Quiz 1 – Higher Algebra I, Math 570, Fall 2008

Date: October 6, 2008.
Time: 11:30 - 13:00.
Instructor: Prof. Eyal Goren.

Instructions: Answer all questions. You can answer a part of a question assuming previous parts (if needed!). Getting a grade of 75 for this exam, means getting full marks (thus grades are multiplied by a factor of 100/75). You can in fact get up to 95 for this exam, which after rescaling gives a grade of 126.666....

1.

(1) (20 points) Let R be a ring and n a positive integer. Prove that the categories ${}_R\mathbf{Mod}$ and ${}_{M_n(R)}\mathbf{Mod}$ are equivalent.
(2) (10 points) Assume now that k is a field. Prove that $M_n(k)$ has a unique module V, up to isomorphism, such that $\dim_k(V) = n$. (V is a k-module via the embedding $k \hookrightarrow M_n(k), r \mapsto \mathrm{diag}(r, r, \dots, r)$.)

2.

(1) (10 points) Consider the following exact diagram of left R-modules,

$$A \xrightarrow{f} B \xrightarrow{g} C ,$$

(the "missing" morphism $A \to C$ is just $g \circ f$). The direct limit of this diagram has a simple expression; find it and prove your statement.
(2) (10 points) Consider the infinite diagram

$$\cdots \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z} \to \cdots ,$$

where all maps are multiplication by 2. Find the inverse and direct limit of this diagram in the category of abelian groups.
(3) (10 points) Consider the infinite diagram

$$\cdots \to \mathbb{Q} \to \mathbb{Q} \to \mathbb{Q} \to \cdots ,$$

where all maps are multiplication by 2. Find the inverse and direct limit of this diagram in the category of abelian groups.

3. Let R be ring and $A \in \mathbf{Mod}_R$.

(1) (15 points) Consider the functor $A \otimes_R (\cdot)$, a covariant functor from ${}_R\mathbf{Mod}$ to abelian groups. Prove that this functor is right exact.
(2) (10 points) Give an example showing that this functor need not be exact.
(3) (10 points) Prove that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} \cong \{0\}$.