Quiz 2 – Higher Algebra I, Math 570, Fall 2008

Date: October 29, 2008.
Time: 10:30 - 12:00.
Instructor: Prof. Eyal Goren.

Instructions: Answer as many questions as you can, but not at the cost of writing untidy solutions. You can answer a question assuming other questions (if needed!).

1. (45 points) Prove that an abelian group is an injective $\mathbb{Z}$-module if and only if it is a divisible abelian group.

2. (20 points) Prove that every abelian group can be embedded in an injective abelian group.

3. (35 points) Let R be a commutative integral domain and F its field of fractions. Prove that F is a flat R-module. (If you follow the proof we gave in class, that used a certain lemma, it is enough to only sketch the proof of the lemma.)

4. (20 points) Let R be a commutative ring and M_1, M_2 two projective R-modules. Prove that $M_1 \otimes_R M_2$ is a projective R-module.

5. (a) (10 points) Let R be a commutative integral domain and assume that R is an injective R-module. Prove that R is a field. (Hint: consider a diagram
$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d] & R \\ & R & \end{tikzcd}$$
, and choose the maps cleverly.).

(b) (10 points) Conversely, prove that if R is a field then R is an injective R-module. (Hint: you may need to use Zorn's lemma. In that case, I will allow you to be very sketchy in verifying that the conditions of Zorn's lemma are satisfied.)

6. (20 points) Let A be an abelian group that is both injective and projective $\mathbb{Z}$-module. Prove that $A = \{0\}$. (Hint: embed A in a free $\mathbb{Z}$-module.)