Remarks on Assignment 1

September 30, 2009

2. Justify why you get the isomorphism $k[\mathbf{Z}/N\mathbf{Z}] \cong k[X]/(x^n - 1)$. Note also that the decomposition that you get at the end using the Chinese Remainder Theorem is an isomorphism of rings, not just of modules.

5. Remember to check exactness at the three terms, not just at the middle one. An alternative approach is to prove that, given any two maps $A \xrightarrow{f} B \xrightarrow{g} C$, then the corresponding sequence $A[S^{-1}] \xrightarrow{f'} B[S^{-1}] \xrightarrow{g'} C[S^{-1}]$ is exact, where this has to be checked of course only at the middle term (note the absence of zeroes at the extremes of the sequence). As an easy exercise, you may prove the following:

Lemma. *Let $F\colon \mathbf{C} \to \mathbf{D}$ be a covariant functor. Then the following are equivalent:*

1. *For each short exact sequence $0 \to A \to B \to C \to 0$, the induced sequence*

$$0 \to FA \to FB \to FC \to 0$$

   is exact.

2. *For each exact sequence $A \to B \to C$, the induced sequence $FA \to FB \to FC$ is exact.*

6. It is important to note that terminal (that is, initial or final) objects in a category are unique up to **unique** isomorphism. That means that given two such objects, they are isomorphic and that this isomorphism is unique. For example, any two sets with the same cardinality are isomorphic, but in general there are many isomorphisms between them. On the other hand, two singletons are isomorphic in a unique way!

7. I think that you underestimate the power of the characterization of equivalence of categories via fully faithfull and essentially-surjective functors. The point of this theorem is that it hides a heavy use of a strong version of the axiom of choice, with which you probably don't want to deal unless you specialize in logic. Some of you defined the functor in the "hard" direction, and almost none of you justified this. The functor that is easy to define is $F\colon \mathbb{M} \to \mathbb{V}$. Defining it in the other way involves choosing a basis for each vectorspace, and this is getting your hands unnecessarily dirty with set theory.

In the second part of the exercise you are given you a module over $k[x_1, x_2]$ which is finite-dimensional as a k-vectorspace. Namely, you are given $X = k[x_1, x_2]/(x_1^2, x_2^2)$. You need to say to which object in the other categories does it correspond. As a k-vectorspace, X is of dimension 4. A basis for it can be given by $\{1, x_1, x_2, x_1x_2\}$. Let V be the k-vectorspace given by X, but forgetting the module structure. Define $T_1, T_2\colon X \to X$ by the following matrices with respect to the given basis:

$$T_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

where T_i has been constructed by looking at how x_i acts on X. Then the data of X corresponds to the triple $(4, T_1, T_2)$ in $\mathbb{M}$, and to (V, x_1, x_2) in $\mathbb{V}$, where x_i are now thought as linear transformations $V \to V$.