Remarks on Assignment 2

October 22, 2009

I have noticed lack of rigor and careful presentation in some of the proofs. It is important not only to write a formally correct proof, but to make it so that the arguments are easy to follow. Make sure that you do this if you want to keep the grader happy.

Also, note that the following remarks may be not completely correct, since they haven't been proofread. So use at your own risk.

10. Be always careful when dealing with free groups. Sometimes they behave in counter-intuitive ways. So make sure that when you write statements about them you can produce proofs for those. If you want to show that a group is infinite, a good strategy is to show that it surjects onto arbitrarily-sized groups (or onto an infinite group). I saw two distinct solutions to this problem, and both are correct. Here is a sketch of them:

1. Define $f, g\colon \mathbf{Z}/2\mathbf{Z} \to S_{2n}$ by

$$f([1]) := \sigma_1 := (1,2)(3,4)\cdots(2n-1,2n),$$

and

$$g([1]) := \sigma_2 := (2,3)(4,5)\cdots(2n-2,2n-1)(2n,1).$$

They are well defined since $\sigma_i^2 = \mathrm{Id}$. One gets $f \star g\colon \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \to S_{2n}$ and $(f \star g)(\sigma_2\sigma_1)$ is seen to have order n. Hence the free product has at least order n. But n is arbitrary, so done.

2. Define

$$D := \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mid x, y \in \mathbf{Z} \right\} \subseteq \mathrm{GL}_2(\mathbf{Z}).$$

Then define $f, g\colon \mathbf{Z}/2\mathbf{Z} \to D$ by mapping $[1] \in \mathbf{Z}/2\mathbf{Z}$ respectively to $A_1 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and $A_2 := \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, which are also seen to be of order two. This gives as before a map $f \star g\colon \mathbf{Z}/2\mathbf{Z} \star \mathbf{Z}/2\mathbf{Z} \to D$, and one shows that:

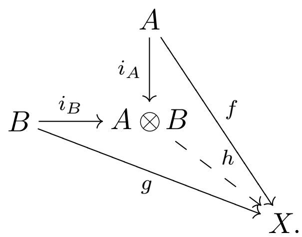
$$(A_1A_2)^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix},$$

so that the image of $f \star g$ generates a copy of $\mathbf{Z}$, hence $f \star g$ has infinite image, and we are done.

11. An important fact of the universal properties with which you are dealing in these exercises is that they assert the existence **and uniqueness** of a function h making some diagram commutative. The uniqueness of h is essential for this to determine uniquely the object that you are trying to define. Therefore, when proving that a certain construction is a universal object, you need to show that the arrows that you define are unique.

In the categories in which you are working, there is always an underlying set, and the morphisms in your categories are usually maps of sets with extra structure. I like to show uniqueness first, and then existence: namely, I try to see how the given maps force h to behave, show that this determines h using the extra structure, and finally showing that the map of sets that I have constructed actually respects the extra structure. Here is an illustration of this method:

Claim. *Let* $\mathbf{C}$ *be the category of rings with unity, with morphisms* $f\colon A \to B$ *taking* 1_A *to* 1_B*. Then the tensor product* $A \otimes_{\mathbf{Z}} B$ *(I will write* $A \otimes B$ *from now on) satisfies the following universal property. There are two maps* $i_A\colon A \to A \otimes B$ *and* $i_B\colon B \to A \otimes B$ *such that, for any pair of maps* $f\colon A \to X$ *and* $g\colon B \to X$ *there exists a unique* $h\colon A \otimes B \to X$ *making the following diagram commutative:*



Proof. I ommit the construction of the tensor product that you have probably seen. I just prove the property. So suppose f, g given. Then one must have $h \circ i_A = f$ and $h \circ i_B = g$. Hence, for all $a \in A$, one must have $h(a \otimes 1) = f(a)$, and for all $b \in B$, $h(1 \otimes b) = g(b)$. This means that, for $a \in A, b \in B$, since h has to be multiplicative (because we are trying to construct a ring homomorphism), we must have:

$$h(a \otimes b) = h((a \otimes 1) \cdot (1 \otimes b)) = h(a \otimes 1)h(1 \otimes b) = f(a)g(b).$$

This forces the image of any elementary tensor. But then, since h has to be additive:

$$h(\sum a_i \otimes b_i) = \sum h(a_i \otimes b_i) = \sum f(a_i)g(b_i).$$

One needs to check that this definition of h (which has been forced by the commutativity) is actually a ring homomorphism, and this is easy to do. □

19. Some of you have constructed a map $f\colon R/I_1 \otimes R/I_2 \to R/(I_1 + I_2)$. You then want to check that it is bijective. But be careful when showing injectivity, since it is not enough to do it only on elementary tensors. Sometimes, when dealing with tensor products, one can reduce to calculating on elementary tensors. This is the case when you want to show that the image of a map is 0, for example. Or that the tensor product itself is zero. But in general, one needs to deal with combinations of tensors. The obvious example is the following: let A be an integral domain, and let $m\colon A \otimes A \to A$ be defined as $m(a \otimes b) := ab$. Then $m(a \otimes b) = 0$ implies that $a = 0$ or $b = 0$ and so that $a \otimes b = 0$. However, m is not injective: $m(a \otimes b - b \otimes a) = 0$ and if $a, b \notin \mathbf{Z}$ the element $a \otimes b - b \otimes a$ will be nonzero.