MATH 570 - Higher Algebra I
Final Exam – December 9, 2008

Instructor: Prof. Eyal Goren

Time: 9:00 - 12:00.

Answer the following questions. You may use the statements made in some questions to solve another, as long as you avoid "circular reasoning".

(1) Let G be a finite group and k an algebraically closed field whose characteristic doesn't divide the order of G. In some of the questions you can use, if needed, that the number of irreducible representations of G is equal to the number of its conjugacy classes, and that the dimensions of the irreducible representations divide the order of G.

   (a) Prove that $k[G]$ is a semi-simple ring.

   (b) Let $V_1, \dots, V_r$ be representatives for the isomorphism classes of irreducible representations of G and let $\chi_1, \dots, \chi_r$ be their characters. Prove the orthogonality of characters:

$$\langle \chi_i, \chi_j \rangle = \delta_{ij}.$$

   (c) Prove that G is abelian if and only if every irreducible representation of G is one dimensional.

   (d) Prove that if G is not abelian it has at least 3 irreducible representations.

(2) This question is about the complex representations of the group D_6 (the symmetries of the hexagon), which is a group with 12 elements. A presentation for this group is

$$\langle x, y : x^2 = 1, y^6 = 1, xyxy = 1 \rangle.$$

(a) Prove that D_6 has 6 conjugacy classes and write them explicitly.

(b) Prove that D_6 has 4 distinct (irreducible) one-dimensional representations.

(c) Let ρ be a non-trivial representation of the cyclic subgroup $H = \langle y \rangle$. Prove that $\mathrm{Ind}_H^G(\rho)$ is irreducible if $\rho(y) \neq \pm 1$.

(d) Write the character table of the group D_6.

(e) Via the embedding $D_6 \subseteq S_6$ the S_6-representation $\mathrm{Perm}^0(S_6) = \{(x_1, \dots, x_6) \in k^6 : x_1 + \cdots + x_6 = 0\}$ becomes a 5-dimensional representation of D_6. Write the character of this representation and decompose it as a sum of irreducible characters.

(3) Prove that a finitely generated $\mathbb{Z}$-module M is projective if and only if it is torsion free. Show that this statement is false if M is not finitely generated. What direction remains true?

(4) Let R be an integral domain, Q a non-zero injective R-module and P a projective R-module. Suppose that $Q \subseteq P$. Prove that R is a field.

(5) Let **G** be the category of groups, **Ab** the category of abelian groups, **F** the category of fields and **R** the category of integral domains.

(a) Let H be the forgetful functor $Ab \to G$. Prove that F has a left adjoint.

(b) Does H have a right adjoint?

(c) Let H be the forgetful functor $F \to R$. Prove that H has a left adjoint.