W(k)$ if and only if all endomorphisms of A lift to $\mathcal{A}$ if and only if Fr_q lifts.

Idea of proof

For $f = Fr_q$, $f^\vee = Ver_q$. But Frobenius and Verschibung act very differently on μ_q and $\underline{\mathbb{Z}/q\mathbb{Z}}$, so the condition that

$$q_{\mathcal{A}}(x, f^\vee(y)) = q_{\mathcal{A}}(f(x), y)$$

can only be satisfied when $q_{\mathcal{A}}$ is trivial.

Moduli Space of Principally Polarized Abelian Varieties

Let $\mathcal{M}$ be the moduli functor of principally polarized abelian varieties with full level n structure ($n \geq 3$, $(n,p) = 1$).

- $\mathcal{M}$ is a fine moduli scheme
- If x is an ordinary closed point $x \in \mathcal{M}(k)$ corresponding to (A, λ), $\widehat{\mathcal{M}}_x = \widehat{\mathcal{M}}_{(A,\lambda)}$

By the Serre-Tate local coordinates theorem,

$$\widehat{\mathcal{M}}_x = \widehat{\mathcal{M}}_{(A,\lambda)} \cong \hat{\mathbb{G}}_m^{\frac{g(g+1)}{2}}.$$

This means that on the ordinary locus, $\mathcal{M}$ is non-singular of dimension $g(g+1)/2$.

Thank you!